



EXPLOSION OF CONTINUOUS-STATE BRANCHING PROCESSES WITH COMPETITION IN A LÉVY ENVIRONMENT

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Abstract

We find sufficient conditions on explosion/non-explosion for continuous-state branching processes with competition in a Lévy random environment. In particular, we identify the necessary and sufficient conditions on explosion/non-explosion when the competition function is a power function and the Lévy measure of the associated branching mechanism is stable.

Keywords: Continuous-state branching processes; competition; random environment; explosion

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1. Introduction

Continuous-state branching processes in a random environment arise as scaling limits of Bienaymé–Galton–Watson processes, which were introduced in Smith [26] and Smith and Wilkinson [27]; see Kurtz [13] for early work on diffusion approximations of branching processes in a random environment. A recent study of the Feller branching processes in a Brownian environment can be found in Böinghoff and Hutzenthaler [6], where the asymptotics of the survival probability are studied for different regimes. The introduction of branching processes under the continuous-state setting allows us to apply the stochastic differential equation (SDE) and Lévy process techniques in its study. We refer to Kyprianou [14] and Li [20] for a comprehensive introduction to continuous-state branching processes and the associated stochastic equations.

To understand the effect of a random environment on the demography of the branching process, a continuous-state branching process with catastrophes was first proposed in Bansaye *et al.* [3] as a continuous-state branching process in a Lévy environment (CBLE) where the random environment is modelled by a Lévy process with sample paths of bounded variation. More general CBLEs were introduced and studied in He *et al.* [12] and Palau and Pardo [24] as unique non-negative strong solutions to certain SDEs driven by Brownian motions and Lévy processes associated to both the branching mechanism and the random environment. We refer

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to Bansaye *et al.* [2] for discussions on the convergence of discrete-state population models to CBLEs.

The quenched Laplace transform for the branching process in a random environment can be expressed using random cumulant semigroups conditional on the environment. He *et al.* [12] showed a necessary and sufficient condition in terms of Grey's condition for the CBLE to become extinct. Bansaye *et al.* [4] also obtained the speed of extinction for CBLEs for which the Lévy environment process fluctuates.

In another development on continuous-state branching processes, Lambert [15] introduced a logistic branching process to incorporate competition among individuals in the continuous-state branching process. Foucart [11] studied the boundary behaviour of continuous-state branching processes with logistic competition and obtained an integral test on explosion/non-explosion. A general competition mechanism was introduced in Ba and Pardoux [1] and Ma [21]. Under the moment condition $\int_0^\infty (z \wedge z^2)\mu(dz) < \infty$ on the Lévy measure μ for the branching mechanism, Ma [21] established the Lamperti transformation between continuous-state branching processes with competition and strong solutions of stochastic equations driven by Lévy processes without negative jumps; see also Berestycki *et al.* [5] for flows of continuous-state branching processes with competition. We refer to Li *et al.* [18] for recent work on ergodic results of continuous-state branching processes with immigration and competition. The continuous-state branching process with immigration and competition in a Lévy random environment was introduced in [24] with its long-term behaviours studied. The extinction and coming down from infinity behaviours have also been studied in Leman and Pardo [16] for CBLEs with competition.

The explosion/non-explosion conditions for continuous-state branching processes are well known: see Grey [10] for an integral test on the Laplace exponent of the associated branching mechanism. An integral test on explosion/non-explosion was further proved in Leman and Pardo [17] for a continuous-state branching process in a Brownian environment with a special branching mechanism that is associated to the Laplace transform of a subordinator and with logistic competition. It was also pointed out that a continuous-state branching process in a Lévy environment is conservative, i.e. the explosion cannot happen, if the Lévy measure μ for the branching mechanism satisfies the moment condition; see Lemma A.1 of [4]. On the other hand, it is known that sufficiently large competition can prevent an explosion from happening; see Foucart [11] and Li *et al.* [19]. Some sufficient conditions on explosion were found in [19] for general continuous-state non-linear branching processes whose competition mechanism is a general function and for which the Lévy measure μ for the branching mechanism satisfies the moment condition. To the best of our knowledge, the explosion/non-explosion conditions for CBLEs with general competition and with general Lévy measure μ have not been studied systematically.

Integral tests on explosion/non-explosion are no longer available for the above-mentioned branching processes with general competition, and as an effective alternative, an approach initiated by Mufa Chen comes into play. Such an approach finds successful applications in characterizing the boundary behaviours of SDEs related to the continuous-state branching processes; see Li *et al.* [19] and Ma *et al.* [22]. In this paper, applying the above-mentioned approach to suitable test functions, we find sufficient conditions on explosion/non-explosion for CBLEs with general competition. In particular, we identify necessary and sufficient conditions on explosion/non-explosion when the competition function is a power function and the jump part of the branching mechanism is an α -stable process for $\alpha \in (0, 2)$, which helps to determine the interplay between competition and large jumps of the branching on the

explosion. As a corollary we also show that Neveu’s CBLE with competition cannot explode. These results suggest that the random environment can neither cause the explosion nor prevent the explosion from happening.

The rest of the paper is arranged as follows. In Section 2 we introduce the CBLE with competition and the approach for showing explosion and non-explosion of CBLEs with competition. Our main results are stated and proved in Section 3.

2. CBLEs with competition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses. Let ϕ be a branching mechanism given by

$$\phi(\lambda) = -b_1\lambda + b_2^2\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z < 1\}})\mu(dz), \quad \lambda \geq 0,$$

where $b_1, b_2 \in \mathbb{R}$ and $(1 \wedge z^2)\mu(dz)$ is a finite measure on $(0, \infty)$. To model the mechanism of a random environment, let $(L(t))_{t \geq 0}$ be a Lévy process with Lévy–Itô decomposition

$$\begin{aligned} L(t) &= \beta t + \sigma B^{(e)}(t) + \int_0^t \int_{[-1,1]} (e^z - 1)\tilde{N}^{(e)}(ds, dz) \\ &\quad + \int_0^t \int_{[-1,1]^c} (e^z - 1)N^{(e)}(ds, dz), \end{aligned}$$

where $\beta \in \mathbb{R}, \sigma \geq 0, (B^{(e)}(t))_{t \geq 0}$ is a Brownian motion, and $N^{(e)}(ds, dz)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $ds\nu(dz)$ satisfying $\int_{\mathbb{R}} (1 \wedge z^2)\nu(dz) < \infty$ and $\tilde{N}^{(e)}(ds, dz) = N^{(e)}(ds, dz) - ds\nu(dz)$.

Let $b_0(y)$ be a competition mechanism, that is, $y \mapsto b_0(y)$ is a continuous non-decreasing function on $[0, \infty)$ with $b_0(0) = 0$. A CBLE with competition can be constructed as the unique strong solution of the following stochastic equation:

$$\begin{aligned} Y_t &= Y_0 + \int_0^t (b_1 Y_s - b_0(Y_s)) ds + \int_0^t \sqrt{2b_2^2 Y_s} dB^{(b)}(s) + \int_0^t \int_0^1 \int_0^{Y_{s-}} z \tilde{N}^{(b)}(ds, dz, du) \\ &\quad + \int_0^t \int_1^\infty \int_0^{Y_{s-}} z N^{(b)}(ds, dz, du) + \int_0^t Y_{s-} dL(s), \end{aligned} \tag{1}$$

where $(B^{(b)}(t))_{t \geq 0}$ is a Brownian motion, $N^{(b)}(ds, dz, du)$ is a Poisson random measure on \mathbb{R}_+^3 with intensity $ds\mu(dz) du$, and $\tilde{N}^{(b)}(ds, dz, du) = N^{(b)}(ds, dz, du) - ds\mu(dz) du$. We also assume that $(B^{(b)}(t))_{t \geq 0}, (B^{(e)}(t))_{t \geq 0}, N^{(b)}(ds, dz, du)$ and $N^{(e)}(ds, dz)$ are independent of each other.

For $u \geq 0$, let

$$\tau_u^- := \inf\{t \geq 0 : Y(t) \leq u\} \quad \text{and} \quad \tau_u^+ := \inf\{t \geq 0 : Y(t) \geq u\}$$

and

$$\tau_0 := \tau_0^- \quad \text{and} \quad \tau_\infty := \lim_{u \rightarrow \infty} \tau_u^+$$

with the convention $\inf \emptyset = \infty$. Throughout this paper, we use the notation

$$\mathbb{P}_{y_0}\{\cdot\} = \mathbb{P}\{\cdot \mid Y_0 = y_0\} \quad \text{and} \quad \mathbb{E}_{y_0}[\cdot] = \mathbb{E}[\cdot \mid Y_0 = y_0], \quad y_0 \geq 0.$$

A $[0, \infty]$ -valued process $(Y_t)_{t \geq 0}$ with càdlàg path is a solution to SDE (1) if it satisfies (1) up to explosion time τ_∞ and $Y_t := \infty$ for all $t \geq \tau_\infty$. It is known that SDE (1) has a unique non-negative strong solution; see Theorem 1 of [24].

Let L be the generator of the process $(Y_t)_{t \geq 0}$. By Itô's formula, we get for $g \in C^2(\mathbb{R})$

$$\begin{aligned}
 Lg(y) &= [\beta y + b_1 y - b_0(y)]g'(y) + \left(\frac{1}{2}\sigma^2 y^2 + b_2^2 y\right)g''(y) \\
 &+ y \int_0^1 [g(y+z) - g(y) - g'(y)z]\mu(dz) + y \int_1^\infty [g(y+z) - g(y)]\mu(dz) \\
 &+ \int_{[-1,1]} [g(ye^z) - g(y) - y(e^z - 1)g'(y)]\nu(dz) \\
 &+ \int_{[-1,1]^c} [g(ye^z) - g(y)]\nu(dz). \tag{2}
 \end{aligned}$$

In this paper we adopt arguments that were first developed by Mufa Chen to classify the boundaries for Markov jump processes via conditions on the generators; see Chen [7, 8] and Theorems 2.25 and 2.27 of Chen [9]. Also see Meyn and Tweedie [23] for more recent results. These techniques are applied in Li *et al.* [19], Ma *et al.* [22], and Ren *et al.* [25] to study the boundary behaviours for SDEs associated to continuous-state branching processes. By a simple modification of the proof of Proposition 2.1 in [25], we have the following proposition on solution Y to SDE (1).

Proposition 1. *If there exist a sequence of strictly positive constants $(d_n)_{n \geq 1}$ and non-negative functions $g_n \in C^2((0, \infty))$ satisfying, for all large enough $n \geq 1$,*

- (i) $\lim_{y \rightarrow \infty} g_n(y) = \infty$,
- (ii) $Lg_n(y) \leq d_n g_n(y)$ for all $y \in [1/n, \infty)$,

then $\mathbb{P}_{y_0}\{\tau_\infty < \infty\} = 0$ for any $y_0 > 0$.

Proposition 2. *If there exist a non-negative bounded and strictly increasing function $g \in C^2((0, \infty))$ and positive constants $d_0, \bar{y} > 0$ satisfying*

$$Lg(y) \geq d_0 g(y) \quad \text{for all } y \geq \bar{y},$$

then $\mathbb{P}_{y_0}\{\tau_\infty < \infty\} > 0$ for any $y_0 > \bar{y}$.

Proof. Taking $X_t = t$ and $\tilde{g}(x, y) = g(y) e^{-d_0 x}$ in Proposition 2.2 in Ren *et al.* [25], we have, for any $m > \bar{y}$,

$$\begin{aligned}
 t \mapsto M_t &:= g(Y_{t \wedge \tau_m^+ \wedge \tau_{\bar{y}}^-}) e^{-d_0 t} - g(Y_0) + \int_0^t g(Y_{s \wedge \tau_m^+ \wedge \tau_{\bar{y}}^-}) d_0 e^{-d_0 s} ds \\
 &- \int_0^t e^{-d_0 s} Lg(Y_s) 1_{\{s \leq \tau_m^+ \wedge \tau_{\bar{y}}^-\}} ds
 \end{aligned}$$

is a local martingale. Then, for any $m > y_0 > \bar{y}$,

$$\begin{aligned} & \mathbb{E}_{y_0} [g(Y_{t \wedge \tau_m^+ \wedge \tau_{\bar{y}}^-}) e^{-d_0 t}] + \int_0^t \mathbb{E}_{y_0} [d_0 e^{-d_0 s} g(Y_{s \wedge \tau_m^+ \wedge \tau_{\bar{y}}^-})] ds \\ &= g(y_0) + \int_0^t \mathbb{E}_{y_0} [e^{-d_0 s} Lg(Y_s) 1_{\{s \leq \tau_m^+ \wedge \tau_{\bar{y}}^-\}}] ds. \end{aligned}$$

Letting $t \rightarrow \infty$, by the assumptions and the dominated convergence theorem we have

$$\begin{aligned} \int_0^\infty \mathbb{E}_{y_0} [d_0 e^{-d_0 s} g(Y_{s \wedge \tau_m^+ \wedge \tau_{\bar{y}}^-})] ds &= g(y_0) + \int_0^\infty \mathbb{E}_{y_0} [e^{-d_0 s} Lg(Y_s) 1_{\{s \leq \tau_m^+ \wedge \tau_{\bar{y}}^-\}}] ds \\ &\geq g(y_0) + \int_0^\infty \mathbb{E}_{y_0} [e^{-d_0 s} d_0 g(Y_s) 1_{\{s \leq \tau_m^+ \wedge \tau_{\bar{y}}^-\}}] ds, \end{aligned}$$

which implies

$$\begin{aligned} g(y_0) &\leq \mathbb{E}_{y_0} \left[\int_{\tau_m^+ \wedge \tau_{\bar{y}}^-}^\infty d_0 e^{-d_0 s} g(Y_{\tau_m^+ \wedge \tau_{\bar{y}}^-}) ds \right] \\ &= \mathbb{E}_{y_0} [g(Y_{\tau_m^+ \wedge \tau_{\bar{y}}^-}) e^{-d_0(\tau_m^+ \wedge \tau_{\bar{y}}^-)}] \\ &= \mathbb{E}_{y_0} [g(Y_{\tau_m^+}) e^{-d_0 \tau_m^+} 1_{\{\tau_m^+ < \tau_{\bar{y}}^-\}}] + \mathbb{E}_{y_0} [g(Y_{\tau_{\bar{y}}^-}) e^{-d_0 \tau_{\bar{y}}^-} 1_{\{\tau_m^+ > \tau_{\bar{y}}^-\}}]. \end{aligned}$$

Since $t \mapsto Y_t$ is right continuous, then $Y_{\tau_{\bar{y}}^-} \leq \bar{y} < y_0 < m \leq Y_{\tau_m^+}$. Notice that g is non-negative bounded and strictly increasing. Then

$$g(y_0) \leq \bar{g} \mathbb{E}_{y_0} [1_{\{\tau_m^+ < \tau_{\bar{y}}^-\}} e^{-d_0 \tau_m^+}] + g(\bar{y}),$$

where $\bar{g} := \sup_y g(y) < \infty$. Letting $m \rightarrow \infty$, we get

$$g(y_0) \leq \bar{g} \mathbb{E}_{y_0} [1_{\{\tau_\infty \leq \tau_{\bar{y}}^-\}} e^{-d_0 \tau_\infty}] + g(\bar{y}).$$

That is,

$$\bar{g} \mathbb{E}_{y_0} [1_{\{\tau_\infty \leq \tau_{\bar{y}}^-\}} 1_{\{\tau_\infty < \infty\}}] \geq g(y_0) - g(\bar{y}),$$

which implies

$$\mathbb{P}_{y_0} \{\tau_\infty < \infty\} \geq \frac{g(y_0) - g(\bar{y})}{\bar{g}} > 0.$$

This proves the desired result. □

3. Main results

In this section we provide the sufficient conditions for explosion and non-explosion of the CBLE with competition. Let $(Y_t)_{t \geq 0}$ be the unique strong solution of (1).

Let $B(p, q)$ denote the Beta function with parameters $p, q > 0$. By integration by parts and L'Hôpital's rule, it is not hard to see the following.

Lemma 1. For any $\delta, y > 0$ and $\alpha \in (0, 1)$, we have

$$\int_0^\infty [(y+z)^{-\delta} - y^{-\delta}]z^{-1-\alpha} dz = -\delta c_{\alpha,\delta} y^{-\alpha-\delta} \tag{3}$$

and

$$\int_0^\infty [\ln(y+z) - \ln y]z^{-1-\alpha} dz = c_{\alpha,0} y^{-\alpha}, \tag{4}$$

where $c_{\alpha,\delta} := \alpha^{-1}B(\alpha + \delta, 1 - \alpha)$.

Remark 1. Note that $c_{\alpha,0} = \pi/(\alpha \sin(\alpha\pi))$.

For two σ -finite measures μ_1 and μ_2 on $(0, \infty)$, we write $\mu_1(dz) \leq \mu_2(dz)$ if $\mu_1(B) \leq \mu_2(B)$ for any Borel set B in $(0, \infty)$. We first present a sufficient condition on explosion of the solution Y to SDE (1).

Theorem 1. Suppose that there exist constants $b_0 \geq 0, q_0 \in \mathbb{R}, \bar{a}, A > 0$, and $\alpha \in (0, 1)$ such that

$$b_0(y) \leq b_0 y^{q_0} \quad \text{for all } y \geq A \quad \text{and} \quad \bar{a} z^{-1-\alpha} \mathbf{1}_{\{z \geq A\}} dz \leq \mathbf{1}_{\{z \geq A\}} \mu(dz).$$

Then $\mathbb{P}_{y_0} \{\tau_\infty < \infty\} > 0$ for large enough $y_0 > 0$ if one of the following conditions holds:

- (i) $b_0 = 0$,
- (ii) $q_0 < 2 - \alpha$ and $b_0 > 0$,
- (iii) $q_0 = 2 - \alpha$ and $0 < b_0 < \bar{a}c_{\alpha,0}$.

Proof. Without loss of generality we can assume that $A > 1$. Given $\delta \in (0, \infty)$, let $g(y) = e^{-y^{-\delta}}$ for $y \geq 0$. Then

$$g'(y) = \delta y^{-\delta-1} g(y) \quad \text{and} \quad g''(y) = [\delta^2 y^{-2\delta-2} - \delta(1 + \delta)y^{-\delta-2}]g(y).$$

It follows that $g'(y) > 0$ and $g''(y) > -\delta(1 + \delta)y^{-\delta-2}g(y)$. By Taylor's formula,

$$\begin{aligned} \int_0^1 [g(y+z) - g(y) - zg'(y)]\mu(dz) &= \frac{1}{2} \int_0^1 z^2 g''(\xi_1)\mu(dz) \\ &\geq -\frac{1}{2}\delta(1 + \delta)y^{-\delta-2}g(y+1) \int_0^1 z^2 \mu(dz) \end{aligned} \tag{5}$$

for some $\xi_1 \in [y, y + 1]$, and

$$\int_1^A [g(y+z) - g(y)]\mu(dz) \geq 0. \quad (6)$$

Moreover, by the assumptions and (3), we have

$$\begin{aligned} & \int_A^\infty [g(y+z) - g(y)]\mu(dz) \\ & \geq \bar{a} \int_A^\infty [g(y+z) - g(y)]z^{-1-\alpha} dz \\ & = \bar{a}g(y) \int_A^\infty [e^{-(y+z)^{-\delta} + y^{-\delta}} - 1]z^{-1-\alpha} dz \\ & \geq \bar{a}g(y) \int_A^\infty [-(y+z)^{-\delta} + y^{-\delta}]z^{-1-\alpha} dz \\ & = \bar{a}g(y) \int_0^\infty [-(y+z)^{-\delta} + y^{-\delta}]z^{-1-\alpha} dz - \bar{a}g(y) \int_0^A [-(y+z)^{-\delta} + y^{-\delta}]z^{-1-\alpha} dz \\ & = \bar{a}g(y)\delta c_{\alpha,\delta}y^{-\alpha-\delta} - \bar{a}g(y)\delta \xi_2^{-1-\delta} \int_0^A z^{-\alpha} dz \\ & \geq g(y)\delta \bar{a}c_{\alpha,\delta}y^{-\alpha-\delta} - g(y)\delta \bar{a}(1-\alpha)^{-1}A^{1-\alpha}y^{-1-\delta}, \end{aligned} \quad (7)$$

where we used the mean-value theorem for the last equality and $\xi_2 \in [y, y + A]$. In view of (5)–(7) we get

$$\begin{aligned} & y \int_0^1 [g(y+z) - g(y) - zg'(y)]\mu(dz) + y \int_1^\infty [g(y+z) - g(y)]\mu(dz) \\ & \geq g(y)\delta \bar{a}c_{\alpha,\delta}y^{1-\alpha-\delta} - g(y)\delta \bar{a}(1-\alpha)^{-1}A^{1-\alpha}y^{-\delta} \\ & \quad - \frac{1}{2}g(y+1)\delta(1+\delta)y^{-\delta-1} \int_0^1 z^2\mu(dz). \end{aligned} \quad (8)$$

On the other hand, it is obvious that for any fixed $\delta > 0$ there exists a large enough $y_\delta > 0$ such that $g''(y) < 0$ for all $y > y_\delta$. Since $|e^z - 1| \leq 3|z|$ for $z \in [-1, 1]$, then by Taylor's formula, for all $y > ey_\delta$,

$$\begin{aligned} & \int_{[-1,1]} [g(ye^z) - g(y) - y(e^z - 1)g'(y)]\nu(dz) \\ & \geq \frac{9}{2}y^2 \left[g''(\xi_3) \int_{-1}^0 z^2\nu(dz) + g''(\xi_4) \int_0^1 z^2\nu(dz) \right] \end{aligned}$$

for some $\xi_3 \in [ye^{-1}, y]$ and $\xi_4 \in [y, ye]$. This together with $g''(y) > -\delta(1 + \delta)y^{-\delta-2}$ yields, for all $y > ey_\delta$,

$$\begin{aligned} & \int_{[-1,1]} [g(ye^z) - g(y) - y(e^z - 1)g'(y)]\nu(dz) \\ & \geq -\frac{9}{2}\delta(1 + \delta)y^2 \left[\xi_3^{-\delta-2} \int_{-1}^0 z^2 \nu(dz) + \xi_4^{-\delta-2} \int_0^1 z^2 \nu(dz) \right] \\ & \geq -\frac{9}{2}\delta(1 + \delta)y^{-\delta} \left[e^{\delta+2} \int_{-1}^0 z^2 \nu(dz) + \int_0^1 z^2 \nu(dz) \right]. \end{aligned} \tag{9}$$

Moreover, since g is strictly increasing and takes values in $[0, 1]$, we have

$$\int_1^\infty [g(ye^z) - g(y)]\nu(dz) \geq 0.$$

Indeed,

$$\begin{aligned} \int_1^\infty [g(ye^z) - g(y)]\nu(dz) &= \int_1^\infty [e^{-(ye^z)^{-\delta}} - e^{-y^{-\delta}}]\nu(dz) \\ &= g(y) \int_1^\infty [e^{-(y^{-\delta}e^{-\delta z})+y^{-\delta}} - 1]\nu(dz) \\ &= g(y) \int_1^\infty [e^{y^{-\delta}(1-e^{-\delta z})} - 1]\nu(dz) \\ &\rightarrow 0 \quad \text{as } y \rightarrow \infty. \end{aligned}$$

It follows that

$$\int_{[-1,1]^c} [g(ye^z) - g(y)]\nu(dz) \geq \int_{-\infty}^{-1} [g(ye^z) - g(y)]\nu(dz) \geq -g(y)\nu((-\infty, -1]). \tag{10}$$

Combining (2) and (8)–(10), we have, for all y large enough,

$$\begin{aligned} Lg(y) &= [\beta y + b_1 y - b_0(y)]\delta y^{-\delta-1}g(y) + \left(\frac{1}{2}\sigma^2 y^2 + b_2^2 y\right) [\delta^2 y^{-2\delta-2} - \delta(1 + \delta)y^{-\delta-2}]g(y) \\ &+ y \int_0^1 [g(y+z) - g(y) - g'(y)z]\mu(dz) + y \int_1^\infty [g(y+z) - g(y)]\mu(dz) \\ &+ \int_{[-1,1]} [g(ye^z) - g(y) - y(e^z - 1)g'(y)]\nu(dz) + \int_{[-1,1]^c} [g(ye^z) - g(y)]\nu(dz) \\ &\geq g(y)\delta \left[(\beta + b_1)y^{-\delta} - b_0 y^{q_0-1-\delta} - \frac{1}{2}\sigma^2(1 + \delta)y^{-\delta} - b_2^2(1 + \delta)y^{-1-\delta} \right. \\ &+ \bar{a}c_{\alpha,\delta}y^{1-\alpha-\delta} - \bar{a}(1 - \alpha)^{-1}A^{1-\alpha}y^{-\delta} \\ &- g(y)^{-1}g(y+1)(1 + \delta)y^{-\delta-1} \int_0^1 z^2 \mu(dz) \\ &\left. - \delta^{-1}\nu((-\infty, -1]) - \frac{9}{2}(1 + \delta)y^{-\delta} e^{y^{-\delta}} e^{\delta+2} \int_{-1}^1 z^2 \nu(dz) \right] \end{aligned}$$

$$\begin{aligned}
 &= g(y)\delta[\bar{a}c_{\alpha,\delta}y^{1-\alpha-\delta} - b_0y^{q_0-1-\delta} - \delta^{-1}\nu((-\infty, -1]) - O(y^{-\delta})] \\
 &=: g(y)\delta G_\delta(y),
 \end{aligned}
 \tag{11}$$

where $O(y^{-\delta}) \rightarrow 0$ as $y \rightarrow \infty$ for any $\delta > 0$.

Since $\alpha < 1$, we can first choose δ small enough such that $1 - \alpha - \delta > 0$. If condition (ii) holds, then $1 - \alpha - \delta > q_0 - 1 - \delta$. Therefore $G_\delta(y) \rightarrow \infty$ as $y \rightarrow \infty$ under condition (i) or (ii). If condition (iii) holds, we can choose δ small enough such that $1 - \alpha - \delta > 0$ and $b_0 < \bar{a}c_{\alpha,\delta}$, then we also have $G_\delta(y) \rightarrow \infty$ as $y \rightarrow \infty$. This together with (11) implies that there is a \bar{y} large enough such that $Lg(y) \geq g(y)$ for all $y \geq \bar{y}$. By Proposition 2 we obtain the desired result. \square

We next present a sufficient condition on non-explosion of process Y .

Theorem 2. *Suppose that there exist constants $b_0 \geq 0$, $q_0 \in \mathbb{R}$, $\bar{a}, A > 0$, and $\alpha \in (0, 2)$ such that*

$$b_0(y)\mathbf{1}_{\{\alpha < 1\}} \geq b_0y^{q_0}\mathbf{1}_{\{\alpha < 1\}} \quad \text{for all } y \geq A \quad \text{and} \quad \mathbf{1}_{\{z \geq A\}}\mu(dz) \leq \bar{a}z^{-1-\alpha}\mathbf{1}_{\{z \geq A\}} dz.$$

Then $\mathbb{P}_{y_0}\{\tau_\infty < \infty\} = 0$ for any $y_0 > 0$ if one of the following conditions holds:

- (i) $\alpha \geq 1$,
- (ii) $q_0 > 2 - \alpha > 1$ and $b_0 > 0$,
- (iii) $q_0 = 2 - \alpha > 1$ and $b_0 \geq \bar{a}c_{\alpha,0}$.

Proof. For $k \geq 2$, we consider the following stochastic equation:

$$\begin{aligned}
 Y_t^{(k)} &= Y_0^{(k)} + \int_0^t (\beta Y_s^{(k)} + b_1 Y_s^{(k)} - b_0(Y_s^{(k)})) ds + \int_0^t \sqrt{2b_2^2 Y_s^{(k)}} dB^{(b)}(s) \\
 &+ \int_0^t \sigma Y_s^{(k)} dB^{(e)}(s) + \int_0^t \int_0^1 \int_0^{Y_{s-}^{(k)}} z\tilde{N}^{(b)}(ds, dz, du) \\
 &+ \int_0^t \int_1^\infty \int_0^{Y_{s-}^{(k)}} zN^{(b)}(ds, dz, du) + \int_0^t \int_{[-1,1]} Y_{s-}^{(k)}(e^z - 1)\tilde{N}^{(e)}(ds, dz) \\
 &+ \int_0^t \int_{(-\infty,-1) \cup (1,k]} Y_{s-}^{(k)}(e^z - 1)N^{(e)}(ds, dz).
 \end{aligned}
 \tag{12}$$

By Theorem 1 in Palau and Pardo [24], for any $k \geq 2$, equation (12) has a unique strong solution $(Y_t^{(k)})_{t \geq 0}$. Clearly, $(Y_t^{(k)})_{t \geq 0}$ consists in truncation of large jumps due to environment. Let L_k be the generator of $(Y_t^{(k)})_{t \geq 0}$. Then

$$\begin{aligned}
 L_k g(y) &= (\beta y + b_1 y - b_0(y))g'(y) + \left(\frac{1}{2}\sigma^2 y^2 + b_2^2 y\right)g''(y) \\
 &+ y \int_0^1 [g(y+z) - g(y) - g'(y)z]\mu(dz) + y \int_1^\infty [g(y+z) - g(y)]\mu(dz)
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{[-1,1]} [g(ye^z) - g(y) - y(e^z - 1)g'(y)]\nu(dz) \\
 & + \int_{(-\infty,-1)\cup(1,k]} [g(ye^z) - g(y)]\nu(dz).
 \end{aligned}
 \tag{13}$$

We first prove that for any fixed $k \geq 2$, process $(Y_t^{(k)})_{t \geq 0}$ does not explode. Without loss of generality we assume $A > 1$.

For $n \geq 9$, let $g_n \in C^2((0, \infty))$ be a non-decreasing function with $g_n(y) = \ln \ln(n^2y)$ for $y \geq 1/(ne)$ and $g_n(y) = 0$ for $y \leq 1/(2ne)$. Then, for any $y \geq 1/n$,

$$g'_n(y) = (\ln n^2y)^{-1}y^{-1} > 0 \quad \text{and} \quad g''_n(y) = -(\ln n^2y)^{-2}y^{-2} - (\ln n^2y)^{-1}y^{-2} < 0.$$

By Taylor’s formula and the above it follows that

$$\int_0^1 [g_n(y+z) - g_n(y) - zg'_n(y)]\mu(dz) \leq 0 \tag{14}$$

and

$$\int_1^A [g_n(y+z) - g_n(y)]\mu(dz) \leq y^{-1}(\ln n^2y)^{-1} \int_1^A z\mu(dz) \tag{15}$$

for $y \geq 1/n$. By the assumption on μ we have

$$\int_A^\infty [g_n(y+z) - g_n(y)]\mu(dz) \leq \bar{a} \int_A^\infty [g_n(y+z) - g_n(y)]z^{-1-\alpha} dz. \tag{16}$$

If $\alpha \geq 1$, by integration by parts and L’Hôpital’s rule we get for $y \geq 1/n$,

$$\begin{aligned}
 & \int_A^\infty [g_n(y+z) - g_n(y)]z^{-1-\alpha} dz \\
 & \leq \int_A^\infty [g_n(y+z) - g_n(y)]z^{-2} dz \\
 & = A^{-1}[\ln \ln n^2(y+A) - \ln \ln n^2y] + \int_A^\infty \frac{1}{z(y+z) \ln n^2(y+z)} dz \\
 & \leq (y \ln n^2y)^{-1} + (\ln n^2y)^{-1} \int_A^\infty \frac{1}{z(y+z)} dz \\
 & = y^{-1}(\ln n^2y)^{-1}[1 + \ln(1+y/A)].
 \end{aligned}
 \tag{17}$$

From (14)–(17) we get for $y \geq 1/n$,

$$\begin{aligned}
 & y \int_0^1 [g_n(y+z) - g_n(y) - zg'_n(y)]\mu(dz) + y \int_1^\infty [g_n(y+z) - g_n(y)]\mu(dz) \\
 & \leq y \int_1^A [g_n(y+z) - g_n(y)]\mu(dz) + y \int_A^\infty [g_n(y+z) - g_n(y)]\mu(dz) \\
 & \leq (\ln n^2y)^{-1} \left[\int_1^A z\mu(dz) + \bar{a}(1 + \ln(1+y/A)) \right].
 \end{aligned}
 \tag{18}$$

On the other hand, since $g_n''(y) \leq 0$ for $y \geq 1/(ne)$,

$$\int_{[-1,1]} [g_n(ye^z) - g_n(y) - y(e^z - 1)g_n'(y)]\nu(dz) \leq 0 \quad (19)$$

for $y \geq 1/n$. Set

$$\gamma_n(y, z) := \ln(\ln n^2 y + z) - \ln(\ln n^2 y) = \ln\left(1 + \frac{z}{\ln n^2 y}\right).$$

Clearly, $y \mapsto \gamma_n(y, z)$ is strictly decreasing and $\lim_{y \rightarrow \infty} \gamma_n(y, z) = 0$ for all $z > 0$. Then we can use the monotone convergence to conclude

$$\begin{aligned} & \int_{(-\infty, -1) \cup (1, k]} [g_n(ye^z) - g_n(y)]\nu(dz) \\ & \leq \int_1^k [\ln(\ln n^2 y + z) - \ln(\ln n^2 y)]\nu(dz) \\ & = \int_1^k \gamma_n(y, z)\nu(dz) \rightarrow 0 \quad \text{as } y \rightarrow \infty. \end{aligned} \quad (20)$$

For all $y \geq 1/n$, by (13) and (18)–(20) and using $b_0 \geq 0$ and $g'' < 0$, we see that if condition (i) holds, then

$$\begin{aligned} L_k g_n(y) & \leq (\beta y + b_1 y)(\ln n^2 y)^{-1} y^{-1} + (\ln n^2 y)^{-1} \left[\int_1^A z \mu(dz) + \bar{a}(1 + \ln(1 + y/A)) \right] \\ & \quad + \int_{(-\infty, -1) \cup (1, k]} [g_n(ye^z) - g_n(y)]\nu(dz) \\ & \leq (\ln n^2 y)^{-1} \left[\beta + b_1 + \int_1^A z \mu(dz) + \bar{a}(1 + \ln(1 + y/A)) \right] + \int_1^k \gamma_n(y, z)\nu(dz) \\ & =: G_{n,k}(y). \end{aligned}$$

Clearly, for any $n \geq 9$, $G_{n,k}(y)$ converges to some constant as $y \rightarrow \infty$ and then $G_{n,k}(y)$ is bounded on $[1/n, \infty)$. Since $g_n(y) \geq 1$ on $[1/n, \infty)$, then for all $n \geq 9$ there exists a constant d_n such that $L_k g_n(y) \leq d_n g_n(y)$. By Proposition 1 we find that $(Y_t^{(k)})_{t \geq 0}$ does not explode for all $k \geq 2$.

We now focus on the case $\alpha < 1$. Write $a = \ln n^2 y$ and $b = \ln n^2(y + z)$. We clearly have $0 < a < b$ for $y \geq 1/n$ and then $\ln b - \ln a \leq a^{-1}(b - a)$ by the concaveness of the logarithm. Thus

$$g_n(y + z) - g_n(y) \leq (\ln n^2 y)^{-1} [\ln(y + z) - \ln y], \quad y \geq 1/n.$$

This combined with (4) implies

$$\begin{aligned} \int_A^\infty [g_n(y + z) - g_n(y)] z^{-1-\alpha} dz & \leq (\ln n^2 y)^{-1} \int_0^\infty [\ln(y + z) - \ln y] z^{-1-\alpha} dz \\ & = (\ln n^2 y)^{-1} c_{\alpha,0} y^{-\alpha}. \end{aligned} \quad (21)$$

By (14)–(16) and (21) we get

$$\begin{aligned}
 & y \int_0^1 [g_n(y+z) - g_n(y) - zg'_n(y)]\mu(dz) + y \int_1^\infty [g_n(y+z) - g_n(y)]\mu(dz) \\
 & \leq (\ln n^2y)^{-1} \left[\int_1^A z\mu(dz) + \bar{a}c_{\alpha,0}y^{1-\alpha} \right].
 \end{aligned} \tag{22}$$

For all $y \geq 1/n$, one can use (13), (19), (20), and (22) to see that

$$\begin{aligned}
 L_k g_n(y) & \leq [\beta y + b_1 y - b_0(y)](\ln n^2y)^{-1}y^{-1} + (\ln n^2y)^{-1} \left[\int_1^A z\mu(dz) + \bar{a}c_{\alpha,0}y^{1-\alpha} \right] \\
 & \quad + \int_{(-\infty, -1) \cup (1, k]} [g_n(ye^z) - g_n(y)]\nu(dz) \\
 & \leq (\ln n^2y)^{-1} \left[\beta + b_1 - b_0(y)y^{-1} + \bar{a}c_{\alpha,0}y^{1-\alpha} + \int_1^A z\mu(dz) \right] + \int_1^k \gamma_n(y, z)\nu(dz) \\
 & =: \tilde{G}_{n,k}(y).
 \end{aligned}$$

Under the assumption $b_0(y) \geq b_0y^{q_0}$ for all $y \geq A$, if either condition (ii) or condition (iii) holds, it is not hard to show that for all $k \geq 2$, $y \mapsto \tilde{G}_{n,k}(y)$ is bounded above on $[1/n, \infty)$, and hence $(Y_t^{(k)})_{t \geq 0}$ does not explode by Proposition 1.

Now, let $(Y_t)_{t \geq 0}$ be the unique strong solution of (1). We proceed to show that $(Y_t)_{t \geq 0}$ does not explode. Clearly, equation (1) can be rewritten as

$$\begin{aligned}
 Y_t & = Y_0 + \int_0^t (\beta Y_s + b_1 Y_s - b_0(Y_s)) ds + \int_0^t \sqrt{2b_2^2 Y_s} dB^{(b)}(s) + \int_0^t \sigma Y_s dB^{(e)}(s) \\
 & \quad + \int_0^t \int_0^1 \int_0^{Y_{s-}} z\tilde{N}^{(b)}(ds, dz, du) + \int_0^t \int_1^\infty \int_0^{Y_{s-}} zN^{(b)}(ds, dz, du) \\
 & \quad + \int_0^t \int_{[-1, 1]} Y_{s-}(e^z - 1)\tilde{N}^{(e)}(ds, dz) + \int_0^t \int_{[-1, 1]^c} Y_{s-}(e^z - 1)N^{(e)}(ds, dz).
 \end{aligned}$$

Define

$$Z(t) := \int_0^t \int_1^\infty zN^{(e)}(ds, dz)$$

and

$$\sigma_k := \inf\{t \geq 0 : Z(t) - Z(t-) \geq k\}.$$

Then $\{\sigma_k\}_{k \geq 2}$ is non-decreasing and $\sigma_k \rightarrow \infty$ almost surely as $k \rightarrow \infty$. On the other hand, by the definition of σ_k , it is easy to see that $(Y_t)_{t \geq 0}$ satisfies (12) on the interval $[0, \sigma_k)$ for all $k \geq 2$. Then the uniqueness of the solution of (12) implies $Y_t = Y_t^{(k)}$ for $t < \sigma_k$. Since $(Y_t^{(k)})_{t \geq 0}$ does not explode for all $k \geq 2$, $\mathbb{P}_{y_0}\{\tau_\infty \geq \sigma_k\} = 1$ for all $k \geq 2$, letting $k \rightarrow \infty$ we have $\mathbb{P}_{y_0}\{\tau_\infty = \infty\} = 1$. This gives the desired result. \square

Remark 2. It follows from Theorems 1 and 2 that the Lévy environment does not seem to be essential for the explosion to happen. Intuitively, this is due to the fact that, in contrast to the

jumps corresponding to the branching mechanism in SDE (1), the jumps in the last terms of (1) arrive at the same rate as the Lévy process for the environment and do not speed up when the process Y takes large values.

In the following corollaries, we consider the special case

$$\mu(dz) = \bar{a}z^{-1-\alpha} dz \quad \text{for constants } \bar{a} > 0, \alpha \in (0, 2) \text{ and for all } z > 0. \quad (23)$$

Combining Theorems 1 and 2, we immediately have the following corollaries.

Corollary 1. *Suppose that (23) holds for $\alpha \geq 1$. Then $\mathbb{P}_{y_0}\{\tau_\infty < \infty\} = 0$ for all $y_0 > 0$.*

Remark 3. Note that the process with $\alpha = 1$ corresponds to Neveu's CBLE with competition whose Lévy measure μ for the branching mechanism does not satisfy the finite moment condition, and the above non-explosion result is not covered in [4] for the CBLE (without competition).

Corollary 2. *Suppose that (23) holds for $\alpha < 1$ and there exist constants $q_0 \in \mathbb{R}$ and $b_0, A \geq 0$ such that $b_0(y) = b_0 y^{q_0}$ for $y \geq A$. Then $\mathbb{P}_{y_0}\{\tau_\infty < \infty\} > 0$ for large enough $y_0 > 0$ if and only if one of the following conditions holds:*

- (i) $b_0 = 0$,
- (ii) $q_0 < 2 - \alpha$ and $b_0 > 0$,
- (iii) $q_0 = 2 - \alpha$ and $0 < b_0 < \bar{a}c_{\alpha,0}$.

Remark 4. Comparing with the integral test in Theorem 1.2 of [17], in which they only considered the special branching mechanism and Brownian environment and the logistic competition, i.e. $b_0(y) = cy^2$ for some $c \geq 0$, the model we consider is more general and our results agree with that in [17]. For example, in the case that $\mu(dz)$ is α -stable with $\alpha \in (0, 1)$ and $b_0(y) = cy^2$, we can immediately conclude from Corollary 2 that the process does not explode if $c > 0$ and the process explodes if $c = 0$, which recovers results for this case by the integral test in [17].

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