BULL. AUSTRAL. MATH. SOC. 16A70
VOL. 43 (1991) [451–462]

COMMUTATIVITY THEOREMS FOR RINGS WITH POLYNOMIAL CONSTRAINTS ON CERTAIN SUBSETS

HIROAKI KOMATSU AND HISAO TOMINAGA

Dedicated to Professor Nobuo Nobusawa on his 60th birthday

We prove several commutativity theorems for unital rings with polynomial constraints on certain subsets, which improve and generalise the recent results of Grosen, and Ashraf and Quadri.

0. INTRODUCTION

Recently Streb [11] gave a classification of non-commutative rings. We have applied the classification to the proof of some commutativity theorems, in [9] and [10]. The classification is effective in our present paper, too. As is easily seen from the proof of [11, Korollar (1)], if R is a non-commutative ring with unity, then there exists a factorsubring of R which is of type (i), (ii), (iii) or (iv):

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime

(ii) $M_\sigma(K) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \in K \right\}$, K is a finite field with a non-trivial automorphism σ.

(iii) A non-commutative ring with no non-zero divisors of zero.

(iv) $S = \langle 1 \rangle + T$, T is a non-commutative subring of S such that $T[T, T] = [T, T]T = 0$.

This result gives the following

META THEOREM. *Let P be a ring property which is inherited by factorsubrings. If no rings of type (i), (ii), (iii) or (iv) satisfy P, then every ring with unity and satisfying P is commutative.*

Meanwhile, in her thesis [4], Grosen generalised some known theorems on the commutativity of a ring R with unity and satisfying polynomial identities by assuming that the identities hold merely for the elements of a certain subset of R rather than for all elements of R. The major purpose of this paper is to prove several commutativity

Received 18 June 1990

theorems which improve and generalise the results of Grosen [4, Section 1] as well as the main results of Ashraf and Quadri [2].

Throughout the present paper, R will represent a unital ring (a ring with unity). We use the following notation:

$C = C(R) =$ the centre of R.
$D = D(R) =$ the commutator ideal of R.
$N = N(R) =$ the set of nilpotent elements in R.
$N^* = N^*(R) = \{x \in R \mid x^2 = 0\}$.
$U = U(R) =$ the multiplicative group of units in R.
$C_R(M) =$ the centraliser of a subset M of R in R.

For $x, y \in R$, we define extended commutators $[x, y]_k$ as follows: let $[x, y]_0 = x$, and proceed inductively: $[x, y]_k = [[x, y]_{k-1}, y]$. Finally, let Q be the intersection of the set of non-units of R with the set of quasi-regular elements of R (that is, $Q = (1 + U)\setminus U$). Obviously, Q contains N and the Jacobson radical of R.

1. Preliminaries

Let n be a positive integer. We consider the following conditions:

$Q_1(n)$ $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1}y^{n+1}$ for all $x, y \in R \setminus Q$.
$Q_2(n)$ $(xy)^n = x^n y^n = y^n x^n$ for all $x, y \in R \setminus Q$.
$Q_3(n)$ $(xy)^n = (yx)^n$ for all $x, y \in R \setminus Q$.
$Q_3'(n)$ $[(xy)^n - (yx)^n, x] = 0$ for all $x, y \in R \setminus Q$.
$Q_4(n)$ $[(xy)^n, x] = 0$ for all $x, y \in R \setminus Q$.
$Q_4^*(n)$ For each $x, y \in R \setminus Q$, there exists a positive integer k such that $[(xy)^n, x]_k = 0$.
$Q_5(n)$ $[(yx)^n, x] = 0$ for all $x, y \in R \setminus Q$.
$Q_6(n)$ $[y^n, x] = 0$ for all $x, y \in R \setminus Q$.
$Q_{10}(n)$ $[x^n, y^n] = 0$ for all $x, y \in R \setminus Q$.
$Q_{11}(n)$ $(xy)^n = x^n y^n$ for all $x, y \in R \setminus Q$.
$Q_{12}(n)$ $[x - x^n, y - y^n] = 0$ for all $x, y \in R \setminus Q$.

In the above conditions, if $R \setminus Q$ is replaced by R, we get the conditions $P_i(n)$ which have been studied by many authors (see, for example [6] and [7]). Furthermore, we consider the following conditions:

$Q(n)$ If $x, y \in R$ and $n[x, y] = 0$ then $[x, y] = 0$.

(C) For each $x, y \in R$, there exist $f(X), g(X) \in X^2\mathbf{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$.

Now, in preparation for proving our main theorems, we state nine lemmas.

Lemma 1.1.

(1) $1 - Q \subseteq U \subseteq R \setminus Q$.

(2) $C_R(R \setminus Q) = C$.

(3) *R is commutative if (and only if) $R \setminus Q$ is commutative.*

Proof: Almost clear. $\square$

Lemma 1.2. *Let $a \in R$. Suppose that for each $x \in R \setminus Q$, there exist integers $n \geqslant 0$, $n_i > 0$ $(i = 1, \cdots, r)$ such that $(n_1, \cdots, n_r) = 1$ and $x^n[a, x^{n_i}] = 0$. Then a is in C.*

Proof: Let $x \in R \setminus Q$. We can choose integers $1 \leqslant k \leqslant r$, $n \geqslant 0$, $n_i > 0$, $m_i \geqslant 0$ $(i = 1, \cdots, r)$ such that

$$-m_1n_1 - \cdots - m_kn_k + m_{k+1}n_{k+1} + \cdots + m_rn_r = 1$$

and $x^n[a, x^{n_i}] = 0$. Then

$$\begin{aligned} x^{n+m_1n_1+\cdots+m_kn_k}[a, x] &= x^n[a, x^{1+m_1n_1+\cdots+m_kn_k}] \\ &= x^n[a, x^{m_{k+1}n_{k+1}+\cdots+m_rn_r}] = 0. \end{aligned}$$

If $x + 1 \in Q$, then $x \in U$ and $[a, x] = 0$. If $x + 1 \notin Q$, then $(x+1)^{n'}[a, x] = 0$ for some positive integer n'. Hence, by [2, Lemma 3], we see that $[a, x] = 0$. Thus $a \in C$, by Lemma 1.1 (2). $\square$

Lemma 1.3. *([4, Lemma 1.3]). Let $a \in R$. Suppose that for some positive integer n, (i) $[a, u^n] = 0$ for all $u \in U$ and (ii) $n[a, c] = 0$ implies $[a, c] = 0$ for all $c \in N$. Then $a \in C_R(N)$. In particular, if $[x, u^n] = 0$ for all $x \in R$, $u \in U$, and if R satisfies $Q(n)$, then $N \subseteq C$.*

Proof: Let $c \in N$, and let k be the least positive integer such that $[a, c^r] = 0$ for all $r \geqslant k$. Suppose $k > 1$. Since $1 + c^{k-1} \in U$, we see that $0 = [a, (1 + c^{k-1})^n] = n[a, c^{k-1}]$, and so $[a, c^{k-1}] = 0$, contradicting the minimality of k. Thus, $k = 1$, so $[a, c] = 0$. $\square$

https://doi.org/10.1017/S0004972700029294 Published online by Cambridge University Press

Lemma 1.4. $Q_1(n) \Rightarrow Q_4(n)$, $Q_2(n) \Rightarrow Q_3(n) \Rightarrow Q_4(n)$, *and* $Q_3'(n) \Rightarrow Q_4^*(n)$.

Proof: Obviously $Q_2(n)$ implies $Q_3(n)$. Now, suppose $Q_1(n)$. Let $x, y \in R \setminus Q$. Then

$$x[x^n, y]y^n = x^{n+1}y^{n+1} - xyx^ny^n = x^{n+1}y^{n+1} - (xy)^{n+1} = 0.$$

In case $y+1 \in Q$, we have $x[x^n, y] = 0$. On the other hand, if $y+1 \notin Q$ then $x[x^n, y](y+1)^n = 0$, and so $x[x^n, y] = 0$ by [2, Lemma 3]. Now, let $x \in R \setminus Q$ and $y \in Q$. Then $x[x^n, y] = -x[x^n, 1-y] = 0$, by the above. We have thus seen that $x[x^n, y] = 0$ for all $x \in R \setminus Q$ and $y \in R$. Hence, for each $x, y \in R \setminus Q$, we get

$$[x, (xy)^n] = x((xy)^n - (yx)^n) = x[x^n, y^n] = 0.$$

Also, this shows that $Q_3(n)$ implies $Q_4(n)$. Finally, suppose $Q_3'(n)$. Then, for each $x, y \in R \setminus Q$, $[[(xy)^n, x], x] = -[x((xy)^n - (yx)^n), x] = 0$. □

Lemma 1.5. *Let n be a positive integer. If R satisfies one of the conditions $Q_4(n)$, $Q_5(n)$ and $Q_6(n)$, then $u^n \in C$ for all $u \in U$, and $D \subseteq N$.*

Proof: First, we consider the case that R satisfies $Q_4(n)$. Let $u, v \in U$. Then

$$[u^n, v] = [(v \cdot v^{-1}u)^n, v] = 0. \tag{1.1}$$

Now, let $x \in R \setminus Q$. If $u^{-1}x \in Q$ then, by (1.1), $-u^{-1}[u^n, x] = [u^n, 1 - u^{-1}x] = 0$, and so $[x, u^n] = 0$. If $u^{-1}x \notin Q$ then

$$[u, x^n] = [u, (u \cdot u^{-1}x)^n] = 0. \tag{1.2}$$

In case $ux^{n-1} \in Q$, we have $-u[u^n, x^{n-1}] = [u^n, 1 - ux^{n-1}] = 0$ (by (1.1)), and so $[x^{n-1}, u^n] = 0$. This together with (1.2) implies that $x^{n-1}[x, u^n] = [x^n, u^n] - [x^{n-1}, u^n]x = 0$. In case $ux^{n-1} \notin Q$, we have $x^{n^2+1}[x, u^n] = [x, x^{n^2+1}u^n] = [x, x(ux^n)^n] = [x, (x \cdot ux^{n-1})^n x] = [x, (x \cdot ux^{n-1})^n]x = 0$ (by (1.2)). Therefore

$$x^{n^2+1}[x, u^n] = 0 \text{ for all } x \in R \setminus Q \text{ and } u \in U.$$

Hence $u^n \in C$ for all $u \in U$, by Lemma 1.2. Now, it is easy to see that R satisfies the polynomial identity

$$[X, (XY)^n]W[(1-X)^n, (1-Y)^n] = 0.$$

Since $X = e_{11}$, $Y = e_{11} + e_{12}$, $W = e_{21}$ fail to satisfy the above identity, [6, Proposition 2] shows that $D \subseteq N$. Similarly, we can prove the statement for $Q_5(n)$ and $Q_6(n)$. □

The following two lemmas are obvious by the proofs of [8, Lemma 1] and [6, Theorem 1], respectively.

LEMMA 1.6. *Let m_1 and m_2 be relatively prime positive integers. Let a, b be elements of a group G. If $a^{m_i}b^{m_i} = b^{m_i}a^{m_i}$, $a^{m_i}(ab)^{m_i} = (ab)^{m_i}a^{m_i}$ and $b^{m_i}(ab)^{m_i} = (ab)^{m_i}b^{m_i}$ $(i = 1, 2)$, then $ab = ba$.*

LEMMA 1.7. *Let $m_1, \cdots, m_r$ be positive integers such that $1 \leqslant m_i \leqslant 6$ $(i = 1, \cdots, r)$. Let $n_1, \cdots, n_r$ be positive integers, and $d = (n_1, \cdots, n_r)$. If R satisfies $P_{m_1}(n_1), \cdots, P_{m_r}(n_r)$ and $Q(d)$, then R is commutative.*

LEMMA 1.8. *Suppose that for each $x, y \in R$, there exist positive integers n_i $(1 \leqslant i \leqslant r)$ such that $(n_1, \cdots, n_r) = 1$ and that for each i, one of the following holds: $(xy)^{n_i} = (yx)^{n_i}$, $[x, (xy)^{n_i}] = 0$ and $[x, (yx)^{n_i}] = 0$. Then R is commutative.*

PROOF: We consider the ring $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime; and put $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then for any positive integer n, $(xy)^n - (yx)^n \neq 0$, $[x, (xy)^n] \neq 0$ and $[x, (yx)^n] \neq 0$. Therefore, in view of Meta Theorem and [**3**, Theorem 3], we can apply the argument employed in the proof of [**10**, Lemmas 1 and 2] to obtain the assertion. $\square$

The next result generalises [**1**, Theorems 2 and 3].

LEMMA 1.9. *Let $n > 0$ (respectively $n > 1$) be an integer such that*

$$(xy)^n - y^n x^n = (yx)^n - x^n y^n \in C_R(x)$$
$$(\text{respectively } (xy)^n - x^n y^n = (yx)^n - y^n x^n \in C_R(x))$$

for all $x, y \in R$. If R satisfies $Q(n(n+1))$ (respectively $Q(n(n-1))$), then R is commutative.

PROOF: Since R satisfies the polynomial identity

$$[(XY)^n - Y^n X^n, X] = 0 \quad (\text{ respectively } [(XY)^n - X^n Y^n, X] = 0),$$

[**6**, Proposition 2] enables us to see that $D \subseteq N$. Now, let $B = \langle \{x^n \mid x \in R\} \rangle$. Obviously,

$$[x^{n+1}, y^n] = \{(xy)^n - y^n x^n\}x - x\{(yx)^n - x^n y^n\} = 0$$
$$(\text{respectively } x[x^{n-1}, y^n]x = x\{(yx)^n - y^n x^n\} - \{(xy)^n - x^n y^n\}x = 0)$$

for all $x, y \in R$; in particular,

$$[u^{n+1}, y^n] = 0 \quad (\text{respectively } [u^{n-1}, y^n] = 0)$$

https://doi.org/10.1017/S0004972700029294 Published online by Cambridge University Press

for all $u \in U$ and $y \in R$. Since R satisfies $Q(n+1)$ (respectively $Q(n-1)$), Lemma 1.3 shows that $[N, B] = 0$, and so $N(B) \subseteq C(B)$. Combining this with $D \subseteq N$, we obtain $D(B) \subseteq C(B)$. Now, let $s, t \in B$. Then

$$(n+1)s^n[s, t^n] = [s^{n+1}, t^n] = 0$$
$$(\text{respectively } (n-1)s^n[s, t^n] = s^2[s^{n-1}, t^n] = s[s^{n-1}, t^n]s = 0).$$

Hence $[s, t^n] = 0$ by $Q(n+1)$ (respectively $Q(n-1)$) and [**2**, Lemma 3]. This together with $Q(n)$ implies the commutativity of B (Lemma 1.7), and so

$$(xy)^n - (yx)^n = -[x^n, y^n] \quad (\text{respectively } [x^n, y^n]) = 0$$

for all $x, y \in R$. Therefore R is commutative, again by Lemma 1.7. ∎

2. Generalisations of Grosen's results

We begin this section with generalising [**4**, Theorem 1.2] and Hongan's conjecture [**8**, p.21] as follows:

Theorem 2.1. *Suppose that for each $x_1, x_2, x_3 \in R \setminus Q$, there exist positive integers m, n such that $(m, n) = 1$ and $[x_i^m, x_j^m] = 0 = [x_i^n, x_j^n]$ $(i, j = 1, 2, 3)$. Then R is commutative.*

Proof: By Lemma 1.6,

(2.1) U is commutative, and so Q is commutative.

Claim 1. $D \subseteq N$, and so N is a commutative ideal, by (2.1).

Proof: If $x, y \in R \backslash Q$, then there exists a positive integer m such that $[x^m, y^m] = 0$; if $x, y \in Q$, then $[x, y] = [1-x, 1-y] = 0$, by (2.1). Now, let $x \in Q$ and $y \notin Q$. Then there exist positive integers m, n such that $(m, n) = 1$ and $[(1-x)^m, y^m] = 0 = [(1-x)^n, y^n]$. Noting that $[(1-x)^m, y^{mn}] = 0 = [(1-x)^n, y^{mn}]$, $1-x \in U$ and $(m, n) = 1$, we can easily see that $[x, y^{mn}] = -[1-x, y^{mn}] = 0$. We have thus seen that for each $x, y \in R$ there exist positive integers h, k such that $[x^h, y^k] = 0$. Hence $D \subseteq N$ by [**5**, Theorem].

Claim 2. $N^* \subseteq C$.

Proof: Let $c \in N^*$ and $x \in R \backslash Q$. If $x + [c, x] \in Q$, then $[c, x] = [c, x + [c, x]] = 0$, by Claim 1. If $x + [c, x] \notin Q$, then there exist positive integers m, n such that $(m, n) = 1$, $[(x + [c, x])^m, x^m] = 0 = [(x + [c, x])^n, x^n]$ and $[(1+c)^m, x^m] = 0 = [(1+c)^n, x^n]$. Noting that $N^2 \subseteq C$ by Claim 1, we can see that $[[c, x^m], x^m] = 0 =$

$[[c, x^n], x^n]$ and $m[c, x^m] = 0 = n[c, x^n]$. Hence $[c, x^{m^2}] = mx^{m^2-m}[c, x^m] = 0$ and $[c, x^{n^2}] = 0$. Hence $N^* \subseteq C$, by Lemma 1.2.

Now, by (2.1) and Claim 1, $D[U, R] \subseteq [U, DR] + [U, D]R = 0$, and so $[U, R] \subseteq N^* \subseteq C$, by Claim 2. Let $u \in U$ and $x \in R \setminus Q$, and choose positive integers m, n such that $(m, n) = 1$ and $[u^m, x^m] = 0 = [u^n, x^n]$. Then $m^2x^{m-1}[u, x]u^{m-1} = mx^{m-1}[u^m, x] = [u^m, x^m] = 0$, and so $m^2x^{mn-1}[u, x] = 0$; similarly $n^2x^{mn-1}[u, x] = 0$. Hence $x^{mn-1}[u, x] = 0$. Now, we see that $U \subseteq C$, by Lemma 1.2. This implies that $Q \subseteq C$, and so, in the hypothesis of our theorem, $R \setminus Q$ may be replaced by R. Hence R is commutative, by [10, Theorem 1]. $\square$

THEOREM 2.2. (1) *Suppose that for each $x, y \in R \setminus Q$, there exist positive integers $n_1, \cdots, n_r$ such that $(n_1, \cdots, n_r) = 1$ and that for each i, either $[x, (xy)^{n_i}] = 0$ or $[x, (yx)^{n_i}] = 0$. Then R is commutative.*

(2) *Suppose that for each $x, y \in R \setminus Q$, there exist positive integers $n_1, \cdots, n_r$ such that $(n_1, \cdots, n_r) = 1$ and $[x, y^{n_i}] = 0\ (i = 1, \cdots, r)$. Then R is commutative.*

PROOF: (1) Let $u \in U$, and $x \in R$. If $u^{-1}x \in Q$ then there exist positive integers $n_1, \cdots, n_r$ such that $(n_1, \cdots, n_r) = 1$ and that for each i,

$$-u^{-1}[x, u^{n_i}] = [1 - u^{-1}x, \{(1 - u^{-1}x) \cdot (1 - u^{-1}x)^{-1}u\}^{n_i}] = 0$$

or

$$-(1 - u^{-1}x)^{-1}u^{-1}[x, u^{n_i}](1 - u^{-1}x) = [1 - u^{-1}x, \{(1 - u^{-1}x)^{-1}u(1 - u^{-1}x)\}^{n_i}] = 0.$$

Hence, in either case, we have $[x, u^{n_i}] = 0$, and therefore $[x, u] = 0$. If $u^{-1}x \notin Q$, then there exist positive integers $n_1, \cdots, n_r$ such that $(n_1, \cdots, n_r) = 1$ and that for each i,

$$[u, x^{n_i}] = [u, (u \cdot u^{-1}x)^{n_i}] = 0$$

or

$$u^{-1}[u, x^{n_i}]u = [u, (u^{-1}x \cdot u)^{n_i}] = 0.$$

Hence, in either case, we have $[u, x^{n_i}] = 0$, and therefore $U \subseteq C$, by Lemma 1.2. This means that $Q \subseteq C$, so that, in the hypothesis, $R \setminus Q$ may be replaced by R, and R is commutative by Lemma 1.8.

(2) Let $u \in U$, and $x \in R \setminus Q$. Then there exist positive integers $n_1, \cdots, n_r$ such that $(n_1, \cdots, n_r) = 1$ and $[x, u^{n_i}] = 0\ (i = 1, \cdots, r)$, so that $[x, u] = 0$. Hence $U \subseteq C$, by Lemma 1.1 (2). Now, as in (1), [10, Theorem 1] enables us to see that R is commutative. $\square$

COROLLARY 2.3. *Suppose that for each $x, y \in R \setminus Q$, there exist positive integers $n_1, \cdots, n_r$ such that $(n_1, \cdots, n_r) = 1$ and that for each i, either $[x, (xy)^{n_i}] = 0$ or $(xy)^{n_i} - (yx)^{n_i} = 0$. Then R is commutative.*

PROOF: Since $[x, (yx)^n] = \{(xy)^n - (yx)^n\}x$ for all $x, y \in R$ and any positive integer n, this is clear by Theorem 2.2. ∎

The next result includes [4, Theorems 1.1, 1.3, 1.4, 1.6 and 1.7].

THEOREM 2.4. *Let $m_1, \cdots, m_r$ be positive integers such that $1 \leqslant m_i \leqslant 6$ $(i = 1, \cdots, r)$. Let $n_1, \cdots, n_r$ be positive integers, and $d = (n_1, \cdots, n_r)$.*

(1) *Suppose that R satisfies $Q_{m_1}(n_1), \cdots, Q_{m_r}(n_r)$ and $Q(d)$. Then R is commutative.*

(2) *Suppose that R satisfies $Q_{m_1}(n_1), \cdots, Q_{m_r}(n_r)$ and $Q_{12}(d)$. Then R is commutative.*

PROOF: (1) In view of Lemmas 1.4 and 1.5, $D \subseteq N$ and $u^{n_i} \in C$ $(i = 1, \cdots, r)$ for all $u \in U$. Hence $u^d \in C$ and $D \subseteq N \subseteq C$ by Lemma 1.3. Accordingly, for each $u \in U$ and $x \in R$, we have $du^{d-1}[u, x] = [u^d, x] = 0$, whence $[u, x] = 0$ follows. Therefore we can easily see that $Q \subseteq C$. This means that R satisfies $P_{m_i}(n_i)$ $(i = 1, \cdots, r)$. Hence R is commutative, by Lemma 1.7.

(2) In view of Lemma 1.4, we may assume that $4 \leqslant m_i \leqslant 6$. Then, by Lemma 1.5, $u^d \in C$ for all $u \in U$, and so by $Q_{12}(d)$ $[u, x] = [u, x^d]$ for all $u \in U$ and $x \in R \setminus Q$; in particular, $[u, v] = 0$ for all $u, v \in U$.

Let $u \in U$, and $x \in R \setminus Q$. If $u^{-1}x \in Q$, then $[u, x] = -u[u, 1 - u^{-1}x] = 0$. Next, we consider the case that $u^{-1}x \notin Q$. If $m_i = 4$ then $[u, x^{n_i}] = [u, (u \cdot u^{-1}x)^{n_i}] = 0$; if $m_i = 5$ then $[u, x^{n_i}] = u[u, u^{-1}x^{n_i}u]u^{-1} = u[u, (u^{-1}x \cdot u)^{n_i}]u^{-1} = 0$. Hence $[u, x^{n_i}] = 0$ $(i = 1, \ldots, r)$. Then $U \subseteq C$, by Lemma 1.2, and so $Q \subseteq C$. This implies that R satisfies $P_{12}(d)$ and $N^* \subseteq C$. Now, R is commutative by [9, Lemma 2]. ∎

THEOREM 2.5. *Let m, n be positive integers. Suppose that R satisfies $Q_4^*(m)$, $Q_{10}(n)$ and $Q(mn)$. Then R is commutative.*

PROOF: Let $u, v \in U$. Then there exists a positive integer k such that $[v^{mn}, u]_k = [(u \cdot u^{-1}v^n)^m, u]_k = 0$. Choose k as minimal, and suppose that $k > 1$. Obviously, $[[[v^{mn}, u]_{k-2}, u], u] = [v^{mn}, u]_k = 0$. Since $[v^{mn}, u^{mn}] = 0$ implies that $[[v^{mn}, u]_{k-2}, u^{mn}] = 0$, we get $mnu^{mn-1}[v^{mn}, u]_{k-1} = mnu^{mn-1}[[v^{mn}, u]_{k-2}, u] = [[v^{mn}, u]_{k-2}, u^{mn}] = 0$. But this forces a contradiction $[v^{mn}, u]_{k-1} = 0$. Therefore k has to be 1, and so

$$[v^{mn}, u] = 0 \text{ for all } u, v \in U. \tag{2.2}$$

CLAIM 1. $[D, U] = 0$.

PROOF: Obviously, R satisfies the polynomial identity

$$[X^n, Y^n]W[(1-X)^n, Y^n]W[X^n, (1-Y)^n]W[(1-X)^n, (1-Y)^n] = 0.$$

https://doi.org/10.1017/S0004972700029294 Published online by Cambridge University Press

However, no $M_2(GF(p))$, p a prime, satisfies this identity, as a consideration of the following elements shows: $X = e_{11}$, $Y = e_{11} + e_{12}$, $W = e_{21}$. Therefore, by [6, Proposition 2], $D \subseteq N$. Since $[N, U] = 0$ by (2.2) and Lemma 1.3, we obtain $[D, U] = 0$.

CLAIM 2. $u^{mn} \in C$ for all $u \in U$.

PROOF: Let $u \in U$, and $x \in R \setminus Q$. Since $[[x^n, u], u] = 0$ by Claim 1, we see that $nu^{n-1}[x^n, u] = [x^n, u^n] = 0$, and so

$$[x^n, u] = 0 \text{ for all } u \in U \text{ and } x \in R \setminus Q. \tag{2.3}$$

If $ux^{n-1} \in Q$ then $[u^n, x^{n-1}] = -u^{-1}[u^n, 1 - ux^{n-1}] = 0$, by (2.3). Since $[[x^{n-1}, u], u] = 0$ by Claim 1, we have $nu^{n-1}[u, x^{n-1}] = [u^n, x^{n-1}] = 0$, namely $[u, x^{n-1}] = 0$. Accordingly, $x^{n-1}[u^m, x] = [u^m, x^n] = 0$, by (2.3). If $ux^{n-1} \notin Q$, then there exists a positive integer k such that $[(x \cdot ux^{n-1})^m, x]_k = 0$. Then, by (2.3),

$$\begin{aligned} x^{mn+1}[u^m, x]_k = [x^{mn+1}u^m, x]_k = [x(ux^n)^m, x]_k &= [(xux^{n-1})^m x, x]_k \\ &= [(xux^{n-1})^m, x]_k x = 0. \end{aligned}$$

We have thus seen that

(2.4) for each $u \in U$ and $x \in R \setminus Q$, there exists a positive integer k such that $x^{mn+1}[u^m, x]_k = 0$.

If $1 + x \in Q$, then $x \in U$ and $[u^m, x]_k = 0$, by (2.4). If $1 + x \notin Q$, then there exists a positive integer $k' \geqslant k$ such that $(1 + x)^{mn+1}[u^m, x]_{k'} = 0$, by (2.4). Hence $[u^m, x]_{k'} = 0$. We have thus seen that for each $u \in U$ and $x \in R \setminus Q$, there exists a positive integer k such that $[u^{mn}, x]_k = 0$. Here, choose k as minimal, and suppose that $k > 1$. Then $[[[u^{mn}, x]_{k-2}, x], x] = [u^{mn}, x]_k = 0$. Noting that $[u^{mn}, x^{mn}] = 0$, we see that $0 = [[u^{mn}, x]_{k-2}, x^{mn}] = mnx^{mn-1}[[u^{mn}, x]_{k-2}, x] = mnx^{mn-1}[u^{mn}, x]_{k-1}$. If $1 + x \in Q$ then $[u^{mn}, x]_{k-1} = 0$, a contradiction. If $1 + x \notin Q$ then $[u^{mn}, (x+1)^{mn}] = 0$ and $[u^{mn}, x+1]_k = [u^{mn}, x]_k = 0$. Hence $mn(x+1)^{mn-1}[u^{mn}, x]_{k-1} = 0$. Combining this with $mnx^{mn-1}[u^{mn}, x]_{k-1} = 0$, by $Q(mn)$ and [2, Lemma 3], we have a contradiction $[u^{mn}, x]_{k-1} = 0$. Thus, k has to be 1, and $[u^{mn}, x] = 0$ for all $u \in U$ and $x \in R \setminus Q$. Accordingly, $u^{mn} \in C$ by Lemma 1.1 (2).

Now, let $u \in U$ and $x \in R$. Since $[[x, u], u] = 0$ by Claim 1, we see that $mnu^{mn-1}[u, x] = [u^{mn}, x] = 0$ (Claim 2). Hence $[u, x] = 0$, and $U \subseteq C$. This implies that $Q \subseteq C$, and so R satisfies $P_{10}(n)$. Then, by [9, Corollary 2], R satisfies the condition (C). Furthermore, since $N^* \subseteq Q \subseteq C$, [9, Lemma 2] shows that R is commutative. ∎

Combining Theorem 2.5 with Lemma 1.4, we readily obtain the following, which includes [4, Theorem 1.8].

https://doi.org/10.1017/S0004972700029294 Published online by Cambridge University Press

COROLLARY 2.6. *Let m, n be positive integers. Suppose that R satisfies $Q_3'(m)$, $Q_{10}(n)$ and $Q(mn)$. Then R is commutative.*

Finally, the next result includes a generalisation of [4, Theorem 1.5] and [2, Theorems 1 and 2].

THEOREM 2.7. *The following conditions are equivalent:*

(0) *R is commutative.*

(1) *There exists a positive integer n such that R satisfies $Q(n(n+1))$ and $(xy)^{n+1} - x^{n+1}y^{n+1} = (yx)^{n+1} - y^{n+1}x^{n+1} \in C$ for all $x, y \in R \setminus Q$.*

(2) *There exists a positive integer n such that R satisfies $Q(n(n+1))$ and $(xy)^n - y^n x^n = (yx)^n - x^n y^n \in C$ for all $x, y \in R \setminus Q$.*

PROOF: It suffices to show that each of (1), (2) implies (0).

(1) $\Rightarrow$ (0). As was shown in the proof of Lemma 1.9,

$$[u^n, y^{n+1}] = 0 \text{ for all } u \in U \text{ and } y \in R \setminus Q. \tag{2.5}$$

This implies that $[u^n, v^{n(n+1)}] = 0$ and $[u^{n+1}, v^{n(n+1)}] = 0$, so

$$[u, v^{n(n+1)}] = 0 \text{ for all } u, v \in U. \tag{2.6}$$

CLAIM 1. $u^{n(n+1)} \in C$ for all $u \in U$, so that $N \subseteq C$ by Lemma 1.3.

PROOF: Let $u \in U$, and $y \in R$. If $uy \in Q$, then (2.6) shows that $[u^{n(n+1)}, y] = -u^{-1}[u^{n(n+1)}, 1 - uy] = 0$. Similarly, if $y \in Q$ or $u^n y \in Q$, then $[u^{n(n+1)}, y] = 0$. Henceforth, we assume that $uy \notin Q$, $u^n y \notin Q$ and $y \notin Q$. Then $y^n \notin Q$, and the hypothesis and (2.5) show that

$$\begin{aligned} y^{(n+1)^2-1}u^{n(n+1)}y &= y^n\left(u^n y^{n+1}\right)^n u^n y = (y^n \cdot u^n y)^{n+1} = y^{n(n+1)}(u^n y)^{n+1} + z \\ &= y^{n(n+1)}u^{n(n+1)}y^{n+1} + y^{n(n+1)}z' + z = y^{(n+1)^2}u^{n(n+1)} + y^{n(n+1)}z' + z \end{aligned}$$

with some $z, z' \in C$, and so $y^{(n+1)^2-1}[u^{n(n+1)}, y]_2 = 0$. We have thus seen that $y^{(n+1)^2-1}[u^{n(n+1)}, y]_2 = 0$ for all $u \in U$, $y \in R$. Hence, by [2, Lemma 3]

$$[u^{n(n+1)}, y]_2 = 0 \text{ for all } u \in U \text{ and } y \in R. \tag{2.7}$$

As was noted above, if $y \in Q$ then $[u^{n(n+1)}, y] = 0$, and so $[u^{n(n+1)}, y^{n+1}] = 0$. This together with (2.5) shows that $[u^{n(n+1)}, y^{n+1}] = 0$ for all $u \in U$ and $y \in R$. Combining the last with (2.7) and $Q(n+1)$, we can easily see that $[u^{n(n+1)}, y] = 0$, namely $u^{n(n+1)} \in C$. Combining the hypothesis with Claim 1, we readily see that R satisfies the polynomial identity

$$[(XY)^{n+1} - X^{n+1}Y^{n+1}, X]W[(1-X)^{n(n+1)}, (1-Y)^{n(n+1)}] = 0.$$

However, no $M_2(GF(p))$, p any prime, satisfies this identity, as a consideration of the following elements shows: $X = e_{11} + e_{21}$, $Y = e_{11} + e_{12}$, $W = e_{21}\,(p = 2)$; $X = e_{11} + e_{12}$, $Y = e_{22} + e_{12}$, $W = e_{21}\,(p \neq 2)$. Therefore, by [6, Proposition 2] and Claim 1, $D \subseteq N \subseteq C$. So, by Claim 1, $0 = [u^{n(n+1)}, y] = n(n+1)u^{n(n+1)-1}[u, y]$, namely $[u, y] = 0$ $(u \in U, y \in R)$ follows. Hence $U \subseteq C$, and so $Q \subseteq C$. Thus, in the hypothesis, we may replace $R \setminus Q$ by R. Now, the commutativity of R is clear, by Lemma 1.9.

(2) $\Rightarrow$ (0). As was shown in the proof of Lemma 1.9,

$$[x^{n+1}, y^n] = 0 \text{ for all } x, y \in R \setminus Q. \tag{2.8}$$

This implies that $[u, v^{n(n+1)}] = 0$ for all $u, v \in U$.

Claim 2. $u^{n(n+1)} \in C$ for all $u \in U$, so that $N \subseteq C$ by Lemma 1.3.

Proof: Let $u \in U$, and $y \in R$. As is easily seen, if $\{uy, u^{n+1}y, y\} \cap Q \neq \emptyset$ then $[u^{n(n+1)}, y] = 0$. Henceforth, we assume that $uy \notin Q$, $u^{n+1}y \notin Q$ and $y \notin Q$. Then the hypothesis and (2.8) show that

$$\begin{aligned} y^{n^2-1}u^{n(n+1)}y &= y^{n-1}\left(u^{n+1}y^n\right)^{n-1}u^{n+1}y = \left(y^{n-1}\cdot u^{n+1}y\right)^n = \left(u^{n+1}y\right)^n y^{n(n-1)} + z \\ &= y^n u^{n(n+1)}y^{n(n-1)} + z'y^{n(n-1)} + z = y^{n^2}u^{n(n+1)} + z'y^{n(n-1)} + z \end{aligned}$$

with some $z, z' \in C$, and so $y^{n^2-1}[u^{n(n+1)}, y]_2 = 0$. Now, by making use of the same argument as in (1) $\Rightarrow$ (0), we see that $u^{n(n+1)} \in C$ for all $u \in U$.

Combining the hypothesis with Claim 2, we readily see that R satisfies the polynomial identity

$$[(XY)^n - Y^nX^n, X]W[(1-X)^{n(n+1)}, (1-Y)^{n(n+1)}] = 0.$$

However, no $M_2(GF(p))$, p any prime, satisfies this identity, as a consideration of the following elements shows: $X = e_{11} + e_{12}$, $Y = e_{11} + e_{21}$, $W = e_{21}\,(p = 2)$; $X = e_{22} + e_{12}$, $Y = e_{11} + e_{12}$, $W = e_{21}\,(p \neq 2)$. Therefore, by [6, Proposition 2] and Claim 2, $D \subseteq N \subseteq C$. The rest of the proof proceeds in the same way as the last part of (1) $\Rightarrow$ (0) did. ∎

References

[1] H. Abu-Khuzam, 'Commutativity results for rings', *Bull. Austral. Math. Soc.* **38** (1988), 191–195.

[2] M. Ashraf and M.A. Quadri, 'On commutativity of rings with some polynomial constraints', *Bull. Austral. Math. Soc.* **41** (1990), 201–206.

https://doi.org/10.1017/S0004972700029294 Published online by Cambridge University Press

[3] L.P. Belluse, I.N. Herstein and S.K. Jain, 'Generalized commutative rings', *Nagoya Math. J.* **27** (1966), 1–5.

[4] J. Grosen, *Rings satisfying polynomial identities or constraints on certain subsets*, Thesis (University of California, Santa Barbara, 1988).

[5] I.N. Herstein, 'A commutativity theorem', *J. Algebra* **38** (1976), 473–478.

[6] Y. Hirano, Y. Kobayashi and H. Tominaga, 'Some polynomial identities and commutativity of s-unital rings', *Math. J. Okayama Univ.* **24** (1982), 7–13.

[7] Y. Hirano, H. Tominaga and A. Yaqub, 'Some polynomial identities and commutativity of s-unital rings. II', *Math. J. Okayama Univ.* **24** (1982), 111–115.

[8] M. Hongan, 'A commutativity theorem for s-unital rings. II', *Math. J. Okayama Univ.* **25** (1983), 19–22.

[9] H. Komatsu and H. Tominaga, 'Chacron's condition and commutativity theorems', *Math. J. Okayama Univ.* **31** (1989), 101–120.

[10] H. Komatsu and H. Tominaga, 'Some commutativity theorems for left s-unital rings', *Resultate Math.* **15** (1989), 335–342.

[11] W. Streb, 'Zur Struktur nichtkommutativer Ringe', *Math. J. Okayama Univ.* **31** (1989), 135–140.

Department of Mathematics
Okayama University
Okayama, 700
Japan