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# A CHARACTERIZATION OF SHARPLY TRANSFERABLE LATTICES

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**1. Introduction.** A lattice $L$ is called *transferable* if and only if, whenever $L$ can be embedded in the ideal lattice $I(K)$ of a lattice $K$, $L$ can be embedded in $K$. $L$ is called *sharply transferable* if and only if, for every lattice embedding $\varphi: L \to I(K)$, there exists an embedding $\psi: L \to K$ such that for $x, y \in L$, $\psi(x) \in \varphi(y)$ if and only if $x \leqq y$. Finite sharply transferable lattices were characterized in [**3**]. In this paper we extend the characterization to the infinite case. We begin by revising some of the terminology of [**3**].

1.1. *Definition.* (a) Let $\langle P; \leqq \rangle$ be a poset and $X, Y \subseteq P$. Then $X$ *dominates* $Y$ (written $X$ Dom $Y$) if and only if, for every $y \in Y$, there exists $x \in X$ such that $y \leqq x$. Dually, $X$ *supports* $Y$ (written $X$ Spp $Y$) if and only if, for every $y \in Y$, there exists $x \in X$ such that $x \leqq y$.

(b) Let $L$ be a lattice, $p \in L$, $U \subseteq L$. Then $\langle p, U \rangle$ is a *join-minimal pair* (JMP) if and only if

(i) $p \leqq \bigvee U$,

(ii) $p \notin U$,

(iii) if $U' \subseteq L$, $p \leqq \bigvee U'$ and $U$ dominates $U'$, then $U \subseteq U'$.

The definition of a *meet-minimal pair* (MMP) is dual.

In [**1**], $X$ Dom $Y$ was written $Y \prec X$, and in [**6**] it was denoted by $Y \ll X$. It is felt that the present terminology is more descriptive, especially with respect to the dual notion. Observe that, if $\langle p, U \rangle$ is a JMP then $U$ is an antichain, every element of $U$ is non-zero and join-irreducible, and $p \nleqq u$ for all $u \in U$. Similar remarks hold for MMP's.

Now consider the following conditions on a lattice $L$.

($R_\vee$) There exists a mapping $\rho: \mathrm{L} \to \omega$ such that, if $\langle p, U \rangle$ is a JMP, then $\rho(p) < \rho(u)$ for each $u \in U$.

($R_\wedge$) There exists a mapping $\sigma: L \to \omega$ such that, if $\langle p, U \rangle$ is an MMP, then $\sigma(p) > \sigma(u)$ for each $u \in U$.

($W$) For all $x, y, u, v \in L$, $x \wedge y \leqq u \vee v$ implies that $[x \wedge y, u \vee v] \cap \{x, y, u, v\} \neq \emptyset$.

($F$) For each $x \in L$, the set $L - [x)$ is finite.

(Here $\omega$ is the set of natural numbers; $[x)$ is the principal dual ideal generated by $x$ and, more generally, if $X \subseteq L$, then $[X)$ is the dual ideal generated by

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$X$; $(x]$ and $(X]$ are the corresponding ideals; if $a \leqq b$ then $[a, b]$ is $[a) \cap (b]$. These and other undefined notations are from [**4**].)

Our principal result is the following:

THEOREM. *A lattice $L$ is sharply transferable if and only if $L$ satisfies $(R_\vee)$, $(R_\wedge)$, $(W)$, and $(F)$.*

A few remarks are in order on comparison with the finite case. The conditions $(R_\vee)$ and $(R_\wedge)$ reduce to conditions $(T_\vee)$ and $(T_\wedge)$, respectively, in the finite case. However, even though $(T_\wedge)$ is the dual of $(T_\vee)$, the condition $(R_\wedge)$ is not the dual of $(R_\vee)$. In fact, in the presence of $(F)$, $(R_\wedge)$ is equivalent to the condition $D'(L) = L$ of [**6**]. The proof of our main result closely parallels the proof in the finite case, the principal difference being the use of the finiteness condition $(F)$ to ensure that certain joins and meets are finite.

Finally, we mention the recent work of R. Freese and J. B. Nation characterizing projective lattices [**2**]. H. Lakser [**7**] has shown that a finite projective lattice is sharply transferable. Using the Freese-Nation characterization and the present characterization of sharp transferability, J. B. Nation has communicated to us a proof that every sharply transferable lattice is projective.

**2. Proof of sufficiency.** We begin with some preliminary observations, mostly without proof. If $x \in L$, let $\mathscr{C}(x)$ denote the set of all antichains $U \subseteq L$ such that $x \leqq \bigvee U$ but $x \nleqq u$ for each $u \in U$. $\mathscr{C}'(x)$ is defined dually.

2.1. LEMMA. *If $L$ satisfies $(F)$, then $\mathscr{C}(x)$ is finite for each $x \in L$. Consequently, if $U \in \mathscr{C}(x)$, then there exists a* JMP $\langle x, U' \rangle$ *such that $U$ dominates $U'$.*

2.2. LEMMA. *If $L$ satisfies $(F)$, then $\mathscr{C}'(x)$ is finite for each $x \in L$. Consequently, if $U \in \mathscr{C}'(x)$, then there exists an* MMP $\langle x, U' \rangle$ *such that $U$ supports $U'$.*

*Proof.* Choose an element $q > x$ and let

$$z = q \vee \bigvee (L - [q)).$$

Let $U$ be an antichain. If $u \in U - [q)$ and $v \in U \cap [z)$, then $v \geqq z \geqq u$, so $u = v \geqq z \geqq q$, contrary to hypothesis. Thus, if $U \cap [z) \neq \emptyset$, then $U \subseteq [q)$, hence $\bigwedge U \geqq q > x$. This shows that if $U \in \mathscr{C}'(x)$ then $U \subseteq L - [z)$ which is finite.

2.3. COROLLARY. *If $K$ is a lattice, $L$ satisfies $(F)$, and $\varphi: L \to K$ is isotone, then $\varphi$ is meet-preserving if and only if $\varphi(x) \geqq \bigwedge \varphi(U)$ for every* MMP $\langle x, U \rangle$.

2.4. LEMMA. *If $L$ satisfies $(F)$, then no element of $L$ can be a member of infinitely many antichains.*

*Proof.* If $U$ is an antichain with $x \in U$, then $U - \{x\} \subseteq L - [x)$.

2.5. *Definition.* Let $\varphi: L \to I(K)$ be an embedding. A mapping $\psi: L \to K$ is called *$\varphi$-normal* if and only if, for $x, y \in L$, $\psi(x) \in \varphi(y)$ if and only if $x \leqq y$.

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2.6. LEMMA. *Let $\varphi: L \to I(K)$ be an embedding. If $L$ satisfies $(R_\vee)$ and $(F)$, then there exists a $\varphi$-normal join-preserving mapping $\psi: L \to K$.*

*Proof.* In view of $(F)$, we can define a $\varphi$-normal mapping $\psi_0$ as follows: given $x \in L$, for each $y \in L$ with $y < x$, we can choose an $\alpha(y) \in \varphi(x) - \varphi(y)$. Then define $\psi_0(x) = \bigvee(\alpha(y)|y < x)$. This is a finite join in view of $(F)$, and $\psi_0$ is obviously $\varphi$-normal. Given the $\varphi$-normal mapping $\psi_n$, we proceed to define $\psi_{n+1}$. For each JMP $\langle p, U\rangle$, since

$$\psi_n(p) \in \varphi(p) \subseteq \bigvee \varphi(U),$$

we can choose for each $u \in U$ an element $\zeta_n(u, p, U) \in \varphi(u)$ such that

$$\psi_n(p) \leqq \bigvee(\zeta_n(u, p, U)|\ u \in U).$$

If $\psi_n(p) \leqq \bigvee \psi_n(U)$, then we restrict the choice to $\zeta_n(u, p, U) = \psi_n(u)$. Finally, for $x \in L$ define

$$\psi_{n+1}(x) = \psi_n(x) \vee \bigvee(\zeta_n(x, p, U)|\ \langle p, U\rangle \text{ is a JMP and } x \in U).$$

By Lemma 2.4 this is a finite join, and $\psi_{n+1}$ is clearly $\varphi$-normal.

Furthermore, if $\langle p, U\rangle$ is any JMP, then clearly $\psi_n(p) \leqq \vee\psi_{n+1}(U)$.

*Claim.* For any $x \in L$, the set $\{\psi_n(x)|n \in \omega\}$ is finite.

Let $\rho: L \to \omega$ be a rank function given by $(R_\vee)$. We will prove the claim by induction on $\rho(x)$. If $\rho(x) = 0$, then there are no JMP's $\langle p, U\rangle$ with $x \in U$, so $\psi_n(x) = \psi_{n+1}(x)$ for all $n$. If $\rho(x) > 0$, let

$$P = \{p \in L|\text{there exists a JMP } \langle p, U\rangle \text{ with } x \in U\}.$$

The set $P$ is finite by Lemma 2.4, and $p \in P$ implies $\rho(p) < \rho(x)$, so by inductive hypothesis we can choose $n_0 \in \omega$ such that for $n \geqq n_0$ and $p \in P$, $\varphi_n(p) = \psi_{n_0}(p)$. Then, if $n > n_0$, $\langle p, U\rangle$ is a JMP and $x \in U$, we have

$$\psi_n(p) = \psi_{n-1}(p) \leqq \bigvee \psi_n(U).$$

Thus, by our restriction on the choice of $\zeta_n$, $\zeta_n(x, p, U) = \psi_n(x)$ for each such $\langle p, U\rangle$, so $\psi_{n+1}(x) = \psi_n(x)$ for all $n > n_0$, proving the claim.

For $x \in L$, let

$$\psi(x) = \bigvee(\psi_n(y)|y \leqq x, n \in \omega).$$

Then $\psi$ is clearly isotone and $\varphi$-normal. To prove $\psi$ is join-preserving it suffices to establish $\psi(\bigvee U) \leqq \bigvee \psi(U)$ for every finite antichain $U \subseteq L$. Let $y \leqq \bigvee U$ and $n \in \omega$. If $y \leqq u \in U$, then

$$\psi_n(y) \leqq \psi(u) \leqq \bigvee \psi(U).$$

If $y \nleqq u$ for all $u \in U$, then by Lemma 2.1 there exists a JMP $\langle y, U'\rangle$ such that $U$ dominates $U'$. Then

$$\psi_n(y) \leqq \bigvee \psi_{n+1}(U') \leqq \bigvee \psi(U') \leqq \bigvee \psi(U).$$

In any case, therefore, we have $\psi_n(y) \leqq \bigvee \psi(U)$ for all $y \leqq \bigvee U$ and $n \in \omega$, so $\psi(\bigvee U) \leqq \bigvee \psi(U)$.

2.7. LEMMA. *Let $L$ satisfy $(R_\wedge)$ and $(F)$, let $\varphi: L \to I(K)$ be an embedding, and let $\psi: L \to K$ be an isotone $\varphi$-normal mapping. Define $\psi^\wedge: L \to K$ by the scheme*

$$\psi^\wedge(x) = \psi(x) \vee \bigvee(\bigwedge \psi^\wedge(U) \mid \langle x, U\rangle \text{ is an MMP}).$$

*Then $\psi^\wedge$ is a meet-preserving $\varphi$-normal mapping. (Note that $\psi^\wedge$ is defined by induction on $\sigma(x)$, where $\sigma: L \to \omega$ is a rank function establishing $(R_\wedge)$. Furthermore, the join in $\psi^\wedge$ is finite by Lemma 2.2.)*

*Proof.* In view of Corollary 2.3, to prove $\psi^\wedge$ is meet-preserving it suffices to prove $\psi^\wedge$ is isotone. We will prove the following statement by induction on $\sigma(x)$:

(1) for all $y \geqq x$, $\psi^\wedge(y) \geqq \psi^\wedge(x)$.

If $\sigma(x) = 0$, this is obvious. If $\sigma(x) > 0$, let $x < y$ and let $\langle x, U\rangle$ be an MMP. We must show $\psi^\wedge(U) \leqq \psi^\wedge(y)$. If there exists $u \in U$ with $y \geqq u$, then since $\sigma(u) < \sigma(x)$, the inductive hypothesis yields

$$\psi^\wedge(y) \geqq \psi^\wedge(u) \geqq \bigwedge \psi^\wedge(U).$$

If $y \ngeq u$ for all $u \in U$, then by Lemma 2.2, there exists an MMP $\langle y, U'\rangle$ such that $U$ supports $U'$. For each $u \in U$, choose $v \in U$ such that $v \leqq u$. Since $\sigma(v) < \sigma(x)$, the inductive hypothesis implies $\psi^\wedge(v) \leqq \psi^\wedge(u)$. Thus, the set $\psi^\wedge(U)$ supports $\psi^\wedge(U')$, so

$$\bigwedge \psi^\wedge(U) \leqq \bigwedge \psi^\wedge(U') \leqq \psi^\wedge(y).$$

This proves (1).

To prove that $\psi^\wedge$ is $\varphi$-normal it suffices to show that $\psi^\wedge(x) \in \varphi(x)$ for all $x \in L$. Again this is obvious if $\sigma(x) = 0$. If $\sigma(x) > 0$ and $\langle x, U\rangle$ is an MMP, then by inductive hypothesis $\varphi^\wedge(u) \in \varphi(u)$ for all $u \in U$, so

$$\bigwedge \psi^\wedge(U) \in \bigwedge \varphi(U) \subseteq \varphi(x).$$

Thus $\psi^\wedge(x) \in \varphi(x)$, since it is a finite join of elements of $\varphi(x)$.

2.8. THEOREM. *If $L$ satisfies $(R_\vee)$, $(R_\wedge)$, $(W)$, and $(F)$, then $L$ is sharply transferable.*

*Proof.* Let $\varphi: L \to I(K)$ be an embedding, and let $\psi$ be the join-preserving mapping given by Lemma 2.6. It then suffices to prove that the mapping $\psi^\wedge$ constructed from $\psi$ in Lemma 2.7 is join-preserving. Let $U \subseteq L$ and $p = \bigvee U$. It remains to show that $\psi^\wedge(p) \leqq \bigvee \psi^\wedge(U)$. Since $\psi$ is join-preserving,

$$\psi(p) \leqq \bigvee \psi(U) \leqq \bigvee \psi^\wedge(U).$$

If $\langle p, U_0 \rangle$ is an MMP, then $\bigwedge U_0 \leqq p = \bigvee U$. Since $U_0 \in \mathscr{C}'(p)$, $(W)$ implies that $\bigwedge U_0 \leqq u$ for some $u \in U$. Therefore

$$\bigwedge \psi^\wedge (U_0) = \psi^\wedge (\bigwedge U_0) \leqq \psi^\wedge (u) \leqq \bigvee \psi^\wedge (U).$$

Thus, $\psi^\wedge (p) \leqq \vee \psi^\wedge (U)$, completing the proof.

The reader familiar with B. Jónsson and J. B. Nation [**6**] should compare the proofs of Lemma 2.7 and Theorem 2.8 with that of Lemma 3.3 of [**6**], observing that their $g_-$ is just the dual of our $\psi^\wedge$.

**3. Necessity of** $(F)$. First we consider the following weaker condition.

3.1. *Definition.* A lattice $L$ is *sectionally finite* if and only if for every $x \in L$, the principal ideal $(x]$ is finite.

Let $A$ be any set and define a *finite partition* of $A$ to be a finite collection of pairwise disjoint finite subsets of $A$ each with more than one element. Let $\mathrm{Part}_{\mathrm{fin}}(A)$ denote the set of all finite partitions of $A$. If $\mathscr{X}, \mathscr{Y} \in \mathrm{Part}_{\mathrm{fin}}(A)$, define $\mathscr{X} \leqq \mathscr{Y}$ to hold if and only if $\mathscr{Y}$ dominates $\mathscr{X}$ with respect to set inclusion. With this ordering, $\mathrm{Part}_{\mathrm{fin}}(A)$ is obviously a sectionally finite lattice. Using Whitman's embedding theorem [**8**] it can be shown that every lattice can be embedded in $I(\mathrm{Part}_{\mathrm{fin}}(A))$ for some $A$ (for details, see [**4**], Theorem IV.4.4 and Corollary IV.4.5). Since sectional finiteness is preserved by sublattices, we have the following.

3.2. LEMMA. *Every transferable lattice is sectionally finite.*

3.3. THEOREM. *If $L$ is sharply transferable, then $L$ satisfies $(F)$.*

*Proof.* Suppose $L$ does not satisfy $(F)$. By Lemma 3.2 we can choose $a_\omega \in L$ which is minimal such that $L - [a_\omega)$ is infinite. Choose distinct elements $a_0, a_1, a_2, \ldots$ in $L - [a_\omega)$ and define the set $K \subseteq L \times (\omega + 1)$:

$$K = \bigcup_{\alpha \leqq \omega} ((L - \bigcup_{\alpha \leqq \beta < \omega} [a_\beta)) \times \{\alpha\}) \cup [a_\omega) \times \{\omega\}.$$

That is, for $\langle x, \alpha \rangle \in L \times \omega$, $\langle x, \alpha \rangle \in K$ if and only if $a_\beta \leqq x$ implies $\beta < \alpha$. Since $L$ is sectionally finite, for each $x \in L$ there exists $\alpha \leqq \omega$ such that $\langle x, \alpha \rangle \in K$.

Thus, for each $x \in L$, define

$$\mu(x) = \bigwedge (\alpha | \langle x, \alpha \rangle \in K).$$

Then the set $K$ with the partial ordering inherited from $L \times (\omega + 1)$ is a lattice with join and meet given by

$$\langle x, \alpha \rangle \vee \langle y, \beta \rangle = \langle x \vee y, \alpha \vee \beta \vee \mu(x \vee y) \rangle$$
$$\langle x, \alpha \rangle \wedge \langle y, \beta \rangle = \langle x \wedge y, \alpha \wedge \beta \rangle.$$

(This is obvious from the fact that for $\langle x, \alpha \rangle \in L \times (\omega + 1)$, $\langle x, \mu(\alpha) \rangle$ is the smallest element of $K$ containing $\langle x, \alpha \rangle$.)

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For $x \in L$, define

$$\varphi(x) = \{ \langle y, \alpha \rangle | \langle y, \alpha \rangle \in K, y \leqq x\}.$$

Projection onto the first factor of $L \times (\omega + 1)$ gives a homomorphism of $K$ onto $L$ (onto because $L$ is sectionally finite), and $\varphi(x)$ is the complete inverse image of the ideal $(x]$ of $L$, hence is an ideal of $K$. This also shows $\varphi: L \to I(K)$ is one-to-one and meet-preserving. To see that $\varphi$ is join-preserving, let $\langle z, \alpha \rangle \in \varphi(x \vee y)$. Then $z \leqq x \vee y$, so

$$\langle z, \alpha \rangle \leqq \langle x, \beta \rangle \vee \langle y, \beta \rangle, \text{ with } \beta = \alpha \vee \mu(x) \vee \mu(y).$$

Thus, $\langle z, \alpha \rangle \in \varphi(x) \vee \varphi(y)$, proving $\varphi(x \vee y) \leqq \varphi(x) \vee \varphi(y)$, the other inclusion being trivial.

Suppose there were a $\varphi$-normal embedding $\psi: L \to K$. Choose an element $b$ which is covered by $a_\omega$. Now, if $x \in L$ and $\psi(x) = \langle y, n \rangle \in \varphi(x)$, then $y \leqq x$. But also $\langle y, n \rangle \in \varphi(y)$, so $\psi(x) \in \varphi(y)$, hence $x \leqq y$ by definition of $\varphi$-normality. Thus $\psi(x) = \langle x, n \rangle$ for some $n \in \omega$. In particular, $\psi(b) = \langle b, n \rangle$, $n \geqq \mu(b)$, and $\psi(a_\omega) = \langle a_\omega, \omega \rangle$. By the minimality of $a_\omega$, $L - [b)$ is finite, hence $b < a_m$ for some $m \geqq n$. Since $a_\omega$ covers $b$ and $a_m \notin [a_\omega)$, we have $b = a_\omega \wedge a_m$. Now $\psi(a_m) = \langle a_m, k \rangle$ for some $k \leqq \omega$, but clearly, $\mu(a_m) > m$, so $k > m \geqq n$. However

$$\psi(a_\omega) \wedge \psi(a_m) = \langle a_\omega, \omega \rangle \wedge \langle a_m, k \rangle = \langle b, k \rangle, \quad k > n$$

which contradicts the assertion that $\psi$ was meet-preserving. This contradiction proves the theorem.

**4. Necessity of** $(W)$. In [**3**], Theorem 4.4, it was proved that every sharply transferable lattice satisfies the following weakening of $(W)$.

$(W')$ $x, y, u, v \in L$ and $u \leqq x \wedge y \leqq u \vee v$ imply

$$[x \wedge y, u \vee v] \cap \{x, y, y, v\} \neq \emptyset.$$

The result of A. Antonius and I. Rival [**1**] implies that a lattice with no infinite chains which satisfies $(SD_\wedge)$ and $(W')$ also satisfies $(W)$. ($(SD_\wedge)$ is the condition that $x \wedge z = y \wedge z$ implies $x \wedge z = (x \vee y) \wedge z$.) As was stated in Remark 4.6 of [**3**], the assumption of no infinite chains can be weakened and, in particular, can be replaced by sectional finiteness. In [**5**] we show that every transferable lattice satisfies $(SD_\wedge)$. Thus we have:

4.1. THEOREM. *A sharply transferable lattice satisfies* $(W)$.

**5. Necessity of** $(R_\vee)$.

5.1. THEOREM. *If $L$ is sharply transferable, then $L$ satisfies* $(R_\vee)$.

*Proof.* For $X \subseteq L \times \omega$, and $n \in \omega$ define

$$X^{(n)} = \{x | \langle x, n \rangle \in X\}.$$

Define $H \subseteq L \times \omega$ to be *closed* if and only if $(H^{(n+1)}] \subseteq H^{(n)}$ for all $n \in \omega$.

Let

$$K_0 = \{H \subseteq L \times \omega | H \text{ is closed}\}.$$

It is obvious that the intersection of any collection of closed sets is closed, so $K_0$ is a lattice under set inclusion. If $X \subseteq L \times \omega$, let $[X]$ denote the smallest element of $K_0$ containing $X$. Then the following are evident:

(i) $[X]^{(n)} = X^{(n)} \cup (\bigcup_{n<m} X^{(m)}]$,

(ii) if $H_1, H_2 \in K_0$, then $H_1 \vee H_2 = [H_1 \cup H_2]$,

(iii) if $H, H_2 \in K_0$, then $H_1 \wedge H_2 = H_1 \cap H_2$.

For $x \in L$, define

$$\varphi_0(x) = \{H \in K_0 | H \subseteq (x] \times \omega\},$$

i.e., $\varphi_0(x)$ is the principal ideal of $K_0$ generated by $(x] \times \omega$. $\varphi_0: L \to I(K_0)$ is obviously an embedding.

Let $K$ be the set of all $H \in K_0$ which are *bounded*, i.e., such that $H \subseteq L \times (n]$ for some $n \in \omega$. Clearly $K$ is an ideal of $K_0$, hence for $x \in L$, $\varphi(x) = \varphi_0(x) \cap K$ is an ideal of $K$.

*Claim* 1. $\varphi: L \to I(K)$ is an embedding.

If $x, y \in L$ and $x \not\leq y$, then $(x] \times \{0\} \in \varphi(x) - \varphi(y)$, proving $\varphi$ is one-to-one. Furthermore,

$$\varphi(x \wedge y) = K \cap \varphi_0(x \wedge y) = K \cap \varphi_0(x) \cap \varphi_0(y) = \varphi(x) \cap \varphi(y),$$

so $\varphi$ is meet-preserving. Thus, to show $\varphi$ is join-preserving, it suffices to establish $\varphi(x \vee y) \subseteq \varphi(x) \vee \varphi(y)$. If $H \in \varphi(x \vee y)$, let $H \subseteq (x \vee y] \times (n]$. But clearly

$$(x \vee y] \times (n] \subseteq (x] \times (n+1] \vee (y] \times (n+1] \in \varphi(x) \vee \varphi(y).$$

This proves Claim 1.

Now let $\psi: L \to K$ be a $\varphi$-normal embedding. Let $J_0(L)$ denote the set of all non-zero join-irreducible elements of $L$. By $(F)$ $J_0(L) \neq \emptyset$ and every element is a finite join of join irreducibles, hence every element $x \in J_0(L)$ has a unique lower cover $x_*$.

*Claim* 2. If $x \in J_0(L)$, then $\langle x, n \rangle \in \psi(x)$ for some $n \in \omega$.

Let $x_*$ be the unique lower cover of $x$. Since $\psi(x) \in \varphi(x)$, we have $\psi(x) \subseteq (x] \times \omega$, so if $\langle x, n \rangle \notin \psi(x)$ for all $n \in \omega$, then $\psi(x) \subseteq (x_*] \times \omega$. Thus, $\psi(x) \in \varphi(x_*)$ which, by $\varphi$-normality, implies $x \leqq x_*$, a contradiction. This proves Claim 2.

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Hence, given $x \in J_0(L)$ we can make the following definition, in view of the boundedness of $\psi(x)$:

$$\rho_0(x) = \bigvee(n \in \omega | \langle x, n \rangle \in \psi(x)).$$

*Claim* 3. If $p \in J_0(L)$ and $\langle p, U \rangle$ is a JMP, then $\rho_0(p) < \rho_0(u)$ for each $u \in U$.

Thus, let $\rho_0(p) = n$, so that $\langle p, n \rangle \in \psi(p)$. Since $\psi$ is join-preserving, we have

$$\psi(p) \subseteq \bigvee\psi(U) = [\bigcap\psi(U)].$$

Let $T = \bigcup\psi(U)$, so $p \in [T]^{(n)}$. Now $p \notin T^{(n)}$ because, if $u \in U$, then $\langle p, n \rangle \in \psi(u)$ and $\psi(u) \in \varphi(u)$ would imply $p \leqq u$, contrary to the definition of JMP.

Hence, by (i),

$$p \in (\bigcup(T^{(m)} | n < m)],$$

so there exists a finite set

$$U' \subseteq \bigcup (T^{(m)} | n < m)$$

such that $p \leqq \bigvee U'$. For each $u' \in U'$, there exists $u \in U$ and $m \in \omega$ such that

$$\langle u', m \rangle \in \psi(u) \in \varphi(u),$$

therefore $u' \leqq u$. Thus, $U$ dominates $U'$, so by definition of JMP, $U \subseteq U'$.

Let $u \in U$. If $u \in T^{(m)}$, then we claim $\langle u, m \rangle \in \psi(u)$. Indeed,

$$\langle u, m \rangle \in \psi(u_0) \in \varphi(u_0)$$

for some $u_0 \in U$, so $u \leqq u_0$. But $U$ is an antichain, so $u = u_0$.

Finally $u \in U$ implies $u \in T^{(m)}$ for some $m > n$, so $q \langle u, m \rangle \in \psi(u)$ by the previous paragraph, proving $\rho_0(u) \geqq m > n$. This proves Claim 3.

In view of the remarks following Definition 1.1, $(R_\vee)$ follows immediately if we define $\rho(x) = \rho_0(x) + 1$ for $x \in J_0(L)$ and $\rho(x) = 0$ for $x \notin J_0(L)$.

## 6. Necessity of $(R_\wedge)$.

6.1. THEOREM. *If $L$ is sharply transferable, then $L$ satisfies $(R_\wedge)$.*

*Proof*. If $X \subseteq L \times \omega$ and $n \in \omega$, let $X^{(n)}$ be as in Section 5, but here define $H \subseteq L \times \omega$ to be *closed* if and only if $[H^{(n)}) \subseteq H^{(n+1)}$ for all $n \in \omega$. Let $K_0$ be the set of all closed sets $H \subseteq L \times \omega$. Again it is obvious that the intersection of any collection of closed sets is closed, so $K_0$ is a lattice under set inclusion.

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Let $K$ be the dual lattice. If $X \subseteq L \times \omega$, let $[X]$ denote the smallest element of $K_0$ containing $X$. Then the following are evident:

(i′) $[X]^{(n)} = X^{(n)} \cup [\bigcup_{m<n} X^{(m)})$.
(ii′) if $H_1, H_2 \in K$, then $H_1 \vee H_2 = H_1 \cap H_2$ in $K$.
(iii′) if $H_1, H_2 \in K$, then $H_1 \wedge H_2 = [H_1 \cup H_2]$ in $K$.

Now, for $x \in L$, define

$$\varphi(x) = \{H \in K \mid \langle x, n \rangle \in H \text{ for some } n \in \omega\}.$$

*Claim* 1. $\varphi$ is an embedding of $L$ into $I(K)$.

For each $n \in \omega$ and $x \in L$, let $\varphi_n(x) = [x) \times [n)$. Clearly $\varphi_n(x) \in K_0$, and $\varphi(x)$ is the union of the increasing chain of principal ideals of $K$ generated by $\varphi_n(x)$, $n \in \omega$, hence $\varphi(x)$ is an ideal.

If $x \nleq y$, then $\varphi_0(x) \in \varphi(x) - \varphi(y)$, so $\varphi$ is one-to-one. Since $\varphi$ is obviously isotone, it suffices to verify that for all $x, y \in L$,

(2) $\quad \varphi(x \vee y) \subseteq \varphi(x) \vee \varphi(y)$, and

(3) $\quad \varphi(x \wedge y) \supseteq \varphi(x) \cap \varphi(y)$.

Suppose $H \in \varphi(x \vee y)$, say $\langle x \vee y, n \rangle \in H$. Then in $K$,

$$H \leqq \varphi_{n+1}(x \vee y) = \varphi_{n+1}(x) \vee \varphi_{n+1}(y).$$

But $\varphi_{n+1}(x) \in \varphi(x)$ and $\varphi_{n+1}(y) \in \varphi(y)$, so $H \in \varphi(x) \vee \varphi(y)$, proving (2).

Suppose $H \in \varphi(x) \cap \varphi(y)$. Then $\langle x, m \rangle$, $\langle y, n \rangle \in H$ for some $m, n \in \omega$. Then $\langle x \wedge y, m + n + 1 \rangle \in H$, so $H \in \varphi(x \wedge y)$. This establishes (3) and completes the proof of Claim 1.

Now suppose $\psi: L \to K$ is a $\varphi$-normal embedding.

*Claim* 2. For $x, y \in L$ and $n \in \omega$, $\langle x, n \rangle \in \psi(y)$ implies $x \geqq y$.

Indeed, $\psi(y) \in \varphi(y)$, so $\langle y, m \rangle \in \psi(y)$ for some $m \in \omega$. Then $\langle x \wedge y, m + n + 1 \rangle \in \psi(y)$, so $\psi(y) \in \varphi(x \wedge y)$. By $\varphi$-normality, $y \leqq x \wedge y \leqq x$, proving Claim 2.

For $x \in L$, define

$$\sigma(x) = \bigwedge (n \in \omega \mid \langle x, n \rangle \in \psi(x)).$$

*Claim* 3. If $\langle p, U \rangle$ is an MMP and $u \in U$, then $\sigma(p) > \sigma(u)$.

Thus, let $\sigma(p) = n$, so that $\langle p, n \rangle \in \psi(p)$. Since $\psi$ is meet-preserving, we have $\psi(p) \geqq \bigwedge \psi(U)$, i.e., $\psi(p) \subseteq [\bigcup \psi(U)]$. Let

$$T = \bigcup \psi(U),$$

so $p \in [T]^{(n)}$. Now $p \notin T^{(n)}$ because, if $u \in U$ and $\langle p, n \rangle \in \psi(u)$, then $p \geqq u$ by Claim 2, contrary to the definition of MMP. Hence, by (i′),

$$p \in [\bigcup (T^{(m)} | m < n)),$$

so there exists a finite set

$$U' \subseteq \bigcup (T^{(m)} | m < n)$$

such that $p \geqq \bigwedge U'$. For each $u' \in U'$ there exists $u \in U$ and $m \in \omega$ such that $\langle u', m \rangle \in \psi(u)$, whence $u' \geqq u$ by (4). Thus, $U$ supports $U'$, so by definition of MMP, $U \subseteq U'$.

Let $u \in U$. If $u \in T^{(m)}$, then we claim $\langle u, m \rangle \in \psi(u)$. Indeed, $\langle u, m \rangle \in \psi(u_0)$ for some $u_0 \in U$, so $u \geqq u_0$ by Claim 2. But $U$ is an antichain, so $u = u_0$.

Finally, $u \in U$ implies $u \in T^{(m)}$ for some $m < n$, so $\langle u, m \rangle \in \psi(u)$ by the previous paragraph, proving $\sigma(u) \leqq m < n$. This proves Claim 3 and the theorem.

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