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Abstract

We prove arithmetic Hilbert–Samuel type theorems for semi-positive singular hermitian line bundles of finite height. This includes the log-singular metrics of Burgos–Kramer–Kühn. The results apply in particular to line bundles of modular forms on some non-compact Shimura varieties. As an example, we treat the case of Hilbert modular surfaces, establishing an arithmetic analogue of the classical result expressing the dimensions of spaces of cusp forms in terms of special values of Dedekind zeta functions.

1. Introduction

Arithmetic intersection theory is an extension of algebraic geometry of schemes over rings of integers of number fields, that incorporates complex geometric tools on the analytic spaces defined by their complex points. Its foundations go back to the work of Arakelov [Ara74], on an extension of intersection theory to this setting. It was vastly generalized in the work of Gillet–Soulé [GS90a, GS90b] and led to the definition of heights of cycles on arithmetic varieties, with respect to hermitian line bundles, by Bost–Gillet–Soulé [BGS94]. This is a counterpart in this theory of the notion of geometric degree of a line bundle, and measures the arithmetic complexity of the equations defining a projective arithmetic variety. One of the major achievements in arithmetic intersection theory was the proof of an analogue of the Grothendieck–Riemann–Roch theorem [GS92], relying on deep results of Bismut and coworkers on analytic torsion. Combining their theorem with work of Bismut and Vasserot [BV89], Gillet and Soulé were able to derive an analogue of the Hilbert–Samuel theorem, relating the covolumes of lattices of global sections of powers of a hermitian ample line bundle to the height of the variety. An alternative approach avoiding the use of analytic torsion was proposed by Abbes and Bouche [AB95]. In view of its diophantine applications, the arithmetic Hilbert–Samuel theorem has been the object of numerous generalizations, as for instance in Zhang [Zha95], Moriwaki [Mor09] and Yuan [Yua08].

Arithmetic intersection theory has also allowed numerical invariants (arithmetic intersection numbers) to be attached to several arithmetic cycles or hermitian vector bundles, for instance those arising from automorphic forms on Shimura varieties. These are involved in the formulation of Kudla's program on generating series of arithmetic intersection numbers and central values of incoherent Eisenstein series, and in conjectures of Maillot–Rössler providing an interpretation of logarithmic derivatives of L-functions at negative integers [MR02]. A technical difficulty in these statements is the fact that, in general, Shimura varieties are not compact, and need to be compactified. This is achieved through the theory of Baily and Borel [BB66] and the theory of toroidal compactifications [AMRT10]. The latter produces pairs (X, D) formed by a projective

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variety and a normal crossings divisor D in X, the boundary of the compactification. Then the analytic component in arithmetic intersection theory requires an extension: the most natural vector bundles on X come equipped with degenerate hermitian structures, whose singularities are localized along D. These are the so-called good metrics, introduced by Mumford in [Mum77]. To deal with this difficulty, the arithmetic intersection theory was extended in the work of Burgos *et al.* [BGKK05]–[BGKK07]. In particular they introduced a variant of the notion of good hermitian metric, namely the notion of log-singular hermitian metric, for which arakelovian heights can be defined. Their formalism was applied with success in the lines of Kudla's program in Bruinier *et al.* [BBGK07].

Contrary to the arithmetic intersection theory of Gillet–Soulé, in the generality of Burgos–Kramer–Kühn there is no analogue of the Grothendieck–Riemann–Roch theorem. The obstruction is of an analytic nature: the analytic torsion forms are not well defined when the hermitian structures are singular. The case of modular curves or, more generally, non-compact hyperbolic curves defined over number fields, was studied by Hahn in his thesis [Hah09] as well as the second author [FiM09a]–[FiM12]. Hahn's approach is in the spirit of the present article (deforming singular metrics to smooth ones) and deals only with the canonical bundle. Freixas' approach is more geometric, relies on Teichmüller theory and the structure of the Deligne–Mumford compactification of the moduli space of curves, and applies to arbitrary powers of the canonical bundle. We also mention work in progress of G. De Gaetano, extending Hahn's approach to powers of the canonical bundle. However, the techniques of these authors do not generalize to higher dimensions.

In this article we prove a general arithmetic Hilbert–Samuel theorem for line bundles in adjoint form, i.e. powers of line bundles twisted by the canonical bundle, and endowed with hermitian metrics with suitable singularities. These are the so-called semi-positive metrics of finite energy, appearing in the work of Boucksom *et al.* [BEGZ10] (see also § 2.3), and they form the biggest possible class of singular semi-positive metrics for which the height can be defined and is a finite real number (see Remark 4.2 below). For instance, this class includes the log-singular metrics of Burgos–Kramer–Kühn. For log-singular hermitian line bundles we are actually able to provide a general statement in not-necessarily adjoint form. Let $\mathscr{X} \to \operatorname{Spec} \mathbb{Z}$ be an arithmetic variety, namely an integral flat projective scheme over $\mathbb{Z}$ of relative dimension n, with smooth generic fibre $\mathscr{X}_{\mathbb{Q}}$. Let $\overline{\mathscr{L}}$ be a line bundle endowed with a semi-positive metric of finite energy. Suppose there exists a model of the canonical sheaf of $\mathscr{X}_{\mathbb{Q}}$, namely an invertible sheaf $\mathscr{K}$ on $\mathscr{X}$ such that $\mathscr{K}_{\mathbb{Q}} = K_{\mathscr{X}_{\mathbb{Q}}}$ (such a model exists possibly after a suitable modification of the model of $\mathscr{X}_{\mathbb{Q}}$). For every $k \geqslant 0$ the cohomology group $H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})$ can be equipped with an L^2 hermitian structure, see § 2.4. Also, the height of $\mathscr{X}$ with respect to $\overline{\mathscr{L}}$, $h_{\overline{\mathscr{L}}}(\mathscr{X})$, is defined in § 4.3. Our first theorem (Theorem 4.3) is stated as follows.

Theorem 1.1. *Suppose there exists an invertible sheaf $\mathscr{K}$ with $\mathscr{K}_{\mathbb{Q}} = K_{\mathscr{X}_{\mathbb{Q}}}$. Let $\overline{\mathscr{L}}$ be a semi-positive line bundle of finite energy. Assume that $\mathscr{L}_{\mathbb{Q}}$ is ample and $\mathscr{L}$ is nef on vertical fibres. Then there is an asymptotic expansion*

$$\widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{L^2} = h_{\overline{\mathscr{L}}}(\mathscr{X}) \frac{k^{n+1}}{(n+1)!} + o(k^{n+1}) \text{ as } k \to +\infty.$$

For log-singular hermitian metrics, we can relax the ampleness assumption and replace $\mathscr{K}$ by an arbitrary log-singular hermitian line bundle. In this situation we have our second theorem (Theorem 4.4).

Theorem 1.2. *Let $D \subset \mathscr{X}_{\mathbb{Q}}$ be a divisor with normal crossings, and $\overline{\mathscr{L}}$ a log-singular hermitian line bundle with singularities along $D(\mathbb{C})$. Assume $\mathscr{L}_{\mathbb{Q}}$ is semi-ample and big, and $\mathscr{L}$ is nef on vertical fibres. Let $\overline{\mathscr{N}}$ be any other log-singular hermitian line bundle. Furthermore, fix a smooth volume form, with respect to which we compute L^2 norms. Then, there is an asymptotic expansion*

$$\widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{N})_{L^2} = h_{\overline{\mathscr{L}}}(\mathscr{X}) \frac{k^{n+1}}{(n+1)!} + o(k^{n+1}) \quad \text{as } k \to +\infty.$$

We note that no assumption is made on the existence of a model of the canonical sheaf. We also recall that the arakelovian heights appearing in the previous theorem enjoy the usual Northcott's type finiteness properties [FiM09b].

The proof exploits recent tools in pluripotential theory. The connection between pluripotential theory and arithmetic intersection theory is already present in the work of Bost [Bos99] and Berman and Boucksom [BB10]. In this last reference, the relation between the energy functional, defining the class of finite energy metrics, and secondary Bott–Chern forms and heights, is exploited. As shown in [BB10] the energy functional can be linked to Donaldson type $\mathcal{L}_k$ functionals, that in the arithmetic setting can be expressed in terms of the covolume of the lattices of sections of the kth powers of an hermitian line bundle. While [BB10] deals with bounded semi-positive metrics, our point is to deal with arbitrary metrics of finite energy. Especially relevant for our purposes is Berndtsson's positivity theorem [Ber09], that is used to prove convexity properties of the $\mathcal{L}_k$ functionals along geodesic paths joining bounded semi-positive metrics. This convexity result is then combined with an easy variant of work of the first author on Bergman kernels [Ber04].

Natural examples of application of Theorems 1.1, 1.2 are provided by integral models of log-canonical sheaves $K_X(D)$ attached to couples (X, D) formed by a smooth projective variety X over $\mathbb{Q}$ together with a divisor with strict normal crossings $D \subset X$. In these situations, under the assumption that $K_X(D)$ is ample, there is a Kähler–Einstein metric of finite energy on $K_X(D)_{\mathbb{C}}$, which is actually log-singular along $D(\mathbb{C})$ [FiM09b]. The content of the main theorem is then the asymptotic behaviour of the covolumes (with respect to the L^2 norms) of the spaces of suitable integral cusp forms. A similar setting arises when considering arithmetic models of toroidal compactifications of non-compact Shimura varieties. In this case, the log-canonical bundles are only semi-ample and big. While for general varieties the Kähler–Einstein metric would only be of finite energy, for these Shimura varieties they are actually log-singular [BGKK05]. Hence, these are examples of application of Theorem 1.2. In § 5 we will focus on the case of Hilbert modular surfaces, establishing an arithmetic analogue of the classical theorem computing the dimension of the space of cusp forms in terms of a special value of a Dedekind zeta function [vdG88, ch. IV, Theorems 1.1 and 4.2]. Let F be a real quadratic number field of prime discriminant $\Delta \equiv 1$ mod 4. For a sufficiently divisible integer $\ell \geqslant 1$, we construct some naive arithmetic model of a toroidal compactification of the Hilbert modular surface attached to $\Gamma_F(\ell) \subset \mathrm{SL}_2(\mathcal{O}_F)$. Here $\Gamma_F(\ell)$ is the principal congruence subgroup of level ℓ of $\mathrm{SL}_2(\mathcal{O}_F)$. Our model $\overline{\mathcal{H}}(\ell)$ is projective over $\operatorname{Spec} \mathbb{Z}$ and smooth over the generic fibre, and comes with a natural semi-ample model $\underline{\omega}$ of the sheaf of Hilbert modular forms of parallel weight 2. Furthermore, $\underline{\omega}_{\mathbb{Q}}$ is big and there is a Kodaira–Spencer isomorphism $\underline{\omega}_{\mathbb{Q}} \simeq K_{\overline{\mathcal{H}}(\ell)_{\mathbb{Q}}}(D)$, where D is the boundary divisor of the toroidal compactification over $\mathbb{Q}$. The sheaf $\underline{\omega}_{\mathbb{C}}$ can be equipped with a Kähler–Einstein metric of finite energy which coincides, through the Kodaira–Spencer isomorphism, with the usual pointwise Petersson metric. As we observed above, it is actually known to be log-singular along $D(\mathbb{C})$. Now, if $\mathscr{K}$ is any model of the canonical sheaf (it will exist for our model $\overline{\mathcal{H}}(\ell)$), the space of global sections $H^0(\overline{\mathcal{H}}(\ell), \underline{\omega}^{\otimes k} \otimes \mathscr{K})$ is an integral structure in the space of Hilbert cusp forms of level ℓ and parallel weight $2k+2$. Its L^2 metric is the Petersson pairing, that we shall indicate by Pet. An application of Theorem 1.2 and the results of Bruinier *et al.* [BBGK07, Theorem 6.4] yields the following statement (Theorem 5.4).

Theorem 1.3. *The following asymptotic formula holds:*

$$\widehat{\deg}\, H^0(\overline{\mathcal{H}}(\ell), \underline{\omega}^{\otimes k} \otimes \mathscr{K})_{\mathrm{Pet}} \\ = -\frac{k^3}{6} d_\ell \zeta_F(-1) \left(\frac{\zeta_F'(-1)}{\zeta_F(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2} \log \Delta \right) + o(k^3),$$

where $d_\ell = [\mathbb{Q}(\zeta_\ell) : \mathbb{Q}][\Gamma_F(1) : \Gamma_F(\ell)]$, *and* ζ_F *is the Dedekind zeta function of* F.

Observe that the result does not depend on the particular model $\mathscr{K}$ of the canonical sheaf, nor on the models of $\overline{\mathcal{H}}(\ell)$ and $\underline{\omega}$ over $\operatorname{Spec} \mathbb{Z}$. A more intrinsic formulation would involve only the spaces of integral sections of finite L^2 norm $H^0(\overline{\mathcal{H}}(\ell), \underline{\omega}^{\otimes (k+1)})_{\mathrm{Pet},L^2}$, with respect to the Petersson norm. Unfortunately, the lack of a good integral model of the toroidal compactification prevents us from doing this. Roughly speaking, the difficulty stems from the fact that we don't know if the flat closure of the boundary of $\overline{\mathcal{H}}(\ell)_{\mathbb{Q}}$ is a Cartier divisor in our naive model over $\operatorname{Spec} \mathbb{Z}$.

Let us finish this introduction by some words about other possible applications. The classical arithmetic Hilbert–Samuel theorem for positive smooth hermitian line bundles is usually applied to produce global integral sections with small sup norm. This requires a comparison of L^2 and sup norms, which is usually proved by Gromov's inequality in the Arakelov geometry literature. In fact, the distortion between the norms coincides with the sup norm of the corresponding Bergman function and all that is needed for the comparison is a bound which is sub-exponential in the power k of the line bundle (this is called the Bernstein–Markov property in the pluripotential literature). In particular it applies to any continuous metric. However, in the full generality of semi-positive metrics of finite energy, such bounds are not available. Indeed, the sup norm is in general not even finite. However, the question makes sense for log-singular metrics if one considers the subspace of sections vanishing along the divisor of singularities. This is the case for cusp forms on Hilbert modular surfaces, as considered above. But the only examples we know of with sub-exponential distortion are given by cusp forms on modular curves [Par88, Xia07]. Already this example shows the interest and non-triviality of the question, since the proof invokes the Ramanujan's bounds for the Fourier coefficients of modular forms, proven by Deligne as a consequence of the Weil conjectures. We hope to come back to this topic in the future.

We briefly review the contents of the article. In § 2 we recall some definitions and facts we need on pluripotential theory and metrics of finite energy. We give an easy version of the local holomorphic Morse inequalities proven by the first author. In § 3 we recall the notion of bounded geodesic paths between bounded semi-positive hermitian metrics, and study the differential of the energy functional along such paths, as well as convexity of $\mathcal{L}_k$ functionals through Berndtsson's positivity theorem. The core of the article is Theorem 3.5 on convergence of $\mathcal{L}_k$ functionals towards energy functionals. In § 4 we introduce the height with respect to a semi-positive line bundle of finite energy, and we prove Theorem 1.1. We then attack the proof of Theorem 1.2, namely the case of log-singular hermitian line bundles, with singularities along rational normal crossing divisors. We conclude with § 5, where we construct some naive integral models of toroidal compactifications of Hilbert modular schemes and prove Theorem 1.3.

2. Pluripotential theory and Bergman measures

In this section we recall some basics on pluripotential theory following [BEGZ10], as well as the construction of the $\mathcal{L}_k$ functionals of [BB10]. We discuss their relation with the Monge–Ampère and the Bergman measures.

2.1 The setting

Let X be a complex manifold of dimension n and L a holomorphic line bundle on X. We fix a smooth hermitian metric h_0 on L, and we suppose that the first Chern form $\omega_0 := \mathrm{c}_1(L, h_0)$ is semi-positive. We introduce the space of ω_0-psh functions. First of all, recall that we call a function $f : X \to \mathbb{R} \cup \{-\infty\}$ psh if it can be locally expressed as the sum of a smooth function and a psh function, in the usual sense of pluripotential theory on $\mathbb{C}^n$. We then define the convex set of ω_0-psh functions to be

$$\mathrm{PSH}(X, \omega_0) := \{\phi : X \to \mathbb{R} \cup \{-\infty\} \text{ psh} \mid \phi \in L^1_{\mathrm{loc}}(X), \omega_\phi := \omega_0 + dd^c\phi \geqslant 0\}.$$

Here $dd^c\phi$ is to be interpreted in the sense of currents. Observe that we can identify $\mathrm{PSH}(X, \omega_0)$ with the space of singular semi-positive metrics on L, by the rule $\phi \mapsto h_\phi := h_0 e^{-\phi}$. The space of bounded ω_0-psh functions $\mathrm{PSH}(X, \omega_0) \cap L^\infty(X)$ will play a prominent role. A smooth ω_0-psh function ϕ is said to be strictly ω_0-psh if the smooth differential form ω_ϕ is strictly positive. One also says that the attached smooth metric h_ϕ is positive. Observe that this notion only makes sense under the smoothness hypothesis.

For the rest of the section, we assume that X is compact, that the line bundle L is ample and that the reference metric h_0 is *strictly positive* (the existence of such a metric is equivalent to the ampleness of L, by Kodaira's embedding theorem).

2.2 The Monge–Ampère operator

By the work of Bedford and Taylor [BT76], the operator sending a smooth ω_0-psh function ϕ to the semi-positive differential form $(\omega_0 + dd^c\phi)^n$ can be extended to $\mathrm{PSH}(X, \omega_0) \cap L^\infty(X)$.

For our purposes it will be convenient to introduce the following normalized Monge–Ampère operator:

$$\mathrm{MA}(\phi) := \frac{1}{\deg(L)} \omega_\phi^n.$$

This is a probability measure putting no mass on pluripolar sets.

2.3 The Aubin–Mabuchi energy functional

By the Bedford–Taylor theory, we can define an energy functional on $\mathrm{PSH}(X, \omega_0) \cap L^\infty(X)$ by the formula

$$\mathcal{E}(\phi) := \frac{1}{(n+1)\deg(L)} \sum_{j=0}^{n} \int_X \phi \omega_\phi^j \wedge \omega_0^{n-j} \quad (\in \mathbb{R}). \tag{2.1}$$

The following proposition summarizes some of its main properties we will need.

Proposition 2.1. (i) *The functional $\mathcal{E}$ is non-decreasing and continuous in the following cases:*

- *along pointwise decreasing sequences of bounded ω_0-psh functions;*
- *along uniformly convergent sequences of bounded ω_0-psh functions.*

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(ii) *The Gâteaux derivative of* $\mathcal{E}$ *on* $\mathrm{PSH}(X,\omega_0)\cap L^\infty(X)$ *is given by*

$$d\mathcal{E}_{|\phi} = \mathrm{MA}(\phi).$$

(iii) *With respect to the convex structure of* $\mathrm{PSH}(X,\omega_0)\cap L^\infty(X)$, $\mathcal{E}$ *is concave. Namely, for given bounded* ω_0*-psh functions* ϕ_0,ϕ_1*, the function* $t\mapsto \mathcal{E}((1-t)\phi_0+t\phi_1)$ *is concave on* $[0,1]$.

Proof. We refer to [BEGZ10, Proposition 2.10], [BB10, Propositions 4.3 and 4.4]. □

The preceding statement suggests a unique monotone and upper semi-continuous extension of $\mathcal{E}$ to $\mathrm{PSH}(X,\omega_0)$ by the rule

$$\mathcal{E}(\phi) = \inf_{\psi\geqslant\phi}\mathcal{E}(\psi)\in[-\infty,+\infty[,$$

where the inf runs over bounded ω_0-psh functions dominating ϕ. It is clear that it remains non-decreasing and concave. Following [BEGZ10, Definition 2.9], the space of *finite energy* functions in $\mathrm{PSH}(X,\omega_0)$ is defined as

$$\mathcal{E}^1(X,\omega_0) = \{\phi\in\mathrm{PSH}(X,\omega_0)\mid \mathcal{E}(\phi)>-\infty\}.$$

Functions of finite energy have full Monge–Ampère mass [BEGZ10, Proposition 2.11]. Moreover, the operator $\mathcal{E}$ extended to $\mathcal{E}^1(X,\omega_0)$ is still continuous along decreasing sequences and formula (2.1) still holds on this space, when the exterior products are interpreted as non-pluripolar products of closed positive currents [BEGZ10, Propositions 2.10–2.17, Corollary 2.18].

2.4 Determinant of the cohomology and the functionals $\mathcal{L}_k$

Let ϕ be an ω_0-psh function of finite energy. For every $k\geqslant 0$, the cohomology group $H^0(X,L^{\otimes k}\otimes K_X)$ is endowed with an L^2 metric. Indeed, given sections s_1,s_2, we write in local coordinates $s_i(z)=f_i(z)\ell_i(z)^k dz_1\wedge\cdots\wedge dz_n$, where the $f_i(z)$ are holomorphic functions and the ℓ_i are local holomorphic sections of L. Then we put

$$\langle s_1,s_2\rangle_{k\phi}(z) := f_1(z)\overline{f_2(z)}h_\phi(\ell_1(z),\ell_2(z))^k|dz_1\wedge\cdots\wedge dz_n|^2,$$

where we follow the notation

$$|dz_1\wedge\cdots\wedge dz_n|^2 = i^n dz_1\wedge d\overline{z}_1\wedge\cdots\wedge dz_n\wedge d\overline{z}_n.$$

Now, the finite energy condition guarantees that the form $\langle s_1,s_2\rangle_{k\phi}$ is an integrable top degree differential form. This is actually a well-known property of finite energy functions. For completeness we recall the argument. First note that ϕ has full Monge–Ampère mass [BEGZ10, Proposition 2.11]. Then, under the working assumption that L is ample and ω_0 is strictly positive, [GZ07, Corollary 1.8] applies and shows that the Lelong numbers of ϕ vanish. By a result of Skoda [Sko72, Proposition 7.1], this implies that for any real number $p\geqslant 0$ (in particular for $p=k$), the function $e^{-p\phi}$ is locally integrable. We may thus define

$$\langle s_1,s_2\rangle_{k\phi,X} = \int_X \langle s_1,s_2\rangle_{k\phi}(z).$$

This gives a non-degenerate hermitian pairing. The determinant line bundle $\det H^0(X,L^{\otimes k}\otimes K_X) = \bigwedge^{N_k} H^0(X,L^{\otimes k}\otimes K_X)$ (with $N_k=\dim H^0(X,L^{\otimes k}\otimes K_X)$) inherits an L^2 metric as well, for which we use the same notation. In other words, given a basis s_k of $H^0(X, L^{\otimes k} \otimes K_X)$ we put

$$\langle s_1 \wedge \cdots \wedge s_{N_k}, s_1 \wedge \cdots \wedge s_{N_k} \rangle_{k\phi,X} := \det(\langle s_i, s_j \rangle_{k\phi,X}).$$

The functional $\mathcal{L}_k$ is defined on $\mathcal{E}^1(X, \omega_0)$ by

$$\mathcal{L}_k(\phi) := -\frac{1}{kN_k} \log \det(\langle s_i^{(k)}, s_j^{(k)} \rangle_{k\phi,X}), \tag{2.2}$$

where the basis $(s_j^{(k)})$ is orthonormal with respect to $\langle \cdot, \cdot \rangle_0$, namely h_0. Observe that the definition does not depend on the choice of orthonormal basis.

Remark 2.2. It is worth stressing that we derived the local integrability of $e^{-p\phi}$ under the ampleness of L and the strict positivity of ω_0. Actually, it is not known whether one can relax these conditions to L nef and big, and ω_0 semi-positive.

2.5 The Bergman measure

As for the operator $\mathcal{E}$, we will be interested in the Gâteaux derivative of the functional $\mathcal{L}_k$. It will be expressed in terms of the Bergman measure, which we proceed to recall. Let ϕ be of finite energy, and take an orthonormal basis $(t_j^{(k)})$ of $H^0(X, L^{\otimes k} \otimes K_X)$, with respect to h_ϕ. Then, according to the notation in § 2.4, the following expression defines a probability measure:

$$\beta_{k\phi}(x) := \frac{1}{N_k} \sum_{j=1}^{k} \langle t_i^{(k)}, t_i^{(k)} \rangle_{k\phi}(x). \tag{2.3}$$

We call it the *Bergman measure*. It acts on measurable functions by integration. We are now in position to compute the derivative of $\mathcal{L}_k$.

Proposition 2.3. *The Gâteaux derivative of $\mathcal{L}_k$ at $\phi \in \mathrm{PSH}(X, \omega_0) \cap L^\infty(X)$ is given by*

$$d\mathcal{L}_{k|\phi} = \beta_{k\phi}.$$

Proof. The proof goes as in [BB10, Lemma 5.1]. □

Remark 2.4. The variational formulas stated in Propositions 2.1–2.3 can be generalized to $\mathcal{E}^1(X, \omega_0)$. This is achieved by approximation by bounded functions. However, in the present article we won't need such a level of generality.

The next statement provides an inequality between the measures $\mathrm{MA}(\phi)$ and $\beta_{k\phi}$, whenever ϕ is a smooth strictly ω_0-psh function. It consists in a special case of the results of Bouche and Tian concerning the asymptotics on the Bergman kernel on the diagonal. In fact, the latter kernel even admits a complete asymptotic expansion; see the survey [Zel09] and references therein. However, the upper bound we need is of elementary nature and inspired by the more general local Morse inequalities proven by the first author [Ber04].

Proposition 2.5 (Local Morse inequalities). *Let ϕ be a smooth and strictly ω_0-psh function. Then there exists a sequence of positive numbers $\delta_k \to 0$ such that*

$$\beta_{k\phi} \leqslant (1 + \delta_k)\, \mathrm{MA}(\phi).$$

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Proof. We refer to [Ber04, Theorem 1.1]. We give a sketch of the argument. The proof exploits the sub-mean inequality of holomorphic functions and the extremal property of the Bergman kernel. Indeed, recall that

$$N_k\beta_{k\phi}(x)=\sup_{s\in H^0(X,L^{\otimes k}\otimes K_X)}\frac{\langle s,s\rangle_{k\phi}(x)}{\langle s,s\rangle_{k\phi,X}}.$$

In a coordinate neighbourhood V centred at x, we trivialize L and write $\langle s,s\rangle_{k\phi}(z)=|f(z)|^2e^{-k\Phi(z)}$, where f is the holomorphic function corresponding to s under the trivialization and Φ is a smooth function on V with $dd^c\Phi=\omega_\phi$. We may choose the trivialization so that

$$\Phi(z)=\Phi_0(z)+o(|z|^2),\quad \Phi_0(z)=\sum_i\lambda_i|z_i|^2. \tag{2.4}$$

Here the λ_i are the eigenvalues of the Hessian of Φ at $z=0$. The strict positivity assumption on ϕ ensures that $\lambda_i>0$ for all i. Now we apply the sub-mean inequality to $|f(z)|^2$ on tori centred at $z=0$ and of radius r, and integrate in $r\in[0,R_k]$ for some small radius R_k. We get

$$|f(0)|^2\prod_j\left(\int_0^{R_k}e^{-\lambda_j r^2}\,d(r^2)\right)\leqslant\int_{|z|\leqslant R_k}|f(z)|^2e^{-k\Phi_0(z)}\,|dz_1\wedge\cdots\wedge dz_n|^2.$$

If we put $R_k=(\log k)/k^{1/2}$ and take (2.4) into account, we can replace Φ_0 by Φ in the previous inequality, up to introducing a factor $(1+\delta_k)$, with $\delta_k\to 0$. Finally, taking into account that the eigenvalues λ_i are strictly positive, the Gaussian integrals can be computed and estimated to derive

$$\langle s,s\rangle_{k\phi}(x)\leqslant(1+\delta_k)\,\mathrm{MA}(\phi)_x\int_{B_{R_k}}\langle s,s\rangle_{k\phi}(z).$$

This concludes the proof. □

Remark 2.6. The proof of the proposition fails whenever ϕ is only assumed to be ω_0-psh, in contrast with strictly ω_0-psh. Indeed, under this weaker assumption there may be null eigenvalues $\lambda_i=0$. The corresponding Gaussian integrals converge to 0 as $k\to\infty$, and therefore one obtains a trivial inequality. However, if we fix a smooth volume form μ, we can obtain an analogous inequality with an error term of the form $\delta_k\mu$. We refer to Remark 3.6 for some difficulties this error term poses.

3. Geodesics in the space of metrics

Following ideas of Mabuchi, Semmes and Donaldson, we consider the problem of joining psh functions in $\mathrm{PSH}(X,\omega_0)$ by *geodesic* paths. The issue of the existence, uniqueness and regularity of geodesics in $\mathrm{PSH}(X,\omega_0)$ for a suitable riemannian structure is a delicate one.[1] The present article deals with a weak notion of geodesic, namely bounded geodesics, good enough to derive consequences for the operators $\mathcal{E}$ and $\mathcal{L}_k$. These bounded geodesics have already been studied by Berndtsson in [Ber13, § 2]. We anticipate the main features: the energy operator $\mathcal{E}$ is affine along geodesics, while the operator $\mathcal{L}_k$ is convex. We stress that the last convexity property relies on positivity results for direct images of line bundles in adjoint form, due to Berndtsson [Ber09]. The

[1] In the literature, one usually restricts to the space of Kähler potentials, namely those smooth ϕ such that $\omega_0+dd^c\phi>0$.

relevant study of bounded geodesics was very recently undertaken in Berman and Berndtsson [BB11], but as a courtesy to the reader we have recalled the proofs of the main properties that we will need.

3.1 Subgeodesics

Let ϕ_0 and ϕ_1 be in $\mathrm{PSH}(X,\omega_0)\cap L^\infty(X)$. Let $\mathcal{T}$ be the Riemann surface with boundary $\mathcal{T}=[0,1]+i\mathbb{R}$ and $\pi_X,\pi_{\mathcal{T}}$ the natural projections from $X\times\mathcal{T}$. With the functions ϕ_0 and ϕ_1 we construct in an obvious manner an $i\mathbb{R}$ invariant function $\phi_{|\partial(X\times\mathcal{T})}$ on the boundary $\partial(X\times\mathcal{T})$. The space of subgeodesics between ϕ_0 and ϕ_1 is by definition

$$\mathcal{K}:=\{\psi \text{ u.s.c. on } X\times\mathcal{T}, \psi\in\mathrm{PSH}(\pi^*\omega_0, X\times\mathring{\mathcal{T}}) \text{ and } \psi_{|\partial(X\times\mathcal{T})}\leqslant\phi_{|\partial(X\times\mathcal{T})}\},$$

where 'u.s.c.' stands for 'upper semi-continuous'. For a subgeodesic ψ, we will usually write $\psi_t(x)$ instead of $\psi(x,t)$.

PROPOSITION 3.1. *Let ψ be a subgeodesic between ϕ_0 and ϕ_1.*

(i) *The function $t\mapsto\mathcal{E}(\psi_t)$ is psh on $(0,1)+i\mathbb{R}$. Moreover, if ψ is locally bounded, then*

$$d_td_t^c\mathcal{E}(\psi_t)=\pi_{\mathcal{T}*}((\pi_X^*\omega_0+dd^c\psi)^{n+1}).$$

Assume moreover that ψ_t does not depend on the imaginary part of t. Then:

(ii) *as a function of $t\in(0,1)$, $t\mapsto\psi_t$ is convex;*

(iii) *if ψ_t is bounded on X uniformly in $t\in[0,1]$ and continuous (as a function of t) at $t=0,1$, then the right derivative of $\mathcal{E}(\psi_t)$ at 0 satisfies*

$$\frac{d}{dt}\bigg|_{t=0^+}\mathcal{E}(\psi_t)\leqslant\int_X\left(\frac{d}{dt}\bigg|_{t=0^+}\psi_t\right)\mathrm{MA}(\psi_0).$$

Proof of Proposition 3.1. For the first item, if ψ is locally bounded the curvature equation is easily checked. In particular it implies the psh property. The general case follows by approximation by bounded functions. We refer to [BBGZ12, Proposition 6.2] for further details.

Assume, until the end of the proof, that ψ_t is independent of the imaginary part of t.

For the second item, the function $u(t)$ is convex on $(0,1)$ by subharmonicity on $(0,1)+i\mathbb{R}$ and independence of $\mathrm{Im}(t)$.

For the last property, we first recall that $\mathcal{E}$ is concave and that $d\mathcal{E}_{|\phi}=\mathrm{MA}(\phi)$ on $\mathrm{PSH}(X,\omega_0)\cap L^\infty(X)$. Therefore, we have for $t>0$

$$\frac{\mathcal{E}(\psi_t)-\mathcal{E}(\psi_0)}{t}\leqslant\int_X\frac{\psi_t-\psi_0}{t}\mathrm{MA}(\psi_0).$$

Now the function ψ_t is convex in $t\in[0,1]$, because it is convex on $(0,1)$ by (ii) and continuous at $t=0,1$. It follows that for $0<t\leqslant 1$, $(\psi_t-\psi_0)/t$ is uniformly bounded above by $\sup_X(\psi_1-\psi_0)<+\infty$. Besides, it is decreasing as $t\searrow 0$, by convexity again. Therefore, by the monotone convergence theorem we conclude

$$\frac{d}{dt}\bigg|_{t=0^+}\mathcal{E}(\psi_t)\leqslant\int_X\left(\frac{d}{dt}\bigg|_{t=0^+}\psi_t\right)\mathrm{MA}(\psi_0),$$

as was to be shown. □

3.2 Bounded geodesics

We maintain the notation of the preceding section. Let ϕ_0, ϕ_1 be bounded ω_0-psh functions and ψ a subgeodesic between them. We say that ψ is a *bounded geodesic* joining ϕ_0 and ϕ_1 if the following conditions are fulfilled:

- ψ_t is independent of the imaginary part of t;
- ψ_t is bounded on X uniformly in $t \in [0,1]$, and converges uniformly to ϕ_0 (respectively ϕ_1) as $t \to 0$ (respectively $t \to 1$).
- ψ solves the degenerate Monge–Ampère equation on $X \times \mathring{\mathcal{T}}$

$$(\pi_X^* \omega_0 + dd^c \psi)^{n+1} = 0. \tag{3.1}$$

Observe that the boundedness assumption permits us to state (3.1) within the frame of Bedford–Taylor theory.

Proposition 3.2. *Let ϕ_t be a geodesic path between $\phi_0, \phi_1 \in \mathrm{PSH}(X, \omega_0) \cap L^\infty(X)$. Then:*

(i) *the function $t \mapsto \mathcal{E}(\phi_t)$ is affine on $[0,1]$;*

(ii) *the following inequality holds:*

$$\mathcal{E}(\phi_1) - \mathcal{E}(\phi_0) \leqslant \int_X \left(\frac{d}{dt}\bigg|_{t=0^+} \phi_t \right) \mathrm{MA}(\phi_0).$$

Proof. First of all, by Proposition 3.1(i) and (3.1), the function $\mathcal{E}(\phi_t)$ is harmonic on $(0,1) + i\mathbb{R}$, and independent of $\mathrm{Im}(t)$ by assumptions on ϕ_t. Therefore it is affine on $(0,1)$. Moreover, ϕ_t uniformly converges to ϕ_0 (respectively ϕ_1) as $t \to 0$ (respectively $t \to 1$). Then, by Proposition 2.1(ii) we deduce that $\mathcal{E}(\phi_t)$ is continuous at $t = 0, 1$. This shows the first assertion. For the second one, we apply Proposition 3.1(iii) and take into account the previous affineness property, that guarantees

$$\frac{d}{dt}\bigg|_{t=0^+} \mathcal{E}(\phi_t) = \mathcal{E}(\phi_1) - \mathcal{E}(\phi_0).$$

The proof is complete. □

The existence of bounded geodesics is discussed in [Ber13, § 2.2]. We provide a more detailed argument for the convenience of the non-expert readers.

Proposition 3.3. *Let $\phi_0, \phi_1 \in \mathrm{PSH}(X, \omega_0) \cap L^\infty(X)$. Then there exists a bounded geodesic ϕ_t between ϕ_0 and ϕ_1.*

Proof. It will be convenient to introduce the annulus $A = \{1 \leqslant |z| \leqslant e\}$, so that $z \to e^z$ defines a locally conformal mapping from $\mathcal{T}$ to A. We have the corresponding notion of (sub)geodesics. Radial functions (respectively (sub) geodesics) on A pull-back to functions on $\mathcal{T}$ independent of $\mathrm{Im}(t)$ (respectively (sub) geodesics). We may thus work on $X \times A$. We consider the upper envelope

$$\phi := \sup\{\psi \text{ subgeodesic on } X \times A \text{ between } \phi_0, \phi_1\}.$$

Observe that the set of subgeodesics under consideration is not empty (for an example, see the barrier below). First we claim that ϕ is a radial subgeodesic. It is an ω_0-psh function on $X \times \mathring{A}$: indeed, the upper semi-continuous regularization ϕ^* is a candidate in the sup, hence $\phi = \phi^*$. In addition, the radial function

$$\widetilde{\phi}_z(x) = \sup_{\theta \in [0, 2\pi]} \phi_{ze^{i\theta}}(x)$$

is also a candidate in the sup, so that $\phi = \widetilde{\phi}$ is radial. Secondly, we claim that ϕ is bounded and uniformly converges to ϕ_0 (respectively ϕ_1) when $|z| \to 1$ (respectively $|z| \to e$). For this, we follow [Ber13, § 2.2] and introduce a barrier

$$\chi_z = \max(\phi_0 - C\log|z|, \phi_1 + C(\log|z| - 1)).$$

For C sufficiently large and because ϕ_0, ϕ_1 are bounded, this barrier is a candidate in the sup. By [Ber13, § 2.2] one has

$$\phi_0 - C\log|z| \leqslant \phi_z \leqslant \phi_0 + C\log|z|$$

and similarly for ϕ_1. These inequalities show that ϕ_z is uniformly bounded in z and uniformly converges to ϕ_0, ϕ_1 when $|z| \to 1, e$.

To conclude, it remains to show that ϕ so defined satisfies the degenerate Monge–Ampère equation (3.1) on $X \times \mathring{A}$. The argument is standard and based on the classical Perron method. We provide the details for the sake of completeness (see also [BB10, proof of Proposition 2.10]). Let B be an open ball, relatively compact in $X \times \mathring{A}$. We already saw that the function ϕ is bounded, in particular on $\overline{B}$. Therefore, by the Bedford–Taylor theory [BT76, Theorem D][2] we can find a $\pi_X^*\omega_{0|B}$-psh function ψ on B, which coincides with ϕ on ∂B. Then we define

$$\widetilde{\phi} = \begin{cases} \psi & \text{on } \overline{B}, \\ \phi & \text{on } (X \times A)\backslash\overline{B}. \end{cases}$$

On the one hand, $\widetilde{\phi} \geqslant \phi$. Indeed, ψ is a decreasing limit of Perron envelopes on B with boundary values decreasing to ϕ, thus $\psi \geqslant \phi_{|\overline{B}}$. On the other hand, $\widetilde{\phi}$ is still a psh function in the sup defining ϕ. Therefore $\phi = \widetilde{\phi}$. Hence ϕ satisfies the degenerate Monge–Ampère equation on B. The proof is now complete. □

3.3 Berndtsson's positivity, convexity of $\mathcal{L}_k$ and convergence to energy

In the previous sections we studied the behaviour of the functional $\mathcal{E}$ along subgeodesics. We now consider the operator $\mathcal{L}_k$. Together with the variational formulas of $\mathcal{E}$ and $\mathcal{L}_k$, we are going to establish a comparison between both, at least for big values of k.

Let the notation be as before. Let ϕ_t be a subgeodesic between bounded functions ϕ_0, ϕ_1. We assume that ϕ_t is uniformly bounded. This is for instance the case for geodesic paths. For every t, ϕ_t defines a semi-positively curved singular bounded metric on L, $h_t = h_0 e^{-\phi_t}$. Then $\det H^0(X, L^{\otimes k} \otimes K_X)$ inherits an L^2 hermitian structure that, we recall, we denote by $\langle \cdot, \cdot \rangle_{k\phi_t, X}$. This family of metrics glues into a singular hermitian metric on the constant sheaf over $\mathcal{T}$ of fibre $\det H^0(X, L^{\otimes k} \otimes K_X)$.[3]

The previous construction can be equivalently seen as the family L^2 metric on $\det \pi_{\mathcal{T}*}(\pi_X^*(L^{\otimes k}) \otimes K_{X\times\mathcal{T}/\mathcal{T}})$ attached to the singular semi-positive metric $\pi_X^*(h_0)e^{-\phi_t}$ on $\pi_X^*(L)$. Observe that $\pi_{\mathcal{T}}$ is a proper submersion. This suggests the use of positivity properties for direct images of hermitian line bundles. The main result we need is due to Berndtsson [Ber09].

[2] Strictly speaking, [BT76, Theorem D] requires a continuous boundary datum. The bounded case follows by approximation by a decreasing sequence of continuous functions (possible by upper semi-continuity of ϕ), by the minimum principle [BT76, Theorem A] and the continuity of Monge–Ampère measures along decreasing sequences of psh functions.

[3] The resulting family is locally integrable in t by Fubini's theorem and the local integrability of $\phi_t \in \mathrm{PSH}(X \times \mathring{\mathcal{T}}, \pi_X^*\omega_0) \subset L^1_{\mathrm{loc}}(X \times \mathring{\mathcal{T}}, \mathbb{R})$.

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Proposition 3.4. *Let ϕ_t be a uniformly bounded subgeodesic between ϕ_0 and ϕ_1.*

(i) *The function $t \mapsto \mathcal{L}_k(\phi_t)$ is psh on $(0,1) + i\mathbb{R}$.*
(ii) *If ϕ_t is a geodesic, then $t \mapsto \mathcal{L}_k(\phi_t)$ is convex and continuous on $[0,1]$.*

Proof. In the case ϕ_t is a smooth subgeodesic, the first property follows from [Ber09, Theorem 1.1]. The general case can be reduced to the smooth case by approximation. Indeed, since L is assumed ample we may, thanks to Demailly's regularization result, write the subgeodesic as a decreasing limit of smooth ones and use the basic fact that a decreasing limit of convex functions on an open interval is convex. Alternatively, the convexity in the singular case follows from the general results in Berndtsson and Păun [BP08]. For the second property, convexity (hence continuity) on the open interval $(0,1)$ is obvious, since the geodesic conditions ensure that ϕ_t does not depend on $\mathrm{Im}(t)$. Also, ϕ_t uniformly converges to ϕ_0 (respectively ϕ_1) as $t \to 0$ (respectively $t \to 1$). By the dominate convergence theorem, $\mathcal{L}_k(\phi_t)$ is continuous at $t = 0, 1$. This concludes the proof. □

We are now in position to state and prove the main theorem of the section.

Theorem 3.5. *Let $\phi \in \mathcal{E}^1(X, \omega_0)$. Then we have*

$$\lim_{k\to+\infty} \mathcal{L}_k(\phi) = \mathcal{E}(\phi).$$

Proof. We begin by showing the inequality

$$\limsup_{k\to+\infty} \mathcal{L}_k(\phi) \leqslant \mathcal{E}(\phi). \tag{3.2}$$

Let us consider a decreasing sequence of smooth functions $\phi_j \searrow \phi$, which always exists. For every j, the quantity $\mathcal{L}_k(\phi_j)$ is well defined and clearly satisfies $\mathcal{L}_k(\phi) \leqslant \mathcal{L}_k(\phi_j)$. By [BB10, Theorem A][4]

$$\lim_k \mathcal{L}_k(\phi_j) = \mathcal{E}(P_X\phi_j),$$

where $P_X\phi_j$ is the ω_0-psh projection of ϕ_j, namely

$$P_X\phi_j = \sup\{\psi \in \mathrm{PSH}(X,\omega_0) \mid \psi \leqslant \phi_j\}.$$

By the very definition of this projection, we have the inequalities $\phi \leqslant P_X\phi_j \leqslant \phi_j$ and the sequence $P_X\phi_j$ is decreasing. Therefore, $P_X\phi_j$ decreases to ϕ and by the monotonicity of the functional $\mathcal{E}$ we have

$$\lim_j \mathcal{E}(P_X\phi_j) = \mathcal{E}(\phi).$$

Alternatively, because L is ample and ω_0 strictly positive, it is actually possible to choose such a sequence of smooth functions ϕ_j in $\mathrm{PSH}(X,\omega_0)$. See [GZ05, Theorem 8.1]. This avoids the use of the projector P_X. In either case, this concludes the proof of (3.2).

Now for the inequality

$$\liminf_{k\to+\infty} \mathcal{L}_k(\phi) \geqslant \mathcal{E}(\phi). \tag{3.3}$$

[4] Strictly speaking, [BB10, Theorem A] applies to $L^{\otimes k}$. The arguments can be easily adapted to obtain the corresponding results for $L^{\otimes k} \otimes K_X$.

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Let us introduce the functional on $\mathcal{E}^1(X, \omega_0)$

$$\mathcal{F}_k(\psi) := \mathcal{L}_k(\psi) - \mathcal{E}(\psi).$$

Observe that $\mathcal{F}_k$ is continuous along decreasing sequences, because $\mathcal{L}_k$ and $\mathcal{E}$ are. Indeed, for $\mathcal{L}_k$ this follows from the dominate convergence theorem, while for $\mathcal{E}$ this property was already invoked previously (Proposition 2.1(i)). Also, it is easily seen to be invariant under translation of ψ by constants. We claim that the inequality

$$\mathcal{F}_k(\phi) \geqslant \delta_k \mathcal{E}(\phi) \tag{3.4}$$

holds, with δ_k being the sequence of Proposition 2.5 applied to $\phi_0 := 0$ (which is tautologically strictly ω_0-psh). This will be enough to conclude. For this, let ψ_j be a sequence of bounded ω_0-psh functions decreasing to ϕ. Let us fix the index j. After possibly making a translation by a constant, we may assume that $\psi_j \leqslant 0$. Let ϕ_t be the bounded geodesic between $\phi_0 = 0$ and $\phi_1 := \psi_j$ (Proposition 3.3). The function $\mathcal{F}_k(\phi_t)$ is convex, because $\mathcal{L}_k(\phi_t)$ is convex (Proposition 3.4) and $\mathcal{E}(\phi_t)$ is affine (Proposition 3.2). Therefore, by convexity and by the variational formulas for $\mathcal{L}_k$ (Proposition 2.3) and $\mathcal{E}$ (Proposition 2.1)

$$\begin{aligned}\mathcal{F}_k(\psi_j) = \mathcal{F}_k(\phi_1) - \mathcal{F}_k(\phi_0) &\geqslant \left.\frac{d}{dt}\right|_{t=0^+} \mathcal{F}_k(\phi_t) \\ &= \int_X \left(\left.\frac{d}{dt}\right|_{t=0^+} \phi_t\right)(\beta_{k\phi_0} - \mathrm{MA}(\phi_0)).\end{aligned} \tag{3.5}$$

Now by Proposition 2.5 we have $\beta_{k\phi_0} - \mathrm{MA}(\phi_0) \leqslant \delta_k \, \mathrm{MA}(\phi_0)$ and by convexity of ϕ_t (Proposition 3.1(ii)) we find

$$\left.\frac{d}{dt}\right|_{t=0^+} \phi_t \leqslant \phi_1 - \phi_0 \leqslant 0.$$

Combined with (3.5) we derive

$$\mathcal{F}_k(\psi_j) \geqslant \delta_k \int_X \left(\left.\frac{d}{dt}\right|_{t=0^+} \phi_t\right) \mathrm{MA}(\phi_0).$$

Finally, we apply Proposition 3.2(ii) to get

$$\mathcal{F}_k(\psi_j) \geqslant \delta_k(\mathcal{E}(\phi_1) - \mathcal{E}(\phi_0)) = \delta_k \mathcal{E}(\psi_j).$$

Therefore, if we let j tend to $+\infty$, we obtain the desired inequality (3.4) and hence (3.3). The proof is complete. □

Remark 3.6. (i) We don't know if the previous theorem holds under the weaker assumption that ω_0 is only semi-positive. As we explained in Remark 2.6, in this case a weak form of Proposition 2.5 holds, with an error term of the form $\delta_k \mu$, where μ is a fixed positive volume form. Following the argument of the proof of the theorem, we would be led to find a lower bound for the integral

$$\int_X \left(\left.\frac{d}{dt}\right|_{t=0^+} \phi_t\right) \mu$$

in terms of the energy of the geodesic at time $t = 1$, i.e. $\mathcal{E}(\phi_1)$. This is an open problem in pluripotential theory that we hope to explore in the future.

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(ii) Our results in the arithmetic setting below suggest that the statement of the theorem should hold in the semi-positive case, as long as we restrict to a large class of functions of finite energy, namely functions with logarithmic singularities along a divisor.

(iii) Along subgeodesics (independent of the imaginary part of the parameter) the operator $\mathcal{E}$ is convex. However, the right property required in the proof is that $-\mathcal{E}$ be convex. In general we can only ensure this along geodesics, which shows the necessity of this notion.

(iv) In view of the previous remark, it is tempting to try the argument of the proof with affine paths $\phi_t = (1-t)\phi_0 + t\phi_1$ instead of geodesics. Indeed, along such paths $-\mathcal{E}$ is actually convex. However, convexity of $\mathcal{L}_k$ along ϕ_t is in general not guaranteed by Berndtsson's theorem.

4. Metrics of finite energy and arithmetic intersection theory

4.1 The geometric setting

Let $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic variety, namely an integral flat projective scheme over $\mathbb{Z}$ of relative dimension n, with smooth generic fibre $\mathscr{X}_{\mathbb{Q}}$. The set of complex points $\mathscr{X}(\mathbb{C})$ has a natural structure of smooth complex analytic space of pure dimension n. We denote by $F_\infty : \mathscr{X}(\mathbb{C}) \to \mathscr{X}(\mathbb{C})$ the antiholomorphic involution given by the action of complex conjugation.

Let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$. We will assume that $\mathscr{L}_{\mathbb{Q}}$ is semi-ample[5] and big over $\mathscr{X}_{\mathbb{Q}}$, although more restrictive hypotheses will be necessary in some cases. Then, the elements of pluripotential theory developed in the preceding sections can be applied to $\mathscr{X}(\mathbb{C})$ and $\mathscr{L}_{\mathbb{C}}$.

Definition 4.1. A semi-positive arakelovian metric of finite energy on $\mathscr{L}$, or simply a metric of finite energy, is a singular hermitian metric on $\mathscr{L}_{\mathbb{C}}$ of the form $h_\phi = h_0 e^{-\phi}$, where:

- h_0 is a smooth hermitian metric on $\mathscr{L}_{\mathbb{C}}$ with semi-positive first Chern form $\omega_0 := \mathrm{c}_1(\overline{\mathscr{L}}_0)$;
- $\phi \in \mathcal{E}^1(\mathscr{X}(\mathbb{C}), \omega_0)$;
- h_ϕ is invariant under the action of complex conjugation F_∞.

Because $\mathscr{L}_{\mathbb{Q}}$ is semi-ample, a smooth metric h_0 with the listed properties always exists, and may be chosen to be invariant under complex conjugation. Furthermore, if $\mathscr{L}_{\mathbb{Q}}$ is ample, we can (and we will) assume that ω_0 is strictly positive. We fix h_0 and $h = h_\phi$ once and for all, and write $\overline{\mathscr{L}}_0$ and $\overline{\mathscr{L}}$ for the corresponding hermitian line bundles.

4.2 Arithmetic degrees and $\mathcal{L}_k$ functionals

We assume given an invertible sheaf $\mathscr{K}$ on $\mathscr{X}$, coinciding over the generic fibre with the canonical sheaf: $\mathscr{K}_{\mathbb{Q}} = K_{\mathscr{X}_{\mathbb{Q}}}$. For every integer $k \geqslant 0$, the module of global sections $M_k := H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})$ is a lattice in the finite-dimensional real vector space $M_{k,\mathbb{R}} := H^0(\mathscr{X}(\mathbb{C}), \mathscr{L}_{\mathbb{C}}^{\otimes k} \otimes K_{\mathscr{X}_{\mathbb{C}}})^{F_\infty}$. According to the discussion in § 2.4, attached to h there is a natural L^2 euclidean structure $\|\cdot\|_2$ on $M_{k,\mathbb{R}}$. We can thus compute the covolume of the lattice with respect to this structure. The *arithmetic degree* of $(M_k, \|\cdot\|_2)$ is by definition

$$\widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{L^2} := -\log \operatorname{vol}\left(\frac{M_{k,\mathbb{R}}}{M_k}\right).$$

We introduce an arithmetic counterpart of the functional $\mathcal{L}_k$ defined in (2.2):

$$\mathcal{L}_k^{\mathrm{ar}}(\phi) := \frac{2}{kN_k}\, \widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{L^2}, \tag{4.1}$$

[5] Recall this means that a sufficiently large power $\mathscr{L}_{\mathbb{Q}}^{\otimes k}$ is generated by global sections.

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where $N_k = \dim H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k} \otimes \mathscr{K}_{\mathbb{C}})$, $k \gg 0$, and we recall that $h = h_\phi = h_0 e^{-\phi}$. Because $\mathscr{L}_{\mathbb{Q}}$ is nef and big, by the Kawamata–Viehweg vanishing theorem [Laz04, Theorem 4.3.1] we have $H^i(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k} \otimes \mathscr{K}_{\mathbb{C}}) = 0$ for $i, k \geqslant 1$. The Riemann–Roch theorem then provides the estimate

$$N_k = \frac{k^n}{n!} \deg \mathscr{L}_{\mathbb{C}} + o(k^n).$$[6]

We bring the reader's attention to the factor 2 in the definition (4.1), included to ensure the compatibility with the analytic $\mathcal{L}_k$ functional dealt with so far. With this normalization, given another ω_0-psh function ϕ' of finite energy, the following relation with the $\mathcal{L}_k$ functional is readily shown:

$$\mathcal{L}_k^{\mathrm{ar}}(\phi) - \mathcal{L}_k^{\mathrm{ar}}(\phi') = \mathcal{L}_k(\phi) - \mathcal{L}_k(\phi'). \tag{4.2}$$

In this expression, the $\mathcal{L}_k$ functional is for the line bundle $\mathscr{L}_{\mathbb{C}}$ on $\mathscr{X}(\mathbb{C})$ and depends on the choice of a fixed orthonormal basis of $M_{k,\mathbb{C}}$, with respect to the L^2 metric attached to h_0. In particular, by choosing $\phi' = \phi_0 = 0$, we thus have

$$\mathcal{L}_k^{\mathrm{ar}}(\phi) = \mathcal{L}_k^{\mathrm{ar}}(\phi_0) + \mathcal{L}_k(\phi). \tag{4.3}$$

4.3 Heights

The height $h_{\overline{\mathscr{L}}_0}(\mathscr{X})$ of $\mathscr{X}$ with respect to $\overline{\mathscr{L}}_0$ (more generally of any cycle in $\mathscr{X}$) has been defined by Bost *et al.* [BGS94, § 3] by means of higher-dimensional arithmetic intersection theory. It is an arithmetic analogue of the degree of a variety. When the metric h is logarithmically singular in the sense of Burgos *et al.* [BGKK07, § 7], the generalized arithmetic intersection theory of [BGKK07, § 7] still allows us to define the height of $\mathscr{X}$ with respect to $\overline{\mathscr{L}}$. It satisfies

$$h_{\overline{\mathscr{L}}}(\mathscr{X}) = h_{\overline{\mathscr{L}}_0}(\mathscr{X}) + \tfrac{1}{2}(n+1)(\deg \mathscr{L}_{\mathbb{C}})\mathcal{E}(\phi), \quad n+1 = \dim \mathscr{X}, \tag{4.4}$$

where the energy $\mathcal{E}(\phi)$ is computed with respect to ω_0 [FiM09b, Proposition 6.5]. We refer to [FiM09b] for a detailed study of these heights. More generally, if h is an arbitrary semi-positive metric of finite energy, we define $h_{\overline{\mathscr{L}}}(\mathscr{X})$ by (4.4). By the properties of the energy functional [BB10, Corollary 4.2 and Remark 4.6], one proves with ease that $h_{\overline{\mathscr{L}}}(\mathscr{X})$ is intrinsically defined, namely it only depends on h and not on the smooth reference metric h_0. Observe however that, while $h_{\overline{\mathscr{L}}}(\mathscr{X})$ can be defined, the height of a cycle in $\mathscr{X}$ is in general meaningless.

Remark 4.2. Actually, formula (4.4) and the monotonicity properties of the energy functional $\mathcal{E}$ show that the height can be extended to singular semi-positive hermitian line bundles $\overline{\mathscr{L}}$, just by declaring

$$\begin{aligned} h_{\overline{\mathscr{L}}_\phi}(\mathscr{X}) &:= \inf_{\psi \geqslant \phi} h_{\overline{\mathscr{L}}_\psi}(\mathscr{X}) \\ &= h_{\overline{\mathscr{L}}_0}(\mathscr{X}) + \tfrac{1}{2}(n+1)(\deg \mathscr{L}_{\mathbb{C}}) \inf_{\psi \geqslant \phi} \mathcal{E}(\psi) \in \mathbb{R} \cup \{-\infty\}. \end{aligned}$$

Here the inf runs over the bounded ω_0-psh functions, invariant under the action of complex conjugation. With this definition, the assignment $\phi \mapsto h_{\overline{\mathscr{L}}_\phi}(\mathscr{X})$ is non-decreasing and continuous along pointwise decreasing sequences of ω_0-psh functions, invariant under complex conjugation. Furthermore, this extension is uniquely determined by these properties. Therefore, the class of semi-positive hermitian metrics of finite energy is the biggest one for which the height can be defined and is a real number.

[6] An elementary argument shows that, by nefness of $\mathscr{L}_{\mathbb{Q}}$ and projectivity of $\mathscr{X}_{\mathbb{Q}}$, the higher cohomology groups are $o(k^n)$. This is enough to obtain the estimate.

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4.4 Arithmetic Hilbert–Samuel theorems

Our first statement is the following arithmetic analogue of the Hilbert–Samuel theorem in adjoint form.

THEOREM 4.3. *Let $\mathscr{X} \to \operatorname{Spec} \mathbb{Z}$ be an arithmetic variety of relative dimension n and $\overline{\mathscr{L}} = (\mathscr{L}, h)$ a semi-positive hermitian line bundle of finite energy. Assume $\mathscr{L}_{\mathbb{Q}}$ is ample and that $\mathscr{L}$ is nef on vertical fibres. Furthermore, suppose there exists an invertible sheaf $\mathscr{K}$ such that $\mathscr{K}_{\mathbb{Q}} = K_{\mathscr{X}_{\mathbb{Q}}}$. Then the following asymptotic expansion holds:*

$$\widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{L^2} = h_{\overline{\mathscr{L}}}(\mathscr{X}) \frac{k^{n+1}}{(d+1)!} + o(k^{n+1}) \text{ as } k \to +\infty. \tag{4.5}$$

Proof. We reduce to the case of the smooth and positive hermitian metric h_0. Proving the theorem is tantamount to estimating the asymptotics of $\mathcal{L}_k^{\mathrm{ar}}(\phi)$, where we recall that $h = h_0 e^{-\phi}$ and ϕ is of finite energy. By the relation (4.3), we have to estimate $\mathcal{L}_k^{\mathrm{ar}}(\phi_0)$, with $\phi_0 = 0$, as well as $\mathcal{L}_k(\phi)$. For the last term $\mathcal{L}_k(\phi)$, we can apply Theorem 3.5:

$$\lim_{k \to +\infty} \mathcal{L}_k(\phi) = \mathcal{E}(\phi).$$

For the term $\mathcal{L}_k^{\mathrm{ar}}(\phi_0)$, the statement is well known, and is a particular instance of [Zha95, Theorem 1.4]. Observe that [Zha95, Theorem 1.4] is stated for sup norms, which are classically related to the L^2 norms by Gromov's inequality. For this, let us fix a smooth hermitian metric on $\mathscr{K}$ and a smooth volume form μ. Then, with respect to these metrics, we can define sup norms $\|\cdot\|_\infty$ on the spaces $H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{\mathbb{C}}$, $k \geqslant 0$. Then, Gromov's inequality asserts there is a positive constant $C > 0$ such that

$$C^{-1} \|s\|_{L^2}^2 \leqslant \|s\|_\infty^2 \leqslant C k^n \|s\|_{L^2}^2$$

for all $s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{\mathbb{C}}$, $k \geqslant 0$. Alternatively, this inequality can also be derived from Proposition 2.5. This allows us to conclude invoking [Zha95, Theorem 1.4]. □

For log-singular hermitian line bundles, which are in particular of finite energy, we allow more general assumptions on $\mathscr{L}$.

THEOREM 4.4. *Let $\mathscr{X} \to \operatorname{Spec} \mathbb{Z}$ be an arithmetic variety of relative dimension n and $D \subset \mathscr{X}_{\mathbb{Q}}$ a divisor with normal crossings. Let $\overline{\mathscr{L}}$ be a semi-positive log-singular hermitian line bundle, with singularities along $D(\mathbb{C})$. Assume $\mathscr{L}_{\mathbb{Q}}$ is semi-ample and big, and that $\mathscr{L}$ is nef on vertical fibres. Also let $\overline{\mathscr{N}}$ be an arbitrary log-singular hermitian line bundle with singularities along $D(\mathbb{C})$. Then there is an asymptotic expansion*

$$\widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{N})_{L^2} = h_{\overline{\mathscr{L}}}(\mathscr{X}) \frac{k^{n+1}}{(n+1)!} + o(k^{n+1}) \text{ as } k \to +\infty,$$

where the L^2 norms are computed with respect to any smooth volume form μ on $\mathscr{X}(\mathbb{C})$, invariant under the action of complex conjugation.

The reader is referred to [BGKK07, § 7] and [BGKK05, Definition 3.29] for the definition and first properties of log-singular line bundles. For a more detailed study of this and related notions in arithmetic intersection theory, the reader can consult [FiM09b].

The proof of Theorem 4.4 occupies the rest of this section, and proceeds in several steps to reduce to Theorem 4.3. The methods are inspired by the proof of [Zha95, Theorem 1.4]. We begin with a lemma, that generalizes Lemma 1.6 of [Zha95].

Lemma 4.5. *Let $\mathscr{X}$ be an arithmetic variety of relative dimension n and $D \subset \mathscr{X}(\mathbb{C})$ a divisor with normal crossings. Let $\overline{\mathscr{L}}$ and $\overline{\mathscr{M}}$ be log-singular hermitian line bundles over $\mathscr{X}$, with singularities along D, and semi-ample on the generic fibre. Let $\overline{\mathscr{P}}$ be any log-singular hermitian line bundle on $\mathscr{X}$, with singularities along D as well. Fix a smooth volume form μ, invariant under the action of complex conjugation, with respect to which we compute L^2 norms. Then there exists a real positive constant $C > 0$ and an integer $R \geqslant 1$ such that for every $k, l \geqslant 0$, the $\mathbb{Z}$-module $H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{M}^{\otimes l} \otimes \mathscr{P})$ contains a set of independent sections of maximal rank whose L^2 norm is bounded by $C^{\delta}(\delta R)!^{n/2}$, with $\delta = 1 + k + l$.*

Proof. Because $\mathscr{L}_{\mathbb{Q}}$ and $\mathscr{M}_{\mathbb{Q}}$ are semi-ample, the bigraded $\mathbb{Q}$-algebra

$$S = \bigoplus_{k,l \geqslant 0} H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k} \otimes \mathscr{M}_{\mathbb{Q}}^{\otimes l})$$

is of finite type. Indeed, S is canonically isomorphic to the graded algebra of sections of the canonical semi-ample line bundle $\mathcal{O}(1)$ on $\mathrm{Proj}(\mathscr{L}_{\mathbb{Q}} \oplus \mathscr{M}_{\mathbb{Q}})$. Similarly, the $\mathbb{Q}$-vector space

$$V = \bigoplus_{k \geqslant 0} H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k} \otimes \mathscr{M}_{\mathbb{Q}}^{\otimes l} \otimes \mathscr{P}_{\mathbb{Q}})$$

is a bigraded S-module of finite type. Let us write

$$S = \mathbb{Q}[s_1, \ldots, s_r],$$
$$V = S v_1 + \cdots + S v_t,$$

where the sections s_i are homogenous of bidegree $(k_i, l_i) \geqslant 1$ and the sections v_j are homogenous of bidegree (e_j, f_j). We can suppose that the sections s_i and v_j are actually integral, by clearing denominators. Given integers $a_1, \ldots, a_r \geqslant 0$, we are thus reduced to bounding the L^2 norms of sections of the form

$$s_1^{a_1} \ldots s_r^{a_r} v_j.$$

For this, it is convenient to write the log-singular metrics in terms of smooth metrics. Namely, we write

$$\| \cdot \|^2_{\overline{\mathscr{L}}} = \| \cdot \|_0^2 g_0, \tag{4.6}$$
$$\| \cdot \|^2_{\overline{\mathscr{M}}} = \| \cdot \|_0'^2 g_0', \tag{4.7}$$
$$\| \cdot \|^2_{\overline{\mathscr{L}}^{\otimes e_j} \otimes \overline{\mathscr{M}}^{\otimes f_j} \otimes \overline{\mathscr{P}}} = \| \cdot \|_j^2 g_j, \quad j = 1, \ldots, t. \tag{4.8}$$

Here the metrics $\| \cdot \|_0$, $\| \cdot \|_0'$ and $\| \cdot \|_j$ are smooth. The functions g_0, g_0' and g_j, $j = 1, \ldots, l$ are smooth on $\mathscr{X}(\mathbb{C}) \backslash D$ and have logarithmic growth along D. We can cover D by a finite number of coordinate polydisks of radius $2\varepsilon < 2/e$, such that the concentric polydisks of radius ε still form a cover. Furthermore, we can suppose that in the corresponding coordinates $z_1, \ldots, z_n$ ($n = \dim \mathscr{X}(\mathbb{C})$), we have

$$g_0, g_0', g_j \leqslant B \left(\prod_{i=1}^{n} \log |z_i|^{-1} \right)^R, \quad j = 1, \ldots, t$$

on the range $|z_i| \leqslant \varepsilon$, $i = 1, \ldots, n$, and for some constant $B > 0$ and integer $R \geqslant 1$.

Let us denote by $\|s_i\|_{0,\infty}$ the supremum norms of the sections s_i computed with respect to the metrics $\|\cdot\|_0$ and $\|\cdot\|_0'$. Then, we have the following inequalities for the L^2 norms:

$$\|s_1^{a_1}\dots s_r^{a_r}v_j\|_{L^2}^2 \leqslant \|s_1\|_{0,\infty}^{2a_1}\dots\|s_r\|_{0,\infty}^{2a_r}\|v_j\|_{j,\infty}^2\cdot\int_{\mathscr{X}(\mathbb{C})} g_0^{\sum_i k_ia_i}g_0'^{\sum_i l_ia_i}g_j\,d\mu. \tag{4.9}$$

We proceed to bound the integral in (4.9). Let us introduce the quantity $\delta = 1+\sum_i(k_i+l_i)a_i$. On a coordinate polydisk as before we have

$$\int_{D(0,\varepsilon)^n} g_0^{\sum_i k_ia_i}g_0'^{\sum_i l_ia_i}g_j\,d\mu \leqslant C'B^\delta\prod_{i=1}^n\int_{D(0,\varepsilon)}(\log|z_i|^{-1})^{R\delta}\,|dz_i\wedge d\overline{z}_i| \tag{4.10}$$

for some constant $C'>0$ determined by the local expression of the volume form μ. A computation in polar coordinates shows that the right-hand side of (4.10) can further be bounded by

$$(4\pi)^nC'\varepsilon^2(B\log\varepsilon^{-1})^{\delta R}(\delta R)!^n.$$

After adjusting C' (depending only on $\|s_i\|_{0,\infty}$ $\|v_j\|_{j,\infty}$, B, R, $\log\varepsilon^{-1}$, the number of polydisks covering D and the integrals of products of g_0, g_0' and g_j on the complement of the polydisks, where these functions are bounded), we finally find

$$\|s_1^{a_1}\dots s_r^{a_r}v_j\|_{L^2}^2\leqslant C'^\delta(\delta R)!^n.$$

We conclude by putting $C=(C'+1)^{1/2}$. □

Remark 4.6. The lemma does not require the divisor D to be defined over $\mathbb{Q}$, nor to have strict normal crossings. Nevertheless, in the sequel it will be crucial that the divisor is defined over $\mathbb{Q}$.

The lemma will be applied in conjunction with the following comparison of arithmetic volumes of nested lattices, which we quote from [Zha95].

Lemma 4.7. *Let $M'\subset M'_{\mathbb{R}}$ and $M\subset M_{\mathbb{R}}$ be lattices in finite-dimensional real vector spaces of dimensions N', N, respectively. Assume $M'_{\mathbb{R}}\subseteq M_{\mathbb{R}}$ and $M'\subseteq M$. Let $\|\cdot\|$ be a euclidean norm on $M_{\mathbb{R}}$, and use the same notation for its restriction to $M'_{\mathbb{R}}$. Then*

$$\widehat{\deg}(M,\|\cdot\|)-\widehat{\deg}(M',\|\cdot\|)\geqslant -\log(N!)-(N-N')\log(\tfrac{1}{2}\lambda_N(M)).$$

Here $\lambda_N(M)$ is defined as

$$\lambda_N(M)=\inf\{\sup\{\|e_1\|,\dots,\|e_N\|\}\mid e_1,\dots e_N\in M \text{ independent}\}.$$

Proof. This is exactly [Zha95, Lemma 1.7]. □

Towards the proof of Theorem 4.4, we first consider the ample case. The argument shows how to deal with hermitian metrics with logarithmic singularities, and it already contains the main ideas needed in the semi-ample case.

Proposition 4.8. *Let $\mathscr{X}\to\operatorname{Spec}\mathbb{Z}$ be an arithmetic variety of relative dimension n and $D\subset\mathscr{X}_{\mathbb{Q}}$ a divisor with normal crossings. Let $\overline{\mathscr{L}}$ be a semi-positive log-singular hermitian line bundle, with singularities along $D(\mathbb{C})$. Assume $\mathscr{L}_{\mathbb{Q}}$ is ample and that $\mathscr{L}$ is nef on vertical fibres. Also let $\overline{\mathscr{N}}$ be an arbitrary log-singular hermitian line bundle with singularities along $D(\mathbb{C})$.*

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Then there is an asymptotic expansion

$$\widehat{\deg}\, H^0(\mathscr{X},\mathscr{L}^{\otimes k}\otimes\mathscr{N})_{L^2}=h_{\overline{\mathscr{L}}}(\mathscr{X})\frac{k^{n+1}}{(n+1)!}+o(k^{n+1})\text{ as }k\to+\infty,$$

where the L^2 norms are computed with respect to any smooth volume form μ on $\mathscr{X}(\mathbb{C})$, invariant under the action of complex conjugation.

Proof. We shall reduce the statement to an application of Theorem 4.3. We proceed in three steps.

The first step is to show that we can suppose that there exists a model $\mathscr{K}$ of the canonical sheaf (this is required in the statement of Theorem 4.3). For this, let us choose an integral, projective and flat model of $\mathscr{X}_{\mathbb{Q}}$ over $\mathbb{Z}$, say $\mathscr{X}'$, affording a model $\mathscr{K}$ of the canonical sheaf of $\mathscr{X}_{\mathbb{Q}}$. Notice that the existence of $\mathscr{X}'$ is guaranteed by the projectivity assumption on $\mathscr{X}$. Define $\widetilde{\mathscr{X}}$ to be the Zariski closure of $\mathscr{X}_{\mathbb{Q}}$ immersed in $\mathscr{X}\times\mathscr{X}'$ through the diagonal map. Hence, we have surjective proper morphisms $\pi:\widetilde{\mathscr{X}}\to\mathscr{X}$ and $\pi':\widetilde{\mathscr{X}}\to\mathscr{X}'$, which become isomorphisms after base change to $\mathbb{Q}$. We introduce $\widetilde{\mathscr{L}}:=\pi^*\mathscr{L}$, $\widetilde{\mathscr{N}}=\pi^*\mathscr{N}$ and $\widetilde{\mathscr{K}}:=\pi'^*\mathscr{K}$. Of course, $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{N}}$ carry the respective pull-back hermitian metric. Also, $\widetilde{\mathscr{L}}$ is still ample on the generic fibre and nef on vertical fibres, and $\widetilde{\mathscr{K}}$ is a model of the canonical sheaf of $\widetilde{\mathscr{X}}_{\mathbb{Q}}\simeq\mathscr{X}_{\mathbb{Q}}$. It is then enough to establish an asymptotic of the form

$$\widehat{\deg}\, H^0(\mathscr{X},\mathscr{L}^{\otimes k}\otimes\mathscr{N})_{L^2}=\widehat{\deg}\, H^0(\widetilde{\mathscr{X}},\widetilde{\mathscr{L}}^{\otimes k}\otimes\widetilde{\mathscr{N}})_{L^2}+O(k^n).\tag{4.11}$$

To see this, let us consider the short exact sequence of coherent sheaves

$$0\longrightarrow\mathcal{O}_{\mathscr{X}}\longrightarrow\pi_*\mathcal{O}_{\widetilde{\mathscr{X}}}\longrightarrow\mathcal{Q}\longrightarrow 0.$$

Here, $\mathcal{Q}$ is supported on a finite number of vertical fibres of the projection $\mathscr{X}\to\operatorname{Spec}\mathbb{Z}$. Now tensor this sequence with $\mathscr{L}^{\otimes k}\otimes\mathscr{N}$ and take global sections, to derive an exact sequence

$$0\to H^0(\mathscr{X},\mathscr{L}^{\otimes k}\otimes\mathscr{N})\to H^0(\widetilde{\mathscr{X}},\widetilde{\mathscr{L}}^{\otimes k}\otimes\widetilde{\mathscr{N}})\to H^0(\mathscr{X},\mathscr{L}^{\otimes k}\otimes\mathscr{N}\otimes\mathcal{Q}).$$

Notice that we used the projection formula in the middle term. The first arrow in this sequence becomes an isometry over $\mathbb{C}$. Moreover, the last term is a finite $\mathbb{Z}$-module and satisfies

$$\log(\sharp H^0(\mathscr{X},\mathscr{L}^{\otimes k}\otimes\mathscr{N}\otimes\mathcal{Q}))=\mathcal{O}(k^{\dim\operatorname{supp}\mathcal{Q}}).\tag{4.12}$$

This is true because $\mathscr{L}$ is nef on vertical fibres and $\mathscr{X}$ is projective. But we recall that the support of $\mathcal{Q}$ is vertical, hence $\dim\operatorname{supp}\mathcal{Q}\leqslant n$. The definition of arithmetic degree together with the asymptotics (4.12) prove the validity of (4.11). This achieves the first reduction step, and we can thus suppose, from now on, that $\mathscr{X}$ carries a model $\mathscr{K}$ of the canonical sheaf $K_{\mathscr{X}_{\mathbb{Q}}}$.

The second step is an elementary observation: since two smooth volume forms are comparable, we can suppose that the fixed volume form μ comes from a smooth hermitian metric on $K_{\mathscr{X}(\mathbb{C})}$. Indeed, for instance by Hadamard's inequality for determinants, we see that a change of smooth volume form will only contribute $O(\operatorname{rk}H^0(\mathscr{X},\mathscr{L}^{\otimes k}\otimes\mathscr{N}))=O(k^n)$ in the asymptotics. Then, the L^2 structure on $H^0(\mathscr{X},\mathscr{L}^{\otimes k}\otimes\mathscr{K})$ constructed according to § 2.4 and the L^2 structure defined using the metric on $\overline{\mathscr{L}}^{\otimes k}\otimes\overline{\mathscr{K}}$ and the volume form μ coincide.

In the third and last step, we show how to replace $\mathscr{K}$ by $\mathscr{N}$ in Theorem 4.3. Let us write $\overline{\mathscr{M}}$ for the trivial hermitian line bundle. Since $\mathscr{L}_{\mathbb{Q}}$ and $\mathscr{M}_{\mathbb{Q}}$ are semi-ample, we can apply Lemma 4.5 with $\overline{\mathscr{P}}$ being $\overline{\mathscr{K}}$ or $\overline{\mathscr{N}}$. We can take common constants C, R for both choices. Furthermore,

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ampleness of $\mathscr{L}_{\mathbb{Q}}$ ensures there exist $k_0, k_1 \geqslant 1$ such that

$$H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k_0} \otimes \mathscr{K}_{\mathbb{Q}} \otimes \mathscr{N}_{\mathbb{Q}}^{-1}(-D)) \neq 0,$$
$$H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k_1} \otimes \mathscr{K}_{\mathbb{Q}}^{-1} \otimes \mathscr{N}_{\mathbb{Q}}(-D)) \neq 0$$

(we considered D with its reduced scheme structure). We choose respective non-vanishing global sections t_0 and t_1. After clearing denominators, we can suppose that

$$t_0 \in H^0(\mathscr{X}, \mathscr{L}^{\otimes k_0} \otimes \mathscr{K} \otimes \mathscr{N}^{-1}),$$
$$t_1 \in H^0(\mathscr{X}, \mathscr{L}^{\otimes k_1} \otimes \mathscr{K}^{-1} \otimes \mathscr{N}).$$

A crucial observation for the sequel is that the pointwise norms $\|t_0\|$ and $\|t_1\|$ are actually continuous on $\mathscr{X}(\mathbb{C})$, and even vanishing along $D(\mathbb{C})$. Indeed, this follows because t_0 and t_1 vanish along D and the metrics are log-singular: the polynomial vanishing of t_0, t_1 along $D(\mathbb{C})$ cancels out the logarithmic singularities of the metrics. Hence $\|t_0\|_\infty, \|t_1\|_\infty < \infty$.

Multiplication by t_0 defines an injective morphism between hermitian modules

$$\iota_0 : H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{N})_{L^2} \hookrightarrow H^0(\mathscr{X}, \mathscr{L}^{\otimes (k+k_0)} \otimes \mathscr{K})_{L^2}$$

whose norm is bounded by $\|t_0\|_\infty$. On the space $H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{N})$ we thus have two norms: its intrinsic L^2 norm as well as the norm induced via ι_0. The corresponding arithmetic degrees differ by a term $O(k^n)$, where the implicit constant depends only on $\|t_0\|_\infty$. This follows from Hadamard's inequality for determinants and the bound on the norm of ι_0. Therefore, in view of applying Lemma 4.7, we can work with the intrinsic L^2 norm instead of the induced norm via ι_0. We also note that the cokernel of ι_0 has dimension $O(k^{n-1})$, again the O term depending on t_0. Analogous considerations can be done for multiplication by t_1:

$$\iota_1 : H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{L^2} \hookrightarrow H^0(\mathscr{X}, \mathscr{L}^{\otimes (k+k_1)} \otimes \mathscr{N})_{L^2}.$$

We combine these observations together with Lemma 4.7 and the estimate provided by Lemma 4.5 (taking $l = 0$). We see there exists a constant $A > 0$ such that

$$\widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes (k+k_0)} \otimes \mathscr{K})_{L^2} - \widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{N})_{L^2} \geqslant -Ak^n \log k,$$
$$\widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes (k+k_1)} \otimes \mathscr{N})_{L^2} - \widehat{\deg}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K})_{L^2} \geqslant -Ak^n \log k.$$

Together with Theorem 4.3, this completes the proof. □

Remark 4.9. The rationality of the divisor D was used in producing non-trivial *integral* sections t_0 and t_1 that vanish along D. The argument breaks down if D is only supposed to be defined over $\mathbb{C}$.

We are now in position to establish Theorem 4.4 in full generality. The argument is an adaptation of [Zha95, Theorem 1.4] to the present setting.

Proof of Theorem 4.4. We fix a very ample line bundle $\mathscr{M}$ on $\mathscr{X}$, and we equip it with a smooth hermitian metric with strictly positive curvature form.

We fix a non-vanishing global section s of $\mathscr{M}$ with sup norm bounded by a positive constant c'. The line bundle $\mathscr{L}_{\mathbb{Q}}$ is big by assumption. Then, Kodaira's lemma [Laz04, Proposition 2.2.6] states there exists $m \geqslant 1$ such that

$$H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes m} \otimes \mathscr{M}_{\mathbb{Q}}^{-1}(-D)) \neq 0.$$

Let t be such a non-vanishing global section. By clearing denominators, we can suppose $t \in H^0(\mathscr{X}, \mathscr{L}^{\otimes m} \otimes \mathscr{M}^{-1})$. Because t vanishes along D and the singularities of the metric on $\overline{\mathscr{L}}$ are of logarithmic type along $D(\mathbb{C})$, the section t has finite sup norm that we denote c''.

Let $k > m$ and j be any positive integers, and i an integer between 0 and $k-1$. We consider $H^0(\mathscr{X}, \mathscr{L}^{\otimes(kj+i)} \otimes \mathscr{N})$. Tensoring with s^j defines a monomorphism

$$\alpha : M_1 = H^0(\mathscr{X}, \mathscr{L}^{\otimes(kj+i)} \otimes \mathscr{N})_{L^2} \hookrightarrow M_2 = H^0(\mathscr{X}, \mathscr{L}^{\otimes(kj+i)} \otimes \mathscr{M}^{\otimes j} \otimes \mathscr{N})_{L^2},$$

whose norm is bounded by c'^j. Similarly, multiplication by t^j gives a monomorphism

$$\beta : M_3 = H^0(\mathscr{X}, \mathscr{L}^{\otimes((k-n)j+i)} \otimes \mathscr{M}^{\otimes j} \otimes \mathscr{N})_{L^2} \hookrightarrow M_1 = H^0(\mathscr{X}, \mathscr{L}^{\otimes(kj+i)} \otimes \mathscr{N})_{L^2},$$

whose norm is bounded by c''^j. Applying Lemma 4.5, Lemma 4.7 and Hadamard's inequality for determinants (as in the proof of Proposition 4.8), we obtain inequalities of the form

$$\begin{aligned} \widehat{\deg}\, \overline{M}_{2,L^2} - \widehat{\deg}\, \overline{M}_{1,L^2} \geqslant & -\log((\operatorname{rk} M_2)!) - (\operatorname{rk} M_2 - \operatorname{rk} M_1) \log(\tfrac{1}{2} C^{\delta} (\delta R)!^{n/2}) \\ & - j (\operatorname{rk} M_1) \log(c') \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \widehat{\deg}\, \overline{M}_{1,L^2} - \widehat{\deg}\, \overline{M}_{3,L^2} \geqslant & -\log((\operatorname{rk} M_1)!) - (\operatorname{rk} M_1 - \operatorname{rk} M_3) \log(\tfrac{1}{2} C^{\delta} (\delta R)!^{n/2}) \\ & - j (\operatorname{rk} M_3) \log(c''). \end{aligned} \tag{4.14}$$

Here, the constant $C > 0$ and the integer $R \geqslant 1$ are independent of k, j. We wrote $\delta = (k+1)(j+1)$. By the Riemann–Roch theorem and Proposition 4.8 applied to the ample line bundles of the form $\mathscr{L}^{\otimes k} \otimes \mathscr{M}$, one k at a time, we derive asymptotics

$$\operatorname{rk} M_l = \frac{(jk+i)^n}{n!} + O(k^n j^n) + o_k(j^n), \quad l = 1, 2, 3, \tag{4.15}$$

$$\widehat{\deg}\, \overline{M}_{l,L^2} = \frac{(jk+i)^{n+1}}{(n+1)!} h_{\overline{\mathscr{L}}}(\mathscr{X}) + O(k^n j^{n+1}) + o_k(j^{n+1}), \quad l = 2, 3. \tag{4.16}$$

We used the notation o_k to mean a little o quantity depending on k. Observe we have implicitly used that $\log(\delta R)! = o(\delta^2) = o_k(j^2)$.

To conclude, let $\varepsilon > 0$ and fix $k > m$ such that $k^n j^{n+1} < \varepsilon k^{n+1} j^{n+1}$ for all $j \geqslant 1$. For j sufficiently big, and since k has been fixed, $o_k(j^{n+1})$ will be bounded by $\varepsilon k^{n+1} j^{n+1}$. Then, from (4.13)–(4.16) we see that

$$\left| \widehat{\deg}\, \overline{M}_{1,L^2} - \frac{(kj+i)^{n+1}}{(n+1)!} h_{\overline{\mathscr{L}}}(\mathscr{X}) \right| \leqslant \kappa_0 \varepsilon (kj+i)^{n+1},$$

where κ_0 is a positive constant independent of i, j. The proof is complete. □

5. Arithmetic volumes of integral cusp forms on Hilbert modular surfaces

In this section we apply Theorem 4.4 to suitable arithmetic models of toroidal compactifications of Hilbert modular surfaces. Combining with the main results of Bruinier *et al.* [BBGK07], we are able to prove an arithmetic analogue of the classical theorem expressing the dimension of the spaces of cusp forms in terms of a special value of a Dedekind zeta function. To shorten the presentation, we refer to the book [vdG88] on Hilbert modular surfaces. We also make use of the results of Rapoport [Rap78] and Chai [Cha90] on arithmetic toroidal compactifications.

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The arithmetic theory of Hilbert modular surfaces is also summarized in Bruinier *et al.* [BBGK07, § 5].

We fix a real quadratic number field F of prime discriminant $\Delta \equiv 1 \mod 4$.[7] Let ℓ be a positive integer. Write $\Gamma_F(\ell) \subset \mathrm{SL}_2(\mathcal{O}_F)$ for the principal level ℓ congruence subgroup. Let $\mathcal{H}(\ell)$ be the smooth algebraic stack over $\operatorname{Spec} \mathbb{Z}[\zeta_\ell, 1/\ell]$ classifying $\mathfrak{d}^{-1}$ polarized abelian surfaces with multiplication by $\mathcal{O}_F$ and principal level ℓ structure.[8] The algebraic stack $\mathcal{H}(\ell)$ carries a universal family of abelian schemes. We consider an arithmetic toroidal compactification $\overline{\mathcal{H}}(\ell)$ of $\mathcal{H}(\ell)$ as constructed by Rapoport [Rap78] and Chai [Cha90]. This depends on the choice of so-called toroidal data, and enjoys the good properties that we now specify. First, it is an irreducible, smooth and proper algebraic stack over $\operatorname{Spec} \mathbb{Z}[\zeta_\ell, 1/\ell]$, with geometrically connected fibres. It comes equipped with a universal semi-abelian scheme extending the universal family over $\mathcal{H}(\ell)$. We denote by $\underline{\omega}_\ell$ the dual of the determinant of the relative Lie algebra of the universal semi-abelian scheme (do not confuse with a canonical bundle!). It is possible to choose toroidal data giving rise to a whole tower $\{\overline{\mathcal{H}}(\ell)\}_{\ell \geqslant 1}$, whose constituents are *projective schemes* (rather than just proper stacks) whenever $\ell \geqslant 3$. The arrows of the tower are of the form

$$\pi_{\ell,\ell'} : \overline{\mathcal{H}}(\ell) \to \overline{\mathcal{H}}(\ell')[1/\ell],$$

whenever $\ell' \mid \ell$, and extend the natural projections 'forgetting level structure' $\mathcal{H}(\ell) \to \mathcal{H}(\ell')[1/\ell]$. The notation $[1/\ell]$ indicates that we inverted the primes dividing ℓ in the structure sheaf of a scheme. Furthermore, we have the compatibility

$$\pi^*_{\ell,\ell'} \underline{\omega}_{\ell'} = \underline{\omega}_\ell. \tag{5.1}$$

For $\ell \geqslant 3$, the sheaves $\underline{\omega}_\ell$ are semi-ample relative to $\operatorname{Spec} \mathbb{Z}[\zeta_\ell, 1/\ell]$ and big over the generic fibre. Observe that bigness is a consequence of the existence of a natural birational morphism to the minimal compactification

$$\pi : \overline{\mathcal{H}}(\ell)_{\mathbb{Q}} \longrightarrow \mathcal{H}(\ell)^*_{\mathbb{Q}},$$

such that $\pi^* \mathcal{O}(1) = \underline{\omega}_{\ell\,\mathbb{Q}}^{\otimes m}$ for some very ample line bundle $\mathcal{O}(1)$ and some integer $m \geqslant 1$. In the sequel, we will consider $\overline{\mathcal{H}}(\ell)$ just as a scheme over $\operatorname{Spec} \mathbb{Z}[1/\ell]$.

To apply our results, we need to extend schematic toroidal compactifications to $\operatorname{Spec} \mathbb{Z}$. For this, we fix from now on relatively prime integers $N, M \geqslant 3$ and put $\ell = NM$. We choose toroidal data so that $\overline{\mathcal{H}}(\ell)$, $\overline{\mathcal{H}}(N)$ and $\overline{\mathcal{H}}(M)$ are schemes and arise in a tower as above (actually only the levels N, M, ℓ are involved).

Lemma 5.1. *There exists an integral, projective and flat scheme $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$, together with invertible sheaves $\mathscr{K}$ and semi-ample $\mathscr{L}$, fulfilling the following properties:*

(i) $\mathcal{X}[1/\ell] = \overline{\mathcal{H}}(\ell)$, $\mathscr{K} \mid_{\mathcal{X}[1/\ell]} = K_{\overline{\mathcal{H}}(\ell)/\mathbb{Z}[1/\ell]}$ and $\mathscr{L} \mid_{\mathcal{X}[1/\ell]} = \underline{\omega}_\ell$;

(ii) *there are proper morphisms*

$$p_N : \mathcal{X}[1/N] \to \overline{\mathcal{H}}(N), \quad p_M : \mathcal{X}[1/M] \to \overline{\mathcal{H}}(M),$$

such that $p_N \mid_{\mathcal{X}[1/\ell]} = \pi_{\ell,N}$ *and* $p_M \mid_{\mathcal{X}[1/\ell]} = \pi_{\ell,M}$. *Furthermore,*

$$\mathscr{L} \mid_{\mathcal{X}[1/N]} = p_N^* \underline{\omega}_N, \quad \mathscr{L} \mid_{\mathcal{X}[1/M]} = p_M^* \underline{\omega}_M.$$

[7] This hypothesis is only required for computations of arithmetic intersection numbers.
[8] As usual, we denote by $\mathfrak{d}^{-1}$ the inverse different ideal of F, namely the inverse fractional ideal of $\mathfrak{d} = (\sqrt{\Delta}) \subset \mathcal{O}_F$.

Proof. Let $r \in \{N, M, \ell\}$. Because $\overline{\mathcal{H}}(r)$ is projective over $\mathbb{Z}[1/r]$, we can choose an integral, flat and projective model over $\mathbb{Z}$, denoted $\widetilde{\mathcal{H}}(r)$. Moreover, we can suppose that for $r = \ell$ the line bundle $K_{\overline{\mathcal{H}}(\ell)/\mathbb{Z}[1/\ell]}$ extends to a line bundle $\widetilde{\mathscr{K}}$ over $\widetilde{\mathcal{H}}(\ell)$. The projections $\pi_{\ell,N}$ and $\pi_{\ell,M}$ induce a natural morphism

$$\varphi : \overline{\mathcal{H}}(\ell) \longrightarrow \widetilde{\mathcal{H}}(N)[1/\ell] \times \widetilde{\mathcal{H}}(M)[1/\ell],$$

since $\widetilde{\mathcal{H}}(N)[1/\ell] = \overline{\mathcal{H}}(N)[1/\ell]$ and $\widetilde{\mathcal{H}}(M)[1/\ell] = \overline{\mathcal{H}}(M)[1/\ell]$. Let Γ denote the graph of φ. We define $\mathcal{X}$ as the Zariski closure of Γ inside $\widetilde{\mathcal{H}}(\ell) \times \widetilde{\mathcal{H}}(N) \times \widetilde{\mathcal{H}}(M)$. There are natural proper morphisms

$$q_r : \mathcal{X} \longrightarrow \widetilde{\mathcal{H}}(r), \quad r = N, M, \ell.$$

Because $\widetilde{\mathcal{H}}(r)[1/r] = \overline{\mathcal{H}}(r)$, the q_r induce proper projections

$$p_r : \mathcal{X}[1/r] \longrightarrow \overline{\mathcal{H}}(r), \quad r = N, M, \ell.$$

We define $\mathscr{K} = q_\ell^*(\widetilde{\mathscr{K}})$. This is an invertible sheaf over $\mathcal{X}$ and provides a model of $K_{\overline{\mathcal{H}}(\ell)/\mathbb{Z}[1/\ell]}$ as required. To construct $\mathscr{L}$, consider the line bundles $\mathscr{L}_r = p_r^* \underline{\omega}_r$ over $\mathcal{X}[1/r]$, for all r. By construction and by the compatibility (5.1), we have

$$\mathscr{L}_r \mid_{\mathcal{X}[1/\ell]} = \underline{\omega}_\ell.$$

Again, these line bundles glue into a single line bundle $\mathscr{L}$ over $\mathcal{X}$. Moreover, because $\underline{\omega}_r$ is semi-ample for all r, it follows that $\mathscr{L}$ is semi-ample too. To finish the proof, one easily checks that $\mathcal{X}$, $\mathscr{L}$, p_N, p_M satisfy the desired properties. □

Notation 5.2. Let $\mathcal{X}$, $\mathscr{L}$, p_N, p_M satisfy the properties stated in the lemma. We abusively write $\overline{\mathcal{H}}(\ell) = \mathcal{X}$, $\underline{\omega}_\ell = \mathscr{L}$ over $\operatorname{Spec} \mathbb{Z}$. We will call the data $(\overline{\mathcal{H}}(\ell), \underline{\omega}_\ell, p_N, p_M)$ a naive integral toroidal compactification over $\operatorname{Spec} \mathbb{Z}$. The lemma shows that there exists such data for which $\overline{\mathcal{H}}(\ell)$ affords a line bundle $\mathscr{K}$ extending $K_{\overline{\mathcal{H}}(\ell)_\mathbb{Q}}$.

Remark 5.3. (i) The scheme $\overline{\mathcal{H}}(\ell)$ and the sheaf $\underline{\omega}_\ell$ over $\operatorname{Spec} \mathbb{Z}$ are not canonically defined, not even in terms of the toroidal data: involved in their construction, there are arbitrary choices of closed embeddings of toroidal compactifications into projective spaces. However, any data $(\overline{\mathcal{H}}(\ell), \underline{\omega}_\ell, p_N, p_M)$ affording a model of $K_{\overline{\mathcal{H}}(\ell)_\mathbb{Q}}$ will be enough for our purposes.

(ii) Naive integral toroidal compactifications are not smooth schemes. But to apply our arithmetic Hilbert–Samuel theorem, it is enough to know generic smoothness, which is the case.

The invertible sheaf $\underline{\omega}_\mathbb{C}$ over $\overline{\mathcal{H}}(\ell)_\mathbb{C}$ can be endowed with the so called (pointwise) Petersson metric [BBGK07, § 2.2]. This is a semi-positive log-singular hermitian metric, with singularities along the normal crossing divisor $(\overline{\mathcal{H}}(\ell) \backslash \mathcal{H}(\ell))(\mathbb{C})$ [BBGK07, Proposition 2.5]. This metric is compatible with pull-back by the natural projections $\pi_{\ell,N}$. The corresponding L^2 metric on global sections $H^0(\overline{\mathcal{H}}(\ell)_\mathbb{C}, \underline{\omega}_{\ell\mathbb{C}}^{\otimes k} \otimes K_{\overline{\mathcal{H}}(\ell)_\mathbb{C}})$ can be identified with the Petersson pairing on cusp forms of parallel weight $2k+2$. For this, recall the Kodaira–Spencer canonical isomorphism $\underline{\omega}_\mathbb{C} \simeq K_{\overline{\mathcal{H}}(\ell)_\mathbb{C}}(D)$, where D is the divisor $D = \overline{\mathcal{H}}(\ell)_\mathbb{C} \backslash \mathcal{H}(\ell)_\mathbb{C}$. We write Pet to refer to the Petersson pairing.

THEOREM 5.4. *Let $\ell = NM$ be the product of two coprime integers $N, M \geqslant 3$. Let $(\overline{\mathcal{H}}(\ell), \underline{\omega}_\ell,$ $p_N, p_M)$ be a naive integral toroidal compactification over* $\operatorname{Spec} \mathbb{Z}$*, affording a line bundle $\mathscr{K}$*

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extending $K_{\overline{\mathcal{H}}(\ell)_{\mathbb{Q}}}$. *Endow* $\underline{\omega}_\ell$ *with the Petersson metric. Then*

$$\widehat{\deg}\, H^0(\overline{\mathcal{H}}(\ell), \underline{\omega}_\ell^{\otimes k} \otimes \mathscr{K})_{\mathrm{Pet}} = -\frac{k^3}{6} d_\ell \zeta_F(-1) \left(\frac{\zeta_F'(-1)}{\zeta_F(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2}\log\Delta \right) + o(k^3),$$

where $d_\ell = [\mathbb{Q}(\zeta_\ell) : \mathbb{Q}][\Gamma_F(1) : \Gamma_F(\ell)]$, *and* ζ_F *is the Dedekind zeta function of* F.

Proof. Because the sheaf $\underline{\omega}_\ell$ is semi-ample and $\underline{\omega}_{\ell\,\mathbb{Q}}$ is big and the Petersson metric is log-singular along a divisor defined over $\mathbb{Q}$, we can apply Theorem 4.4. Hence, it is enough to check the formula

$$h_{\overline{\underline{\omega}}_\ell}(\overline{\mathcal{H}}(\ell)) = -d_\ell \zeta_F(-1) \left(\frac{\zeta_F'(-1)}{\zeta_F(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2}\log\Delta \right). \tag{5.2}$$

Let $r \in \{N, M\}$. By the properties of naive toroidal compactifications and the compatibility of the Petersson metric with pull-back by $\pi_{\ell,N}$ and $\pi_{\ell,M}$, we see that

$$p_r^* \overline{\underline{\omega}}_r = \overline{\underline{\omega}}_\ell \quad \text{on } \overline{\mathcal{H}}(\ell)[1/r].$$

By the functoriality properties of heights, we derive an equality in $\mathbb{R}/\sum_{p|r} \mathbb{Q}\log p$

$$\begin{aligned} h_{\overline{\underline{\omega}}_\ell}(\overline{\mathcal{H}}(\ell)) &= h_{p_r^*\overline{\underline{\omega}}_r}(\overline{\mathcal{H}}(\ell)) = (\deg p_{r\,\mathbb{Q}}) h_{\overline{\underline{\omega}}_r}(\overline{\mathcal{H}}(r)) \\ &= -[\Gamma_F(r) : \Gamma_F(\ell)] d_r \zeta_F(-1) \cdot \left(\frac{\zeta_F'(-1)}{\zeta_F(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2}\log\Delta \right). \end{aligned} \tag{5.3}$$

In the last equality we appealed to [BBGK07, Theorem 6.4]. Notice that $[\Gamma_F(r) : \Gamma_F(\ell)] d_r = d_\ell$. Because N, M are coprime, the relation (5.3) for $r = N, M$ (with values in $\mathbb{R}/\sum_{p|r} \log p$) implies the desired equality (5.2) (with values in $\mathbb{R}$). We conclude by Theorem 4.4. □

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