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R-ORDERS IN A SPLIT ALGEBRA HAVE FINITELY MANY NON-ISOMORPHIC IRREDUCIBLE LATTICES AS SOON AS R HAS FINITE CLASS NUMBER

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Let R be a Dedekind domain with quotient field K and Λ an R-order in the finite-dimensional separable K-algebra A. If K is an algebraic number field with ring of integers R, then the Jordan–Zassenhaus theorem states that for every left A-module L, the set $S_L(M)=\{M: M=\Lambda\text{-lattice}, KM\cong L\}$ splits into a finite number of nonisomorphic Λ-lattices (cf. Zassenhaus [5]). The same statement holds if $R=k[x]$, $K=k(X)$, where k is a finite field and X an indeterminate over k (cf. Higman–MacLaughlin [1]). (This is true more general for orders in separable algebras over $\mathscr{A}$-fields (cf. Weil [6]).) The proofs of these theorems are based on the fact that in both cases, R has finite class number and finite residue class degrees. It follows from the results of Maranda [2] that a Jordan–Zassenhaus theorem is valid locally as soon as the residue class degrees of R are finite. Here we shall show that for any Dedekind domain and any R_p-order Λ_p in the split K-algebra A, R_p being the localization of R at some prime ideal p of K, there are only finitely many nonisomorphic irreducible Λ_p-lattices (using a result of Roggenkamp [4]). With a theorem of Maranda [2], we can globalize this fact in case R has finite class number. Simple examples show that a Jordan–Zassenhaus type theorem need not hold though the number of nonisomorphic irreducible Λ-lattices is finite as soon as R has an infinite residue class field.

LEMMA. *Let R be a local Dedekind domain with quotient field K and Λ an R-order in the separable finite-dimensional K-algebra A. Then there are only finitely many different maximal R-orders in A containing Λ.*

Proof. By $\hat{X}$ we denote the completion of an R-module X. Since the maximal R-orders in A containing Λ are in a one-to-one correspondence with the maximal $\hat{R}$-orders in $\hat{A}$ containing $\hat{\Lambda}$, it suffices to prove the lemma in case R is a complete discrete rank one valuation ring. Let $\{\Gamma_i\}_{i=1,2\ldots}$ be an infinite set of maximal R-orders in A containing Λ. Then we have a descending chain of Λ-lattices

$$\Gamma_1 \supset \Gamma_1 \cap \Gamma_2 \supset \cdots \supset \bigcap_{i=1}^{n} \Gamma_i \supset \cdots \supset \bigcap_{i=1,2,\ldots} \Gamma_i.$$

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If $\pi^s\Gamma_1 \subset \Lambda$, where π is the informizing parameter, then we have the descending chain of Λ-modules

$$\Lambda \supset \pi^s\Gamma_1 \supset \pi^s(\Gamma_1 \cap \Gamma_2) \supset \cdots \supset \pi^s\Lambda.$$

However, $\Lambda/\pi^s\Lambda$ is an artinian ring, and thus this chain has to terminate; i.e. there exists $n_0 \in \mathbf{N}$ such that

$$\Gamma_0 = \bigcap_{i=1}^{n_0} \Gamma_i = \bigcap_{i=1}^{n_0+s} \Gamma_i \quad \text{for } s = 1, 2, \ldots.$$

In particular, we conclude that Γ_0 is contained in infinitely many maximal R-orders. However, if $A = \bigoplus_{i=1}^t A_i$ is the decomposition of A into simple K-algebras, then Γ_0—as the intersection of maximal orders—decomposes accordingly, say $\Gamma_0 = \bigoplus_{i=1}^t \Gamma_{0_i}$. At least one Γ_{0_i} is contained in infinitely many maximal orders. Thus we may assume A to be simple, even central simple, since Γ_0 is the intersection of maximal orders. Now we have the following situation:

$A = (D)_n$ is a central simple K-algebra, D a central skewfield over A. (We view D as embedded into $(D)_n$ diagonally, and this embedding is fixed in the sequel.)

$$\Gamma_0 = \bigcap_{i=1}^{n} \Sigma_i,$$

where Σ_i are maximal R-orders in A,

$$\Gamma_0 \subset \Gamma_i, \quad i = 1, 2, \ldots,$$

where the $\{\Gamma_i\}$ are different maximal R-orders in A. Ω is the unique maximal R-order in D.

Let Γ be a maximal R-order in A. We put

$$\Omega_{ij}(\Gamma) = \{\omega \in D: \omega \text{ occurs at the } (i,j)\text{-position of some } \gamma \in \Gamma \subset (D)_n\}.$$

Then $\Omega_{ij}(\Gamma)$ is a two-sided Ω-ideal in D.

Claim: Γ is uniquely determined by $\{\Omega_{ij}\}_{1 \le i, j \le n}$.

Proof. Let

$$e_i = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & \cdots & \\ & & & \vdots & 0 & \\ & & & & & \ddots \\ & & & i & & & 0 \end{pmatrix}^{n \times n} \cdots\ i.$$

Then $e_i \in \Gamma$, $1 \le i \le n$. In fact, let $M_i = \Gamma e_i$. Then $\mathrm{End}_\Gamma(M_i) \cong \Omega$, and one shows easily that $e_i \in \mathrm{End}_\Omega(M_i)$. However, M_i is a progenerator for the category of

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Γ-lattices; i.e. $\mathrm{End}_\Omega(M_i)=\Gamma$ and $e_i \in \Gamma$. We shall show next, that $\omega \in \Omega_{ij}$ implies $\omega E_{ij} \in \Gamma$, where E_{ij} is the matrix with 1 at the (i,j)-position and zeros elsewhere.

Let $\gamma \in \Gamma$ be the element where ω stands at the (i,j)-position. Then $e_i\gamma e_j=\omega E_{ij} \in \Gamma$. Thus, γ is uniquely determined by $\{\Omega_{ij}\}_{1\le i,j\le n}$. This proves the claim.

Now we continue with the proof of the lemma. Since all maximal R-orders in A are conjugate by a regular element in A, we may assume that $\Sigma_1=(\Omega)_n$. Thus, there exists a positive integer t such that

$$(\omega_0^t\Omega)_n \subset \Gamma_i, \quad i = 1, 2, \ldots,$$

where $\omega_0\Omega=\mathrm{rad}\ \Omega$. We shall show that this cannot happen for infinitely many Γ_i. By the claim, there exists an index (k, l) and an infinite subset of maximal orders $\{\Gamma_{i_\rho}\}_{\rho=1,2\ldots}\subset\{\Gamma_i\}_{i=1,2\ldots}$ such that

$$\Omega_{k1}(\Gamma_{i_\rho}) = \omega_0^{-t_\rho}\Omega,$$

where t_i is a strictly increasing chain of positive integers. We now choose ρ such that $t_\rho>2t$. Because of the claim we have

$$\omega_0^{-t_\rho}E_{k1} \in \Gamma_{i_\rho}.$$

But $\Gamma_{i_\rho}\supset(\omega_0^t\Omega)_n$, and thus

$$\omega_0^tE_{jk}\omega_0^{-t_\rho}E_{k1}\omega_0^tE_{l1} \in \Gamma_{i_\rho};$$

i.e. $\omega_0^{-1}E_{jj} \in \Gamma$; i.e. $\omega_0^{-1}1 \in \Gamma$. But the reduced norm of $\omega_0^{-1}1$ is not integral over R as is easily seen. Thus we have obtained a contradiction, since every element in Γ is integral over R and hence its reduced norm must be integral. This proves the lemma.

THEOREM 1. *Let R be a discrete rank one valuation ring with quotient field K and assume that A is split by K. If Λ is an R-order in A, then there are only finitely many nonisomorphic irreducible Λ-lattices.*

Proof. We have shown in Roggenkamp [4], that if A is split by K, there is a one-to-one correspondence between the nonisomorphic irreducible Λ-lattices and the different maximal R-orders in A containing Λ. Now the result follows from the lemma.

THEOREM 2. *Let R be a Dedekind domain with quotient field K. Assume that K has finite class number. If Λ is an R-order in the split K-algebra A, then there are only finitely many nonisomorphic irreducible Λ-lattices.*

Proof. We recall that two Λ-lattices lie in the same genus if they are locally isomorphic. By Theorem 1 there are only finitely many genera of irreducible Λ-lattices. However, Maranda [2] has shown that there is a one-to-one correspondence between the ideal classes of K and the number of nonisomorphic Λ-lattices in the same genus as an irreducible Λ-lattice. Whence the statement follows.

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REMARKS. (i) We observe that Theorems 1 and 2 can also be formulated for orders in any algebra, if one only considers lattices which span absolutely irreducible modules over the algebra (cf. Roggenkamp [4]).

(ii) From Theorem 1 we cannot conclude that the Jordan–Zassenhaus theorem is valid for the category of Λ-lattices. In fact if A has two simple components A_1 and A_2 and if M and N are irreducible Λ-lattices such that $A_1M\neq 0$ and $A_1N\neq 0$, then there are infinitely many nonisomorphic extensions of M by N provided

(1) $\mathrm{Ext}^1_\Lambda(M, N)$ decomposes as R-module;

(2) R has an infinite residue class field. (This is an immediate consequence of a formula of Reiner [3, Lemma 6] cf. example below.)

(iii) Theorems 1 and 2 are not valid any longer, if we drop the hypothesis that K splits A (cf. example below).

EXAMPLE. (i) Let R be a discrete rank one valuation ring with infinite residue class field and uniformizing parameter π. Let T be a finite separable extension field of the quotient field K of R and denote by S the ring of integers in T. Assume furthermore that $\pi S=P_1P_2$, where P_1 and P_2 are different prime ideals in T. By "^" we denote the π-adic completion. Then

$$\hat{T} = \hat{T}_1 \oplus \hat{T}_2 \quad \text{and} \quad \hat{S} = \hat{S}_1 \oplus \hat{S}_2.$$

We write

$$S_1 = \bigoplus_{i=1}^{n} \hat{R}\omega_i^{(1)}, \quad \omega_1^{(1)} = 1,$$

$$S_2 = \bigoplus_{i=1}^{n} \hat{R}\omega_i^{(2)}, \quad \omega_1^{(2)} = 1.$$

Assume that $n\geq 3$. Then we consider the following $\hat{R}$-order in $\hat{A}=T_1 \oplus T_2$:

$$\hat{\Lambda} = \left\{\left(\sum_{i=1}^{n} r_i\omega_i^{(1)}, \sum_{i=3}^{n} r_i'\omega_i^{(2)}+(r_1+\hat{\pi}r_1')\omega_1^{(2)}+(r_2+\hat{\pi}r_2')\omega_2^{(2)}\right), \hat{r}_i, \hat{r}_i' \in \hat{R}\right\}.$$

Let e_1 and e_2 be the central idempotents in $\hat{A}$. We put $\hat{M}_1=\hat{\Lambda}e_1$, $\hat{M}_2=\hat{\Lambda}e_2$, and claim

$$\mathrm{Ext}^1_{\hat{\Lambda}}(\hat{M}_1, \hat{M}_2) \simeq \hat{R}/\hat{\pi}\hat{R} \oplus \hat{R}/\hat{\pi}\hat{R}.$$

To show this, we consider the exact sequence

$$0 \to \mathrm{Ker}\,\varphi \to \hat{\Lambda} \xrightarrow{\varphi} \hat{M}_1 \to 0,$$

where $\varphi\colon \lambda \mapsto \lambda e_1$, $\lambda \in \Lambda$. Then

$$\mathrm{Ker}\,\varphi = \hat{\pi}\hat{R}\omega_1^{(2)} \oplus \hat{\pi}\hat{R}\omega_2^{(2)} \oplus \sum_{i=3}^{n} \oplus \hat{R}\omega_i^{(2)},$$

and

$$\mathrm{Ext}^1_{\hat{\Lambda}}(\hat{M}_1, \hat{M}_2) \simeq \mathrm{Hom}_{\hat{\Lambda}}(\mathrm{Ker}\,\varphi, \hat{M}_2)/\mathrm{Im}\,\mathrm{Hom}(\varphi, 1_{\hat{M}_2}) \simeq \hat{R}/\hat{\pi}\hat{R} \oplus \hat{R}/\hat{\pi}\hat{R}.$$

The above-mentioned formula of Reiner states now: Among the exact sequences

$$0 \to \hat{M}_2 \to \hat{X} \to \hat{M}_1 \to 0$$

there are $1+\text{card}\,(\hat{R}/\hat{\pi}\hat{R})$ nonisomorphic $\hat{\Lambda}$-modules $\hat{X}$; in particular there are infinitely many such nonisomorphic $\hat{X}$. Now we put $\Lambda = T \cap \hat{\Lambda}$; then Λ is an R-order, and all the Λ-lattices $X = \hat{X} \cap T$ are nonisomorphic and irreducible. Whence Theorems 1 and 2 break down if one omits the hypothesis that K splits A.

(ii) A similar example shows that under the hypotheses of Theorem 1, the Jordan–Zassenhaus theorem cannot be valid. Let R be a discrete rank one valuation ring with infinite residue class field. Consider the K-algebra

$$A = K \oplus (K)_2,$$

and in this the R-order

$$\Lambda = \{(r, rE_2 + (\pi r_{ij})), r \in R, r_{ij} \in R\},$$

where E_2 denotes the two-dimensional identity matrix. Let

$$M_1 = \Lambda e_1^{(1)}, \qquad M_2 = \Lambda e_1^{(2)},$$

when $e_1^{(1)} = (1, 0)$ and $e_1^{(2)} = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$. Then it is easily seen that

$$\operatorname{Ext}^1_\Lambda (M_1, M_2) \cong R/\pi R \oplus R/\pi R,$$

and again an application of Reiner's result shows that there are infinitely exact sequences

$$0 \to M_2 \to X \to M_1 \to 0$$

with nonisomorphic middle terms. If Γ is the maximal R-order $\Gamma = R \oplus (R)_2$, then $\pi\Gamma \subset \Lambda$, and this example shows that there are infinitely many nonisomorphic Λ-lattices between $\Gamma M_1 \oplus \Gamma M_2$ and $\pi\Gamma M_1 \oplus \pi\Gamma M_2$; Γ being a principal ideal ring.

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