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ON ALMOST LOCALLY CONNECTED SPACES

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Abstract

In this paper it is shown that almost local connectedness is hereditary for the subspace that is the union of regular open sets and is preserved under almost-open (in the sense of Singal) θ-continuous surjections.

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1. Introduction

Recently, V. J. Mancuso [3] has introduced and investigated the concept of almost locally connected spaces. In [3], among others, the following theorems have been established:

THEOREM A. *Let f: $X \to Y$ be an almost-open, almost-continuous and connected surjection. If X is almost locally connected and Y is almost-regular, then Y is almost locally connected.*

THEOREM B. *Let f: $X \to Y$ be an open, almost-continuous and connected surjection. If X is almost locally connected, then so is Y.*

The main purpose of the present paper is to improve the previous theorems. In Section 4 it will be shown that almost local connectedness is preserved under almost-open and almost-continuous surjections. By making use of this result we

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shall show in Section 5 that the word "regular open" in Theorem 3.8 of [3] can be replaced by "the union of regular open sets".

2. Preliminaries

Throughout this paper spaces mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of a space X. The closure of S and the interior of S in X are denoted by $\mathrm{Cl}_X(S)$ and $\mathrm{Int}_X(S)$ (or simply $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$), respectively. A subset S is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(S)) = S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S)) = S$). The family of all regular open sets of a space X is denoted by $\mathrm{RO}(X)$. A function $f\colon X \to Y$ is said to be *almost-continuous* [12] (resp. *θ-continuous* [1], *weakly-continuous* [2]) if for each point $x \in X$ and each open set V of Y containing $f(x)$ there exists an open set U of X containing x such that $f(U) \subset \mathrm{Int}_Y(\mathrm{Cl}_Y(V))$ (resp. $f(\mathrm{Cl}_X(U)) \subset \mathrm{Cl}_Y(V)$, $f(U) \subset \mathrm{Cl}_Y(V)$).

REMARK 2.1. It is known that *continuity* $\Rightarrow$ *almost-continuity* $\Rightarrow$ *θ-continuity* $\Rightarrow$ *weak-continuity* and none of these implications is reversible ([6], [12]).

3. Almost locally connected spaces

DEFINITION 3.1. A space X is said to be *almost locally connected* (simply a.l.c.) [3] if for each $x \in X$ and each $G \in \mathrm{RO}(X)$ containing x there exists an open connected set V such that $x \in V \subset G$.

We shall begin by giving a characterization of a.l.c. spaces which will be used in the subsequence.

THEOREM 3.2. *The following statements are equivalent for a space X:*

(1) *X is a.l.c.*

(2) *The components of regular open sets in X are regular open in X.*

(3) *For each $x \in X$ and each $G \in \mathrm{RO}(X)$ containing x, there exists a regular open connected set V such that $x \in V \subset G$.*

PROOF. (1) $\Rightarrow$ (2): Let $G \in \mathrm{RO}(X)$ and C be a component of G. By Theorem 3.5 of [3], C is open in X and $C \subset \mathrm{Int}_X(\mathrm{Cl}_X(C))$. On the other hand, since C is connected, so is $\mathrm{Int}_X(\mathrm{Cl}_X(C))$. Since C is a component of G, $\mathrm{Int}_X(\mathrm{Cl}_X(C)) \subset C$. Therefore, we have $C = \mathrm{Int}_X(\mathrm{Cl}_X(C))$ which shows that $C \in \mathrm{RO}(X)$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are easy and the proofs are thus omitted.

DEFINITION 3.3. A space X is said to be *nearly-compact* [11] if every regular open cover of X has a finite subcover.

COROLLARY 3.4. *A nearly-compact a.l.c. space has a finite number of components.*

PROOF. Let X be a nearly-compact a.l.c. space. Since $X \in \mathrm{RO}(X)$, by Theorem 3.2 the family of components of X is a regular open cover of X. Therefore, X has a finite number of components.

A space X is said to be *weakly-Hausdorff* [13] if every point of X is the intersection of regular closed sets.

THEOREM 3.5. *A nearly-compact weakly-Hausdorff space X is a.l.c. if and only if every regular open cover of X is refined by a cover consisting of a finite number of regular open connected sets.*

PROOF. Let X be a nearly-compact a.l.c. space and $\mathcal{V} = \{V_\alpha \mid \alpha \in \nabla\}$ a regular open cover of X. By Theorem 3.2, for each $\alpha \in \nabla$ the components $C_{\alpha(j)}$ of V_α are regular open in X, where $\alpha(j) \in \nabla(\alpha)$. Since X is nearly-compact, there exist a finite subset ∇_0 of ∇ and a finite subset $\nabla_0(\alpha)$ of $\nabla(\alpha)$ for each $\alpha \in \nabla_0$ such that

$$X = \bigcup \left\{C_{\alpha(j)} \mid \alpha(j) \in \nabla_0(\alpha), \alpha \in \nabla_0\right\}.$$

Thus, the family $\{C_{\alpha(j)} \mid \alpha(j) \in \nabla_0(\alpha), \alpha \in \nabla_0\}$ is a desirable refinement of $\mathcal{V}$.

Conversely, under the condition that X is a weakly-Hausdorff space and the hypothesis holds, we shall show that X is a.l.c. Let $x \in X$ and $x \in G \in \mathrm{RO}(X)$. Since X is weakly-Hausdorff, for each $y \in X - G$ there exists $U_y \in \mathrm{RO}(X)$ such that $y \in U_y$ and $x \notin U_y$. Then $G \cup \{U_y \mid y \in X - G\}$ is a regular open cover of X. By the hypothesis, it has a refinement $\{V_\alpha \mid \alpha \in \nabla\}$ consisting of a finite number of regular open connected sets. There exists an $\alpha_0 \in \nabla$ such that $x \in V_{\alpha_0}$. If $V_{\alpha_0} \subset U_y$ for some $y \in X - G$, then $x \in U_y$. This is a contradiction. Therefore, we obtain $x \in V_{\alpha_0} \subset G$. This shows that X is a.l.c.

4. Preservation theorems

A function $f\colon X \to Y$ is said to be *almost-open* (simply a.o.R.) [10] if $f(U) \subset \mathrm{Int}_Y(\mathrm{Cl}_Y(f(U)))$ for every open set U of X. A function $f\colon X \to Y$ is said to be *connected* [8] if for each connected set C of X, $f(C)$ is connected in Y. The following theorem is an improvement of Theorem B.

THEOREM 4.1. *Let f: $X \to Y$ be an a.o.R., weakly-continuous and connected surjection. If X is a.l.c., then so is Y.*

PROOF. Let $y \in Y$ and $y \in G \in \mathrm{RO}(Y)$. It follows from Theorem 3.4 of [6] that $f^{-1}(G) \in \mathrm{RO}(X)$. Since X is a.l.c., for $x \in f^{-1}(y)$ there exists an open connected set U of X such that $x \in U \subset f^{-1}(G)$. Since f is a.o.R., we have $f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U)))$. Since f is connected, $f(U)$ is connected and so is $\mathrm{Int}(\mathrm{Cl}(f(U)))$. Moreover, we obtain

$$y \in f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U))) \subset G.$$

This shows that Y is a.l.c.

We shall show the main theorem of this paper which is an improvement of Theorem A and Theorem B. For this purpose we need some lemmas.

DEFINITION 4.2. A function f: $X \to Y$ is said to be *almost-open* (simply a.o.S.) [12] if for each $U \in \mathrm{RO}(X)$ $f(U)$ is open in Y.

REMARK 4.3. It is known in [7] that "a.o.S." neither implies "a.o.R.", nor does "a.o.R." imply "a.o.S.".

LEMMA 4.4. *Let f: $X \to Y$ be a weakly-continuous surjection and X_0 be a subset of X. If $f(X_0)$ is open in Y, then the function f_0: $X_0 \to f(X_0)$, defined by $f_0(x) = f(x)$ for each $x \in X_0$, is weakly-continuous.*

PROOF. Put $Y_0 = f(X_0)$. Let $x \in X_0$ and V_0 be an open set of Y_0 containing $f_0(x)$. Since Y_0 is open in Y, V_0 is open in Y. By weak-continuity of f, there exists an open set U of X containing x such that $f(U) \subset \mathrm{Cl}_Y(V_0)$. Put $U_0 = U \cap X_0$, then U_0 is an open set of X_0 containing x and $f_0(U_0) \subset \mathrm{Cl}_Y(V_0) \cap Y_0 = \mathrm{Cl}_{Y_0}(V_0)$. This shows that f_0 is weakly-continuous.

LEMMA 4.5. *Let f: $X \to Y$ be an a.o.S. weakly-continuous surjection. If U is a regular open connected set of X, then $f(U)$ is open connected in Y.*

PROOF. Since f is a.o.S. and $U \in \mathrm{RO}(X)$, $f(U)$ is open in Y. It follows from Lemma 4.4 that the function f_0: $U \to f(U)$ is weakly-continuous. Since U is a connected set of X, by Theorem 3 of [4] $f_0(U) = f(U)$ is connected.

THEOREM 4.6. *Let f: $X \to Y$ be an a.o.S. θ-continuous surjection. If X is a.l.c., then so is Y.*

Proof. Let $y \in Y$ and $y \in G \in \mathrm{RO}(Y)$. Every a.o.S. function is weakly-open [7, Lemma 1.4]. Hence, it follows from Theorem 4.4 of [6] that $f^{-1}(G) \in \mathrm{RO}(X)$. Since X is a.l.c., for $x \in f^{-1}(y)$ by Theorem 3.2 there exists a regular open connected set U of X such that $x \in U \subset f^{-1}(G)$. Every θ-continuous function is weakly-continuous. Therefore, by Lemma 4.5 $f(U)$ is open connected in Y and $y \in f(U) \subset G$. This shows that Y is a.l.c.

Corollary 4.7. *Almost local connectedness is preserved under a.o.S. almost-continuous surjections.*

Proof. This is an immediate consequence of Theorem 4.6.

Remark 4.8. The previous corollary shows that the hypothesis "connected" on f in Theorems A and B and also "almost-regular" on Y in Theorem A can be removed.

In this paper, for simplicity, we call the set X with the topology having $\mathrm{RO}(X)$ as a basis the *semi-regularization*, denoted by X_s, of a space X.

Corollary 4.9. *A space X is a.l.c. if and only if the semi-regularization X_s is locally connected.*

Proof. *Necessity.* Let X be a.l.c. The identity function i_X: $X \to X_s$ is a.o.S. and continuous. Thus, by Corollary 4.7 X_s is a.l.c. and it is locally connected by Proposition 3.3 of [3].

Sufficiency. Let X_s be locally connected. The identity function j_X: $X_s \to X$ is open and almost-continuous. It follows from Corollary 4.7 that X is a.l.c.

Theorem 4.10. *Let f: $X \to Y$ be an almost-continuous surjection. If X is compact a.l.c. and Y is Hausdorff, then Y is a.l.c.*

Proof. Since f is almost-continuous and X is compact, $Y = f(X)$ is nearly-compact [11, Theorem 3.2]. Moreover, since Y is Hausdorff, it is almost-regular [11, Theorem 2.4]. It follows from Theorems 4.9 and 2.5 of [5] that f_s is a continuous function of a compact space X_s onto a Hausdorff space Y_s. Therefore, f_s is closed. Since X is a.l.c., by Corollary 4.9 X_s is locally connected and hence $f_s(X_s) = Y_s$ is locally connected. It follows from Corollary 4.9 that Y is a.l.c.

5. Subspaces of a.l.c. spaces

Theorem 3.8 of [3] states that if A is a regular open set of an a.l.c. space X then the subspace A is a.l.c. The following theorem shows that the hypothesis "regular open" on A in this result can be replaced by "the union of regular open sets".

THEOREM 5.1. *If X is an a.l.c. space and A is the union of arbitrarily many regular open sets, then the subspace A is a.l.c.*

PROOF. It follows from Corollary 4.9 that X_s is locally connected. Since the identity function i_X: $X \to X_s$ is a.o.S., $i_X(A)$ is open in X_s and hence the subspace $i_X(A)$ is locally connected. The identity function j_X: $X_s \to X$ is open and almost-continuous. Since A is open in X, by Lemma 4.4 the induced identity function $(j_X)_0$: $i_X(A) \to A$ is open weakly-continuous. Moreover, every open weakly-continuous function is almost-continuous [12, Theorem 2.3]. Therefore, it follows from Corollary 4.7 that A is a.l.c.

A subset K of a Hausdorff space X is said to be *H-closed relative to X* [9] if for every cover $\{ \nabla_\alpha \mid \alpha \in \nabla \}$ of K by open sets of X there exists a finite subset ∇_0 of ∇ such that $K \subset \bigcup \{ \mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla \}$.

COROLLARY 5.2. *If a space X is a.l.c. Hausdorff and K is H-closed relative to X, then $X - K$ is a.l.c.*

PROOF. Let $x \in X - K$. Since X is Hausdorff, for each $y \in K$ there exist regular open sets $V(y)$ and $W(y)$ containing x and y, respectively, such that $V(y) \cap \mathrm{Cl}(W(y)) = \varnothing$. Since $\{W(y) \mid y \in K\}$ is a cover of K by open sets of X, there exists a finite subset K_0 of K such that

$$K \subset \bigcup \{\mathrm{Cl}_X(W(y)) \mid y \in K_0\}.$$

Put $V_x = \bigcap \{V(y) \mid y \in K_0\}$, then $x \in V_x \in \mathrm{RO}(X)$ and $V_x \subset X - K$. It follows from Theorem 5.1 that $X - K$ is a.l.c.

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