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Abstract

We prove that the complete L-function associated to any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ has infinitely many simple zeros.

1. Introduction

In [Boo16], the first author showed that the complete L-functions associated to classical holomorphic newforms have infinitely many simple zeros. The purpose of this paper is to extend that result to the remaining degree-2 automorphic L-functions over $\mathbb{Q}$, that is, those associated to cuspidal Maass newforms. This also extends work of the second author [Cho13] which established a quantitative estimate for the first few Maass forms of level 1. When combined with the holomorphic case from [Boo16], we obtain the following theorem.

Theorem 1.1. *Let $\mathbb{A}_\mathbb{Q}$ denote the adèle ring of $\mathbb{Q}$, and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Then the associated complete L-function $\Lambda(s,\pi)$ has infinitely many simple zeros.*

The basic idea of the proof is the same as in [Boo16], which is in turn based on the method of Conrey and Ghosh [CG88]. Let f be a primitive Maass cuspform of weight $k \in \{0,1\}$ for $\Gamma_0(N)$ with nebentypus character ξ, and let $L_f(s)$ be the finite L-function attached to f:

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

We define

$$D_f(s) = L_f(s) \frac{d^2}{ds^2} \log L_f(s) = \sum_{n=1}^{\infty} c_f(n) n^{-s}.$$

Then it is easy to see that $D_f(s)$ has a pole at some point if and only if $L_f(s)$ has a simple zero there.

For $\alpha \in \mathbb{Q}$ and $j \geqslant 0$ we define the additive twists

$$L_f(s,\alpha,\cos^{(j)}) = \sum_{n=1}^{\infty} \lambda_f(n) \cos^{(j)}(2\pi n\alpha) n^{-s}, \quad D_f(s,\alpha,\cos^{(j)}) = \sum_{n=1}^{\infty} c_f(n) \cos^{(j)}(2\pi n\alpha) n^{-s},$$

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where $\cos^{(j)}$ denotes the jth derivative of the cosine function. Let $q \nmid N$ be a prime and χ_0 the principal character mod q. Then we have the following expansions of the trigonometric functions in terms of Dirichlet characters:

$$\cos\left(\frac{2\pi n}{q}\right) = 1 - \frac{q}{q-1}\chi_0(n) + \frac{\sqrt{q}}{q-1} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \overline{\epsilon_\chi}\chi(n),$$

$$\sin\left(\frac{2\pi n}{q}\right) = \frac{\sqrt{q}}{q-1} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi(-1)=-1}} \overline{\epsilon_\chi}\chi(n),$$

where ϵ_χ denotes the root number of the Dirichlet L-function $L(s,\chi)$. In particular, we have

$$D_f\left(s, \frac{1}{q}, \cos\right) = D_f(s) - \frac{q}{q-1}D_f(s,\chi_0) + \frac{\sqrt{q}}{q-1} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \overline{\epsilon_\chi} D_f(s,\chi),$$

where

$$D_f(s,\chi) = \sum_{n=1}^{\infty} c_f(n)\chi(n)n^{-s}$$

is the corresponding multiplicative twist.

By the non-vanishing results for automorphic L-functions [JS77], all non-trivial poles of $D_f(s)$ and $D_f(s,\chi)$ for $\chi \neq \chi_0$ are located in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. However, for the case of the principal character, since

$$L_f(s,\chi_0) = \sum_{n=1}^{\infty} \lambda_f(n)\chi_0(n)n^{-s} = (1 - \lambda_f(q)q^{-s} + \xi(q)q^{-2s})L_f(s),$$

$D_f(s,\chi_0)$ has a pole at every simple zero of the local Euler factor polynomial, $1 - \lambda_f(q)q^{-s} + \xi(q)q^{-2s}$, at which $L_f(s)$ does not vanish.

Since f is cuspidal, the Rankin–Selberg method implies that the average of $|\lambda_f(q)|^2$ over primes q is 1, that is,

$$\lim_{x\to\infty} \frac{\sum_{q \text{ prime},\, q \leqslant x} |\lambda_f(q)|^2}{\#\{q \text{ prime} : q \leqslant x\}} = 1. \tag{1}$$

To see this, write

$$-\frac{L_f'}{L_f}(s) = \sum_{n=1}^{\infty} \Lambda(n)a_n n^{-s},$$

where Λ is the von Mangoldt function and $a_n = 0$ unless n is prime or a prime power. Then, by [LY07, Lemma 5.2], we have

$$\sum_{n \leqslant x} \Lambda(n)|a_n|^2 \sim x \quad \text{as } x \to \infty. \tag{2}$$

By the estimate of Kim and Sarnak [Kim03], we have $|a_n| \leqslant n^{7/64} + n^{-7/64}$, so the contribution of composite n to (2) is $O(x^{23/32})$. Since $a_q = \lambda_f(q)$ for primes q, this implies that

$$\sum_{\substack{q \text{ prime} \\ q \leqslant x}} (\log q)|\lambda_f(q)|^2 \sim x,$$

and (1) follows by partial summation and the prime number theorem.

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In particular, there are infinitely many $q \nmid N$ such that $|\lambda_f(q)| < 2$. For any such q, it follows that $D_f(s, \chi_0)$ has infinitely many poles on the line $\Re(s) = 0$. In view of the above, $D_f(s, 1/q, \cos)$ inherits these poles when they occur. On the other hand, under the assumption that $L_f(s)$ has at most finitely many non-trivial simple zeros, we will show that $D_f(s, 1/q, \cos)$ is holomorphic apart from possible poles along two horizontal lines. The contradiction between these two implies the main theorem.

1.1 Overview

We begin with an overview of the proof. First, by [DFI02, (4.36)], f has the Fourier–Whittaker expansion

$$f(x+iy) = \sum_{n=1}^{\infty} \big(\rho(n) W_{k/2,\nu}(4\pi ny) e(nx) + \rho(-n) W_{-k/2,\nu}(4\pi ny) e(-nx)\big),$$

where $W_{\alpha,\beta}$ is the Whittaker function defined in [DFI02, (4.20)] and $\nu = \sqrt{\frac{1}{4} - \lambda}$, where λ is the eigenvalue of f with respect to the weight-k Laplace operator. When $k = 1$, the Selberg eigenvalue conjecture holds, so that $\nu \in i[0, \infty)$. When $k = 0$ the conjecture remains open, but we have the partial result of Kim and Sarnak [Kim03] that $\nu \in (0, \frac{7}{64}] \cup i[0, \infty)$.

Since f is primitive, it is an eigenfunction of the operator Q_{sk} defined in [DFI02, (4.65)], so that

$$\rho(-n) = \epsilon \frac{\Gamma((1+k)/2 + \nu)}{\Gamma((1-k)/2 + \nu)} \rho(n) = \epsilon \nu^k \rho(n)$$

for some $\epsilon \in \{\pm 1\}$. Further, we have $\rho(n) = \rho(1)\lambda_f(n)/\sqrt{n}$. Choosing the normalization $\rho(1) = \pi^{-k/2}$ and writing $e(\pm nx) = \cos(2\pi nx) \pm i \sin(2\pi nx)$, we obtain the expansion

$$f(x+iy) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}} (V_f^+(ny)\cos(2\pi nx) + iV_f^-(ny)\sin(2\pi nx)), \tag{3}$$

where

$$V_f^{\pm}(y) = \pi^{-k/2}(W_{k/2,\nu}(4\pi y) \pm \epsilon \nu^k W_{-k/2,\nu}(4\pi y)) = \begin{cases} 4\sqrt{y} K_\nu(2\pi y) & \text{if } k = 0 \text{ and } \epsilon = \pm 1, \\ 0 & \text{if } k = 0 \text{ and } \epsilon = \mp 1, \\ 4y K_{\nu \pm \epsilon/2}(2\pi y) & \text{if } k = 1. \end{cases} \tag{4}$$

Let $\bar{f}(z) := \overline{f(-\bar{z})}$ denote the dual of f. Since f is primitive, it is also an eigenfunction of the operator $\overline{W}_k$ defined in [DFI02, (6.10)], so we have

$$f(z) = \eta \left(i \frac{|z|}{z}\right)^k \bar{f}\left(-\frac{1}{Nz}\right) \tag{5}$$

for some $\eta \in \mathbb{C}$ with $|\eta| = 1$.

Next we define a formal Fourier series $F(z)$ associated to $D_f(s)$ by replacing $\lambda_f(n)$ in the above by $c_f(n)$:

$$F(x+iy) = \sum_{n=1}^{\infty} \frac{c_f(n)}{\sqrt{n}} (V_f^+(ny)\cos(2\pi nx) + iV_f^-(ny)\sin(2\pi nx)).$$

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We expect $F(z)$ to satisfy a relation similar to the modularity relation (5). To make this precise, we first recall the functional equation for $L_f(s)$. Define

$$\gamma_f^{\pm}(s) = \Gamma_{\mathbb{R}}\left(s + \frac{1 \mp (-1)^k \epsilon}{2} + \nu\right)\Gamma_{\mathbb{R}}\left(s + \frac{1 \mp \epsilon}{2} - \nu\right). \tag{6}$$

Then the complete L-function $\Lambda_f(s) := \gamma_f^+(s) L_f(s)$ satisfies

$$\Lambda_f(s) = \eta \epsilon^{1-k} N^{1/2-s} \Lambda_{\bar{f}}(1-s), \tag{7}$$

with η as above.

We define a completed version of $D_f(s)$ by multiplying by the same Γ-factor: $\Delta_f(s) := \gamma_f^+(s) D_f(s)$. Then, differentiating the functional equation (7), we obtain

$$\Delta_f(s) + (\psi_f'(s) - \psi_{\bar{f}}'(1-s))\Lambda_f(s) = \eta \epsilon^{1-k} N^{1/2-s} \Delta_{\bar{f}}(1-s), \tag{8}$$

where $\psi_f(s) := (d/ds) \log \gamma_f^+(s)$. In § 2, we take a suitable inverse Mellin transform of (8). Under the assumption that $\Lambda_f(s)$ has at most finitely many simple zeros, this yields a pseudo-modularity relation for F of the form

$$F(z) + A(z) = \eta \left(i \frac{|z|}{z}\right)^k \overline{F}\left(-\frac{1}{Nz}\right) + B(z), \tag{9}$$

for certain auxiliary functions A and B, where $\overline{F}(z) := \overline{F(-\bar{z})}$. Roughly speaking, A is the contribution from the correction term $(\psi_f'(s) - \psi_{\bar{f}}'(1-s))\Lambda_f(s)$ in (8), and B comes from the non-trivial poles of $\Delta_f(s)$.

The main technical ingredient needed to carry this out is the following pair of Mellin transforms involving the K-Bessel function and trigonometric functions [GR15, 6.699(3) and 6.699(4)]:

$$\begin{aligned}\int_0^\infty x^{\lambda+1} K_\mu(ax) \sin(bx) \frac{dx}{x} &= 2^\lambda b \Gamma\left(\frac{2+\lambda+\mu}{2}\right)\Gamma\left(\frac{2+\lambda-\mu}{2}\right) \\ &\quad \cdot {}_2\mathrm{F}_1\left(\frac{2+\lambda+\mu}{2}, \frac{2+\lambda-\mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right)\end{aligned} \tag{10}$$

and

$$\begin{aligned}\int_0^\infty x^{\lambda+1} K_\mu(ax) \cos(bx) \frac{dx}{x} &= \frac{2^{\lambda-1}}{a^{\lambda+1}} \Gamma\left(\frac{1+\lambda+\mu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu}{2}\right) \\ &\quad \cdot {}_2\mathrm{F}_1\left(\frac{1+\lambda+\mu}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2}\right),\end{aligned} \tag{11}$$

where

$${}_2\mathrm{F}_1(a, b; c; z) = \sum_{j=1}^\infty \frac{a(a+1)\cdots(a+j-1) \cdot b(b+1)\cdots(b+j-1)}{c(c+1)\cdots(c+j-1)} \frac{z^j}{j!} \tag{12}$$

is the Gauss hypergeometric function. The origin of these hypergeometric factors is explained in the introduction to [BT18], and the need to analyze them is the main difference between this paper and the holomorphic case from [Boo16] (for which the corresponding factors are elementary functions).

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Specializing (9) to $z=\alpha+iy$ for $\alpha\in\mathbb{Q}^\times$, we have

$$F(\alpha+iy)+A(\alpha+iy)=\eta\left(i\frac{|\alpha+iy|}{\alpha+iy}\right)^k\overline{F}\left(-\frac{1}{N(\alpha+iy)}\right)+B(\alpha+iy). \tag{13}$$

We will take the Mellin transform of (13). Without difficulty the reader can guess that the transform of $F(\alpha+iy)$ will be a combination of $D_f(s,\alpha,\cos)$ and $D_f(s,\alpha,\sin)$. The calculation of the other terms is non-trivial, but ultimately we obtain the following proposition, which will play the role of [Boo16, Proposition 2.1].

Proposition 1.2. *Suppose that $\Lambda_f(s)$ has at most finitely many simple zeros. Then, for every $M\in\mathbb{Z}_{\geqslant 0}$ and $a\in\{0,1\}$,*

$$P_f(s;a,0)\Delta_f(s,\alpha,\cos^{(a+k)}) \\ -\eta(-\operatorname{sgn}\alpha)^k(N\alpha^2)^{s-1/2}\sum_{m=0}^{M-1}\frac{(2\pi N\alpha)^m}{m!}P_f(s;a,m)\Delta_{\bar f}\left(s+m,-\frac{1}{N\alpha},\cos^{(a+m)}\right)$$

is holomorphic for $\Re(s)>\frac{3}{2}-M$ except for possible poles for $s\pm\nu\in\mathbb{Z}$, where

$$P_f(s;a,m)=\frac{\gamma_f^{(-)^a}(1-s)}{\gamma_f^{(-)^a}(1-s-2\lfloor m/2\rfloor)}\begin{cases}\dfrac{s+2\lfloor m/2\rfloor-(-1)^a\epsilon\nu}{2\pi} & \text{if } k=1 \text{ and } 2\nmid m,\\ 0 & \text{if } k=0 \text{ and } (-1)^a=-\epsilon,\\ 1 & \text{otherwise,}\end{cases}$$

and

$$\Delta_f(s,\alpha,\cos^{(a)})=\gamma_f^{(-)^a}(s)D_f(s,\alpha,\cos^{(a)}).$$

1.2 Proof of Theorem 1.1

Assuming Proposition 1.2 for the moment, we can complete the proof of Theorem 1.1 for the case of π corresponding to a Maass cusp form, f. First, as noted above, we may choose a prime $q\nmid N$ for which $D_f(s,1/q,\cos)$ has infinitely many poles on the line $\Re(s)=0$. Then, by Dirichlet's theorem on primes in an arithmetic progression, for any $M\in\mathbb{Z}_{>0}$ there are distinct primes $q_0,q_1,\ldots,q_{M-1}$ such that $q_j\equiv q \pmod N$ and $D_{\bar f}(s,-q_j/N,\cos^{(a)})=D_{\bar f}(s,-q/N,\cos^{(a)})$ for all j, a.

Let m_0 be an integer with $0\leqslant m_0\leqslant M-1$. By the Vandermonde determinant, there exist rational numbers $c_0,c_1,\ldots,c_{M-1}$ such that

$$\sum_{j=0}^{M-1}c_jq_j^{-m}=\begin{cases}1 & \text{if } m=m_0\\ 0 & \text{if } m\neq m_0\end{cases}\quad\text{for all } m\in\{0,1,\ldots,M-1\}.$$

We fix $\delta\in\{0,1\}$ and apply Proposition 1.2 with $a\equiv\delta+m_0 \pmod 2$ and $\alpha=1/q_j$ for $j=0,1,\ldots,M-1$. Multiplying by $(-1)^kc_j(q_j^2/N)^{s-1/2}$, summing over j and replacing s by $s-m_0$, we find that

$$\sum_{j=0}^{M-1}(-1)^kc_j\left(\frac{q_j^2}{N}\right)^{s-m_0-1/2}P_f(s-m_0;\delta+m_0,0)\Delta_f\left(s-m_0,\frac{1}{q_j},\cos^{(\delta+m_0+k)}\right) \\ -\eta\frac{(-2\pi N)^{m_0}}{m_0!}P_f(s-m_0;\delta+m_0,m_0)\Delta_{\bar f}\left(s,-\frac{q}{N},\cos^{(\delta)}\right)$$

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is holomorphic on $\{s \in \Omega : \Re(s) > \frac{3}{2} + m_0 - M\}$, where we set

$$\Omega = \{s \in \mathbb{C} : s \pm \nu \notin \mathbb{Z}\}.$$

Since $D_f(s - m_0, 1/q_j, \cos^{(\delta+m_0+k)})$ is holomorphic on $\{s \in \Omega : \Re(s) < m_0 - \frac{1}{2}\}$, choosing $m_0 = 2 + \delta + (1-\epsilon)/2$ and M arbitrarily large, we conclude that $D_{\bar{f}}(s, -q/N, \cos^{(\delta)})$ is holomorphic on Ω.

Next we apply Proposition 1.2 again with $a = k$, $\alpha = 1/q$ and $M = 2$. When $k = 1$ or $k = 0$ and $\epsilon = 1$, we see that $D_f(s, 1/q, \cos)$ is holomorphic on $\{s \in \Omega : \Re(s) = 0\}$. This is a contradiction, and Theorem 1.1 follows in these cases.

The remaining case is that of odd Maass forms of weight 0. The above argument with $\delta = 1$ shows that $D_f(s, -q/N, \sin)$ is entire apart from possible poles for $s \pm \nu \in \mathbb{Z}$. Applying Proposition 1.2 with $a = 1$, $\alpha = -q/N$ and $M = 3$, we find that

$$\begin{aligned}-\Delta_f\Big(s, -\frac{q}{N}, \sin\Big) + \eta\Big(\frac{q^2}{N}\Big)^{s-1/2}\Big[\Delta_{\bar{f}}\Big(s, \frac{1}{q}, \sin\Big) - 2\pi q\Delta_{\bar{f}}\Big(s+1, \frac{1}{q}, \cos\Big)\\ - \frac{(2\pi q)^2}{2!}P_f(s;1,2)\Delta_{\bar{f}}\Big(s+2, \frac{1}{q}, \sin\Big)\Big]\end{aligned}$$

is holomorphic on $\{s \in \Omega : \Re(s) > -\frac{5}{2}\}$. Since $D_{\bar{f}}(s, 1/q, \sin)$ is holomorphic on the lines $\Re(s) = -1$ and $\Re(s) = 1$, we see that $D_{\bar{f}}(s, 1/q, \cos)$ is holomorphic on $\{s \in \Omega : \Re(s) = 0\}$. This is again a contradiction, and concludes the proof.

2. Proof of Proposition 1.2

Using expansion (3), we take the Mellin transform of (5) along the line $z = (\omega + i)y$. First, the left-hand side becomes, for $\Re(s) \gg 1$,

$$\begin{aligned}&\int_0^\infty f(\omega y + iy)y^{s-1/2}\frac{dy}{y}\\ &\quad= \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}}\int_0^\infty (V_f^+(ny)\cos(2\pi n\omega y) + iV_f^-(ny)\sin(2\pi n\omega y))y^{s-1/2}\frac{dy}{y}\\ &\quad= G_f(s,\omega)L_f(s),\end{aligned} \tag{14}$$

where, by (4), (10) and (11),

$$\begin{aligned}&G_f(s,\omega)\\ &\quad= \int_0^\infty (V_f^+(y)\cos(2\pi\omega y) + iV_f^-(y)\sin(2\pi\omega y))y^{s-1/2}\frac{dy}{y}\\ &\quad= \begin{cases}(2\pi i\omega)^{(1-\epsilon)/2}\gamma_f^+(s)\ {}_2\mathrm{F}_1\Big(\frac{s+(1-\epsilon)/2+\nu}{2}, \frac{s+(1-\epsilon)/2-\nu}{2}; 1-\frac{\epsilon}{2}; -\omega^2\Big) & \text{if } k = 0,\\ \gamma_f^+(s)\ {}_2\mathrm{F}_1\Big(\frac{s+(1+\epsilon)/2+\nu}{2}, \frac{s+(1-\epsilon)/2-\nu}{2}; \frac{1}{2}; -\omega^2\Big) & \\ \quad + 2\pi i\omega\gamma_f^-(s+1)\ {}_2\mathrm{F}_1\Big(\frac{s+(3-\epsilon)/2+\nu}{2}, \frac{s+(3+\epsilon)/2-\nu}{2}; \frac{3}{2}; -\omega^2\Big) & \text{if } k = 1.\end{cases}\end{aligned} \tag{15}$$

Note that we have $G_{\bar{f}}(s,\omega) = \overline{G_f(\bar{s}, -\omega)}$.

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On the other hand, the Mellin transform of the right-hand side of (5) is, for $-\Re(s) \gg 1$,

$$\eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k \int_0^\infty \bar{f}\left(-\frac{\omega}{N(\omega^2+1)y}+\frac{i}{N(\omega^2+1)y}\right)y^{s-1/2}\frac{dy}{y}.$$

Making the substitution $y \mapsto (N(\omega^2+1)y)^{-1}$, this becomes

$$\begin{aligned}\eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k &(N(1+\omega^2))^{1/2-s}\int_0^\infty \bar{f}(-\omega y+iy)y^{1/2-s}\frac{dy}{y}\\ &= \eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k (N(1+\omega^2))^{1/2-s}G_{\bar{f}}(1-s,-\omega)L_{\bar{f}}(1-s).\end{aligned} \tag{16}$$

By (5), (14) and (16) must continue to entire functions and equal each other. In particular, taking $\omega \to 0$, we recover the functional equation (7). Equating (14) with (16) and dividing by (7), we discover the functional equation for the hypergeometric factor $H_f(s,\omega) := G_f(s,\omega)/\gamma_f^+(s)$:

$$H_f(s,\omega) = \epsilon^{1-k}\left(i\frac{|\omega+i|}{\omega+i}\right)^k (1+\omega^2)^{1/2-s}H_{\bar{f}}(1-s,-\omega). \tag{17}$$

Next, for $z = x+iy \in \mathbb{H}$, define

$$A(z) = \frac{1}{2\pi i}\int_{\Re(s)=1/2}(\psi'(s+\nu)+\psi'(s-\nu))H_f(s,x/y)\Lambda_f(s)y^{1/2-s}\,ds$$

and

$$B(z) = \frac{1}{2\pi i}\int_{\Re(s)=1/2}X_f(s)\Lambda_f(s)H_f(s,x/y)y^{1/2-s}\,ds - \sum_\rho \Lambda_f'(\rho)H_f(\rho,x/y)y^{1/2-\rho}, \tag{18}$$

where the sum runs over all simple zeros of $\Lambda_f(s)$, and

$$X_f(s) = \frac{\pi^2}{4}\left[\csc^2\left(\frac{\pi}{2}\left[s+\frac{1+(-1)^k\epsilon}{2}+\nu\right]\right)+\csc^2\left(\frac{\pi}{2}\left[s+\frac{1+\epsilon}{2}-\nu\right]\right)\right].$$

Lemma 2.1.

$$F(z)+A(z) = \eta\left(i\frac{|z|}{z}\right)^k\overline{F}\left(-\frac{1}{Nz}\right)+B(z) \quad \textit{for all } z \in \mathbb{H}.$$

Proof. Fix $z = x+iy \in \mathbb{H}$, and put $\omega = x/y$. Applying Mellin inversion as in (14), we have

$$F(z) = \frac{1}{2\pi i}\int_{\Re(s)=2}D_f(s)G_f(s,\omega)y^{1/2-s}\,ds$$

and

$$\begin{aligned}&\eta\left(i\frac{|z|}{z}\right)^k\overline{F}\left(-\frac{1}{Nz}\right)\\ &\quad= \eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k\cdot\frac{1}{2\pi i}\int_{\Re(s)=2}G_{\bar{f}}(s,-\omega)D_{\bar{f}}(s)\big(N(1+\omega^2)y\big)^{s-1/2}\,ds\\ &\quad= \eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k\cdot\frac{1}{2\pi i}\int_{\Re(s)=-1}H_{\bar{f}}(1-s,-\omega)\Delta_{\bar{f}}(1-s)\big(N(1+\omega^2)y\big)^{1/2-s}\,ds.\end{aligned}$$

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Applying (17) and (8), and using the fact that $\psi'_{\bar f}(1-s)$ is holomorphic for $\Re(s) \leqslant \frac{1}{2}$, the last line becomes

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Re(s)=-1} \eta\epsilon^{1-k} H_f(s,\omega)\Delta_{\bar f}(1-s)(Ny)^{1/2-s}\,ds \\
&\quad = \frac{1}{2\pi i}\int_{\Re(s)=-1} H_f(s,\omega)[\Delta_f(s) + (\psi'_f(s) - \psi'_{\bar f}(1-s))\Lambda_f(s)]y^{1/2-s}\,ds \\
&\quad = \frac{1}{2\pi i}\int_{\Re(s)=-1} H_f(s,\omega)[\Delta_f(s) + \psi'_f(s)\Lambda_f(s)]y^{1/2-s}\,ds \\
&\qquad - \frac{1}{2\pi i}\int_{\Re(s)=1/2} H_f(s,\omega)\psi'_{\bar f}(1-s)\Lambda_f(s)y^{1/2-s}\,ds.
\end{aligned}$$

Shifting the contour of the first integral to the right and using that $\psi'_f(s)$ is holomorphic for $\Re(s) \geqslant \frac{1}{2}$, we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Re(s)=2} H_f(s,\omega)\Delta_f(s)y^{1/2-s}\,ds - \frac{1}{2\pi i}\int_{\mathcal{C}} H_f(s,\omega)(\Delta_f(s) + \psi'_f(s)\Lambda_f(s))y^{1/2-s}\,ds \\
&\quad + \frac{1}{2\pi i}\int_{\Re(s)=1/2} (\psi'_f(s) - \psi'_{\bar f}(1-s))H_f(s,\omega)\Lambda_f(s)y^{1/2-s}\,ds,
\end{aligned}$$

where $\mathcal{C}$ is the contour running from $2-i\infty$ to $2+i\infty$ and from $-1+i\infty$ to $-1-i\infty$. Note that

$$\Delta_f(s) + \psi'_f(s)\Lambda_f(s) = \Lambda_f(s)\frac{d^2}{ds^2}\log\Lambda_f(s),$$

which has a pole at every simple zero ρ of $\Lambda_f(s)$, with residue $-\Lambda'_f(\rho)$. Hence,

$$-\frac{1}{2\pi i}\int_{\mathcal{C}} H_f(s,\omega)(\Delta_f(s) + \psi'_f(s)\Lambda_f(s))y^{1/2-s}\,ds = \sum_{\rho}\Lambda'_f(\rho)H_f(\rho,\omega)y^{1/2-\rho}.$$

Next, writing $\psi_{\mathbb{R}}(s) = (\Gamma'_{\mathbb{R}}/\Gamma_{\mathbb{R}})(s)$, we have

$$\psi_f(s) = \psi_{\mathbb{R}}\biggl(s + \frac{1-(-1)^k\epsilon}{2} + \nu\biggr) + \psi_{\mathbb{R}}\biggl(s + \frac{1-\epsilon}{2} - \nu\biggr).$$

Applying the reflection formula and Legendre duplication formula in the form

$$\psi'_{\mathbb{R}}(s) = \frac{\pi^2}{4}\csc^2\biggl(\frac{\pi s}{2}\biggr) - \psi'_{\mathbb{R}}(2-s) \quad \text{and} \quad \psi'_{\mathbb{R}}(s) + \psi'_{\mathbb{R}}(s+1) = \psi'(s),$$

we derive

$$\psi'_f(s) - \psi'_{\bar f}(1-s) = \psi'(s+\nu) + \psi'(s-\nu) - X_f(s).$$

Thus,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Re(s)=1/2} (\psi'_f(s) - \psi'_{\bar f}(1-s))H_f(s,\omega)\Lambda_f(s)y^{1/2-s}\,ds \\
&\quad = A(z) - \frac{1}{2\pi i}\int_{\Re(s)=1/2} X_f(s)H_f(s,\omega)\Lambda_f(s)y^{1/2-s}\,ds.
\end{aligned}$$

Rearranging terms completes the proof. □

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Lemma 2.2. *For any* $\alpha \in \mathbb{Q}^\times$,

$$\frac{1}{\Gamma(s+\nu)\Gamma(s-\nu)} \int_0^\infty A(\alpha+iy) y^{s-1/2} \frac{dy}{y}$$

continues to an entire function of s.

Proof. Define $\Phi(s) = \psi'(s+\nu) + \psi'(s-\nu)$. Then we have $\Phi(s) = \int_1^\infty \phi(x) x^{1/2-s}\, dx$, where $\phi(x) = \cosh(\nu \log x) \log x / \sinh((1/2)\log x)$. Applying (15) and the change of variables $y \mapsto xt$, we have

$$\begin{aligned} \Phi(s) G_f(s,\omega) &= \int_1^\infty \int_0^\infty \phi(x)(V_f^+(y)\cos(2\pi\omega y) + iV_f^-(y)\sin(2\pi\omega y))\left(\frac{y}{x}\right)^{s-1/2} \frac{dy}{y}\, dx \\ &= \int_0^\infty \left(\int_1^\infty \phi(x)(V_f^+(tx)\cos(2\pi\omega tx) + iV_f^-(tx)\sin(2\pi\omega tx))\, dx\right) t^{s-1/2} \frac{dt}{t}. \end{aligned}$$

Hence, writing $\omega = \alpha/y$, we have

$$\begin{aligned} A(\alpha+iy) &= \frac{1}{2\pi i} \int_{\Re(s)=2} \Lambda_f(s)\Phi(s) H_f(s,\omega) y^{1/2-s}\, ds \\ &= \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}} \frac{1}{2\pi i} \int_{\Re(s)=2} \Phi(s) G_f(s,\omega)(ny)^{1/2-s}\, ds \\ &= \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}} \int_1^\infty \phi(x)(V_f^+(nxy)\cos(2\pi\alpha nx) + iV_f^-(nxy)\sin(2\pi\alpha nx))\, dx, \end{aligned}$$

so that

$$\begin{aligned} &\int_0^\infty A(\alpha+iy) y^{s-1/2}\frac{dy}{y} \\ &\quad = \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}} \int_1^\infty \phi(x) \int_0^\infty (V_f^+(nxy)\cos(2\pi\alpha nx) + iV_f^-(nxy)\sin(2\pi\alpha nx)) y^{s-1/2}\frac{dy}{y}\, dx \\ &\quad = \sum_{n=1}^\infty \lambda_f(n) n^{-s} \int_1^\infty \phi(x) x^{1/2-s}(\widetilde{V}_f^+(s)\cos(2\pi\alpha nx) + i\widetilde{V}_f^-(s)\sin(2\pi\alpha nx))\, dx, \end{aligned}$$

where

$$\widetilde{V}_f^\pm(s) = \int_0^\infty V_f^\pm(y) y^{s-1/2}\frac{dy}{y} = \begin{cases} \gamma_f^\pm(s) & \text{if } k=1 \text{ or } \epsilon = \pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

A case-by-case inspection of (6) shows that $\widetilde{V}_f^\pm(s)/(\Gamma(s+\nu)\Gamma(s-\nu))$ is entire for both choices of sign.

Define $\phi_j = \phi_j(x,s)$ for $j \geqslant 0$ by

$$\phi_0 = \phi \quad \text{and} \quad \phi_{j+1} = x\frac{\partial \phi_j}{\partial x} - \left(s + j - \frac{1}{2}\right)\phi_j.$$

Then, applying integration by parts m times, we see that

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$$\int_1^\infty \phi(x)\cos(2\pi\alpha n x)x^{1/2-s}\,dx = \sum_{j=0}^{m-1}\frac{\cos^{(j+1)}(2\pi\alpha n)}{(2\pi\alpha n)^{j+1}}\phi_j(1,s) + \int_1^\infty \frac{\cos^{(m)}(2\pi\alpha n x)}{(2\pi\alpha n)^m}\phi_k(x,s)x^{1/2-m-s}\,dx$$

and

$$\int_1^\infty \phi(x)\sin(2\pi\alpha n x)x^{1/2-s}\,dx = \sum_{j=0}^{m-1}\frac{\sin^{(j+1)}(2\pi\alpha n)}{(2\pi\alpha n)^{j+1}}\phi_j(1,s) + \int_1^\infty \frac{\sin^{(m)}(2\pi\alpha n x)}{(2\pi\alpha n)^m}\phi_k(x,s)x^{1/2-m-s}\,dx.$$

Thus,

$$\begin{aligned}\int_0^\infty & A(\alpha+iy)y^{s-1/2}\frac{dy}{y}\\ &= \widetilde{V}_f^+(s)\Bigg(\sum_{j=0}^{m-1}\frac{\phi_j(1,s)L(f,s+j+1,\alpha,\cos^{(j+1)})}{(2\pi\alpha)^{j+1}}\\ &\quad+\frac{1}{(2\pi\alpha)^m}\sum_{n=1}^\infty\frac{a_f(n)}{n^{s+m}}\int_1^\infty \cos^{(m)}(2\pi\alpha n x)\phi_m(x,s)x^{1/2-m-s}\,dx\Bigg)\\ &\quad+i\widetilde{V}_f^-(s)\Bigg(\sum_{j=0}^{m-1}\frac{\phi_j(1,s)L(f,s+j+1,\alpha,\sin^{(j+1)})}{(2\pi\alpha)^{j+1}}\\ &\quad+\frac{1}{(2\pi\alpha)^m}\sum_{n=1}^\infty\frac{a_f(n)}{n^{s+m}}\int_1^\infty \sin^{(m)}(2\pi\alpha n x)\phi_m(x,s)x^{1/2-m-s}\,dx\Bigg).\end{aligned}$$

It follows from [BK11, Proposition 3.1] that $L_f(s,\alpha,\cos)$ and $L_f(s,\alpha,\sin)$ continue to entire functions. We see by induction that $\phi_m(x,s) \ll_m \big((1+|s|)(1+|\nu|)\big)^m x^{-1}$ uniformly for $x \geqslant 1$, and thus the integral terms above are holomorphic for $\Re(s) > \frac{1}{2}-m$. Choosing m arbitrarily large, the lemma follows. $\square$

Lemma 2.3. *For any $\sigma \geqslant 0$ and any $l \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\frac{y^l}{l!}(V_{\bar f}^{\pm})^{(l)}(y) \ll_\sigma 2^l y^{-\sigma} \quad \text{for } y > 0.$$

Proof. In view of (19), since $|\Re(\nu)| < \frac{1}{2}$, for any $\sigma \geqslant 0$ we have the integral representation

$$V_{\bar f}^{\pm}(y) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma+1/2}\widetilde{V}_{\bar f}^{\pm}(s)y^{1/2-s}\,ds.$$

Differentiating l times, we obtain

$$\frac{y^l}{l!}(V_{\bar f}^{\pm})^{(l)}(y) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma+1/2}\binom{\frac{1}{2}-s}{l}\widetilde{V}_{\bar f}^{\pm}(s)y^{1/2-s}\,ds.$$

Using the estimate

$$\left|\binom{\frac{1}{2}-s}{l}\right| = \left|\binom{s-\frac{1}{2}+l}{l}\right| \leqslant 2^{|s-1/2|+l},$$

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we have

$$\frac{y^l}{l!}(V_{\bar f}^{\pm})^{(l)}(y) \leqslant 2^l y^{-\sigma}\cdot\frac{1}{2\pi}\int_{\Re(s)=\sigma+1/2} 2^{|s-1/2|}\left|\widetilde{V}_{\bar f}^{\pm}(s)\,ds\right| \ll_\sigma 2^l y^{-\sigma},$$

where the last inequality is justified by Stirling's formula. □

Lemma 2.4. *Let $\alpha\in\mathbb{Q}^\times$ and $z=\alpha+iy$ for some $y\in(0,|\alpha|/2]$. Then, for any integer $T\geqslant 0$, we have*

$$\begin{aligned}&\left(i\frac{|z|}{z}\right)^k\overline{F}\left(-\frac{1}{Nz}\right)\\ &\quad= O_{\alpha,T}(y^{T-1}) + (i\,\mathrm{sgn}(\alpha))^k\sum_{t=0}^{T-1}\frac{(2\pi iN\alpha)^t}{t!}\\ &\qquad\cdot\sum_{a\in\{0,1\}}\frac{i^{-a}}{2\pi i}\int_{\Re(s)=2}P_f(s;a+t,t)\Delta_{\bar f}\left(s+t,-\frac{1}{N\alpha},\cos^{(a)}\right)\left(\frac{y}{N\alpha^2}\right)^{1/2-s}ds. \qquad (20)\end{aligned}$$

Proof. Let $z=\alpha+iy$, $\beta=-1/N\alpha$ and $u=y/\alpha$. Then

$$-\frac{1}{Nz}=\frac{\beta}{1+u^2}+i\frac{|\beta u|}{1+u^2},$$

so that

$$\begin{aligned}\left(i\frac{|z|}{z}\right)^k\overline{F}\left(-\frac{1}{Nz}\right) &= \left(i\,\mathrm{sgn}(\alpha)\frac{|1+iu|}{1+iu}\right)^k\overline{F}\left(\frac{\beta}{1+u^2}+i\frac{|\beta u|}{1+u^2}\right)\\ &= \left(i\,\mathrm{sgn}(\alpha)\frac{|1+iu|}{1+iu}\right)^k\sum_{n=1}^{\infty}\frac{c_{\bar f}(n)}{\sqrt{n}}\left(V_{\bar f}^{+}\left(\frac{|\beta nu|}{1+u^2}\right)\cos\left(\frac{2\pi\beta n}{1+u^2}\right)\right.\\ &\qquad\left.+iV_{\bar f}^{-}\left(\frac{|\beta nu|}{1+u^2}\right)\sin\left(\frac{2\pi\beta n}{1+u^2}\right)\right).\end{aligned}$$

By Lemma 2.3, for any $\sigma\geqslant 0$ and any $l_0\in\mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned}V_{\bar f}^{\pm}\left(\frac{|\beta nu|}{1+u^2}\right) &= \sum_{l=0}^{\infty}\frac{1}{l!}(V_{\bar f}^{\pm})^{(l)}(|\beta nu|)\left(\frac{\beta nu^3}{1+u^2}\right)^l\\ &= \sum_{l=0}^{l_0-1}\frac{1}{l!}(V_{\bar f}^{\pm})^{(l)}(|\beta nu|)\left(\frac{\beta nu^3}{1+u^2}\right)^l + O_\sigma\left(|\beta nu|^{-\sigma}\sum_{l=l_0}^{\infty}\left(\frac{2u^2}{1+u^2}\right)^l\right)\\ &= \sum_{l=0}^{l_0-1}\frac{1}{l!}(V_{\bar f}^{\pm})^{(l)}(|\beta nu|)\left(\frac{\beta nu^3}{1+u^2}\right)^l + O_{\alpha,\sigma,l_0}(|nu|^{-\sigma}u^{2l_0}).\end{aligned}$$

Similarly, for any $a\in\{0,1\}$, we have

$$\begin{aligned}\cos^{(a)}\left(\frac{2\pi\beta n}{1+u^2}\right) &= \sum_{j=0}^{\infty}\frac{1}{j!}\cos^{(j+a)}(2\pi\beta n)\left(-\frac{2\pi\beta nu^2}{1+u^2}\right)^j\\ &= \sum_{j=0}^{j_0-1}\frac{1}{j!}\cos^{(j+a)}(2\pi\beta n)\left(-\frac{2\pi\beta nu^2}{1+u^2}\right)^j + O\left(\frac{1}{j_0!}\left|\frac{2\pi\beta nu^2}{1+u^2}\right|^{j_0}\right)\\ &= \sum_{j=0}^{j_0-1}\frac{1}{j!}\cos^{(j+a)}(2\pi\beta n)\left(-\frac{2\pi\beta nu^2}{1+u^2}\right)^j + O_{\alpha,j_0}((nu^2)^{j_0}),\end{aligned}$$

by the Lagrange form of the error in Taylor's theorem. Taking $j_0=2(l_0-l)$ and applying Lemma 2.3 with σ replaced by $\sigma+2(l_0-l)$, we obtain

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$$
\begin{aligned}
&V_{\bar f}^{(-)^a}\left(\frac{|\beta n u|}{1+u^2}\right)\cos^{(a)}\left(\frac{2\pi\beta n}{1+u^2}\right)\\
&\quad=\sum_{j+2l<2l_0}\frac{(-2\pi)^j}{j!l!}(V_{\bar f}^{(-)^a})^{(l)}(|\beta n u|)\cos^{(j+a)}(2\pi\beta n)u^l\left(\frac{\beta n u^2}{1+u^2}\right)^{j+l}+O_{\alpha,\sigma,l_0}(|nu|^{-\sigma}u^{2l_0}).
\end{aligned}
$$

Next, defining

$$
b_{j,k,l,m}=\begin{cases}\dbinom{j+l-1+\left\lfloor\frac{m}{2}\right\rfloor+\frac{k}{2}}{\left\lfloor\frac{m}{2}\right\rfloor} & \text{if } k=1 \text{ or } k=0 \text{ and } 2\mid m,\\ 0 & \text{otherwise,}\end{cases}
$$

we have

$$
\begin{aligned}
\left(\frac{|1+iu|}{1+iu}\right)^k(1+u^2)^{-j-l}=(1-iu)^k(1+u^2)^{-j-l-k/2}&=\sum_{m=0}^{\infty}b_{j,k,l,m}(-iu)^m\\
&=\sum_{m=0}^{m_0-1}b_{j,k,l,m}(-iu)^m+O\bigg(\sum_{m=m_0}^{\infty}2^{j+l+m/2}|u|^m\bigg)\\
&=\sum_{m=0}^{m_0-1}b_{j,k,l,m}(-iu)^m+O_{j,l,m_0}(|u|^{m_0}).
\end{aligned}
$$

Taking $m_0=2l_0-j-2l$ and applying Lemma 2.3 with σ replaced by $\sigma+j$, we obtain

$$
\begin{aligned}
&\left(i\operatorname{sgn}(\alpha)\frac{|1+iu|}{1+iu}\right)^k V_{\bar f}^{(-)^a}\left(\frac{|\beta n u|}{1+u^2}\right)\cos^{(a)}\left(\frac{2\pi\beta n}{1+u^2}\right)\\
&\quad=(i\operatorname{sgn}(\alpha))^k\sum_{j+2l+m<2l_0}\frac{(-2\pi)^j(-i)^m}{j!l!}b_{j,k,l,m}(\beta n u)^{j+l}(V_{\bar f}^{(-)^a})^{(l)}(|\beta n u|)\cos^{(j+a)}(2\pi\beta n)u^{j+2l+m}\\
&\qquad+O_{\alpha,\sigma,l_0}(|nu|^{-\sigma}u^{2l_0}).
\end{aligned}
$$

Recalling the definition of u, multiplying by $c_{\bar f}(n)/\sqrt{n}$ and summing over n and both choices of a, the error term converges if $\sigma\geqslant 1$, to give

$$
\begin{aligned}
&\sum_{a\in\{0,1\}}i^{-a}\left(i\frac{|\alpha+iy|}{\alpha+iy}\right)^k\sum_{n=1}^{\infty}\frac{c_{\bar f}(n)}{\sqrt{n}}V_{\bar f}^{(-)^a}\left(\frac{ny}{N(\alpha^2+y^2)}\right)\cos^{(a)}\left(\frac{2\pi\beta n}{1+(y/\alpha)^2}\right)\\
&\quad=\sum_{j+2l+m<2l_0}(i\operatorname{sgn}(\alpha))^k\sum_{a\in\{0,1\}}i^{-a}\sum_{n=1}^{\infty}\frac{c_{\bar f}(n)}{\sqrt{n}}\frac{(2\pi i)^j}{j!l!}b_{j,k,l,m}\left(\frac{ny}{N\alpha^2}\right)^{j+l}\\
&\qquad\cdot(V_{\bar f}^{(-)^a})^{(l)}\left(\frac{ny}{N\alpha^2}\right)\cos^{(j+a)}(2\pi\beta n)\left(\frac{y}{i\alpha}\right)^{j+2l+m}+O_{\alpha,\sigma,l_0}(y^{2l_0-\sigma})\\
&\quad=\sum_{j+2l+m<2l_0}(i\operatorname{sgn}(\alpha))^k\sum_{a\in\{0,1\}}i^{-a}\sum_{n=1}^{\infty}\frac{c_{\bar f}(n)}{\sqrt{n}}\frac{(-2\pi)^j}{j!l!}b_{j,k,l,m}\left(\frac{ny}{N\alpha^2}\right)^{j+l}\\
&\qquad\cdot(V_{\bar f}^{(-)^{a+j}})^{(l)}\left(\frac{ny}{N\alpha^2}\right)\cos^{(a)}(2\pi\beta n)\left(\frac{y}{i\alpha}\right)^{j+2l+m}+O_{\alpha,\sigma,l_0}(y^{2l_0-\sigma}).
\end{aligned}
$$

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Taking the Mellin transform of a single term of the sum over j, l, m and making the change of variables $y \mapsto N\alpha^2 y/n$, we get

$$
\begin{aligned}
&(i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a} \int_0^\infty \sum_{n=1}^\infty \frac{c_{\bar f}(n)}{\sqrt{n}} \frac{(-2\pi)^j}{j!l!} b_{j,k,l,m} \left(\frac{ny}{N\alpha^2}\right)^{j+l} \\
&\quad \cdot (V_{\bar f}^{(-)^{a+j}})^{(l)}\left(\frac{ny}{N\alpha^2}\right) \cos^{(a)}(2\pi\beta n) \left(\frac{y}{i\alpha}\right)^{j+2l+m} y^{s-1/2}\frac{dy}{y} \\
&= (i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a} (N\alpha^2)^{s-1/2} (-iN\alpha)^{j+2l+m} \frac{(-2\pi)^j}{j!} b_{j,k,l,m} \\
&\quad \cdot \sum_{n=1}^\infty \frac{c_{\bar f}(n)\cos^{(a)}(2\pi\beta n)}{n^{s+j+2l+m}} \int_0^\infty \frac{y^l}{l!} (V_{\bar f}^{(-)^{a+j}})^{(l)}(y) y^{s+2j+2l+m-1/2}\frac{dy}{y} \\
&= (i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a} (N\alpha^2)^{s-1/2} (-iN\alpha)^{t} \frac{(-2\pi)^j}{j!} b_{j,k,l,m} \\
&\quad \cdot D_{\bar f}(s+t,\beta,\cos^{(a)}) \binom{\frac12 - s - t - j}{l} \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j),
\end{aligned}
$$

where $t = j+2l+m$.

Next we fix $t \in \mathbb{Z}_{\geqslant 0}$ and sum over all (j,l,m) satisfying $j+2l+m=t$. When $k=0$, $b_{j,k,l,m}$ vanishes unless m is even. Hence, defining

$$
I_k(m) = \begin{cases} 1 & \text{if } k=1 \text{ or } 2 \mid m, \\ 0 & \text{otherwise,} \end{cases}
$$

we get

$$
\begin{aligned}
&(i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a} (N\alpha^2)^{s-1/2}(-iN\alpha)^t \sum_{j+2l+m=t} I_k(t-j) \frac{(-2\pi)^j}{j!} \binom{j+l-1+\lfloor\frac m2\rfloor + \frac k2}{\lfloor \frac m2 \rfloor} \\
&\quad \cdot D_{\bar f}(s+t,\beta,\cos^{(a)}) \binom{\frac12 - s-t-j}{l} \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j) \\
&= (i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a} (N\alpha^2)^{s-1/2}(-iN\alpha)^t \sum_{j=0}^{t} I_k(t-j) \frac{(-2\pi)^j}{j!} D_{\bar f}(s+t,\beta,\cos^{(a)}) \\
&\quad \cdot \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j) \sum_{l=0}^{\lfloor (t-j)/2\rfloor} \binom{j+\lfloor\frac{t-j}{2}\rfloor + \frac k2 - 1}{\lfloor\frac{t-j}{2}\rfloor - l} \binom{\frac12 - s - t - j}{l} \\
&= (i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a} (N\alpha^2)^{s-1/2}(-iN\alpha)^t \sum_{j=0}^{t} I_k(t-j) \frac{(-2\pi)^j}{j!} D_{\bar f}(s+t,\beta,\cos^{(a)}) \\
&\quad \cdot \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j) \binom{\lfloor\frac{t-j}{2}\rfloor + \frac{k-1}{2} - s - t}{\lfloor \frac{t-j}{2}\rfloor},
\end{aligned}
$$

by the Chu–Vandermonde identity.

We now consider two cases according to the weight, k. When $k=0$, the inner sum vanishes identically when $(-1)^{a+t} = -\epsilon$, so we may assume that $(-1)^{a+t}=\epsilon$. Thus, in this case, we have

$$
(N\alpha^2)^{s-1/2}(iN\alpha)^t i^{-a} D_{\bar f}(s+t,\beta,\cos^{(a)}) \sum_{\substack{j\leqslant t \\ j\equiv t \ (\mathrm{mod}\ 2)}} \frac{(2\pi)^j}{j!} \gamma_{\bar f}^{(-)^{a+t}}(s+t+j) \binom{\frac{t-j}{2} - \frac12 - s - t}{\frac{t-j}{2}}.
$$

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Put $t=2n+b$, with $b\in\{0,1\}$. Then, writing $j=2r+b$, the above becomes

$$\begin{aligned}&(N\alpha^2)^{s-1/2}(iN\alpha)^t i^{-a}\Delta_{\bar f}(s+t,\beta,\cos^{(a)})\\ &\quad\cdot\sum_{r=0}^{n}\frac{(2\pi)^{2r+b}}{(2r+b)!}\frac{\Gamma_{\mathbb{R}}(s+t+2r+b+\nu)\Gamma_{\mathbb{R}}(s+t+2r+b-\nu)}{\Gamma_{\mathbb{R}}(s+t+b+\nu)\Gamma_{\mathbb{R}}(s+t+b-\nu)}\binom{n-r-\frac12-s-t}{n-r}\\ &=(N\alpha^2)^{s-1/2}(iN\alpha)^t i^{-a}\Delta_{\bar f}(s+t,\beta,\cos^{(a)})(-1)^n\\ &\quad\cdot\sum_{r=0}^{n}\left(\frac{2\pi}{2r+1}\right)^b\frac{(-4)^r r!^2}{(2r)!}\binom{-(s+t+b+\nu)/2}{r}\binom{-(s+t+b-\nu)/2}{r}\binom{s+t-\frac12}{n-r}.\end{aligned}$$

Applying [BK11, Lemma A.1(ii)–(iii)], we get

$$\begin{aligned}&(N\alpha^2)^{s-1/2}(iN\alpha)^t i^{-a}\Delta_{\bar f}(s+t,\beta,\cos^{(a)})\\ &\quad\cdot\left(\frac{2\pi}{2n+1}\right)^b\frac{4^n n!^2}{(2n)!}\binom{(s+t-1-b+\nu)/2}{n}\binom{(s+t-1-b-\nu)/2}{n}\\ &=(N\alpha^2)^{s-1/2}\frac{(2\pi iN\alpha)^t}{t!}i^{-a}\frac{\gamma_f^{(-)^{a+t}}(1-s)}{\gamma_f^{(-)^{a+t}}(1-s-2n)}\Delta_{\bar f}(s+t,\beta,\cos^{(a)}).\end{aligned}$$

Turning to $k=1$, we have

$$\begin{aligned}&i\,\mathrm{sgn}(\alpha)(N\alpha^2)^{s-1/2}(-iN\alpha)^t\sum_{a\in\{0,1\}}i^{-a}\sum_{j=0}^{t}\frac{(-2\pi)^j}{j!}D_{\bar f}(s+t,\beta,\cos^{(a)})\\ &\quad\cdot\gamma_{\bar f}^{(-)^{a+j}}(s+t+j)\binom{\lfloor\frac{t-j}{2}\rfloor-s-t}{\lfloor\frac{t-j}{2}\rfloor}\\ &=i\,\mathrm{sgn}(\alpha)(N\alpha^2)^{s-1/2}(-iN\alpha)^t\sum_{a\in\{0,1\}}i^{-a}D_{\bar f}(s+t,\beta,\cos^{(a)})\\ &\quad\cdot\sum_{j=0}^{t}\frac{(-2\pi)^j}{j!}\gamma_{\bar f}^{(-1)^{a+j}}(s+t+j)\binom{\lfloor\frac{t-j}{2}\rfloor-s-t}{\lfloor\frac{t-j}{2}\rfloor}.\end{aligned}$$

Writing $j=2r-c$ with $c\in\{0,1\}$, this is

$$\begin{aligned}&i\,\mathrm{sgn}(\alpha)(N\alpha^2)^{s-1/2}(-iN\alpha)^t\sum_{a\in\{0,1\}}i^{-a}\Delta_{\bar f}(s+t,\beta,\cos^{(a)})\\ &\quad\cdot\sum_{c\in\{0,1\}}\sum_{2r-c\leqslant t}\frac{(-2\pi)^{2r-c}}{(2r-c)!}\binom{n-r+\lfloor\frac{b+c}{2}\rfloor-s-t}{n-r+\lfloor\frac{b+c}{2}\rfloor}\\ &\quad\cdot\frac{\Gamma_{\mathbb{R}}(s+t+2r-c+(1-(-1)^{a+c}\epsilon)/2+\nu)\Gamma_{\mathbb{R}}(s+t+2r-c+(1+(-1)^{a+c}\epsilon)/2-\nu)}{\Gamma_{\mathbb{R}}(s+t+(1-(-1)^a\epsilon)/2+\nu)\Gamma_{\mathbb{R}}(s+t+(1+(-1)^a\epsilon)/2-\nu)}\\ &=i\,\mathrm{sgn}(\alpha)(N\alpha^2)^{s-1/2}(-iN\alpha)^t\sum_{a\in\{0,1\}}i^{-a}\Delta_{\bar f}(s+t,\beta,\cos^{(a)})\sum_{c\in\{0,1\}}(-1)^{n+bc}\\ &\quad\cdot\sum_{2r-c\leqslant t}\frac{(-4)^r r!^2}{(2r)!}\binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{r-c\frac{1-(-1)^a\epsilon}{2}}\binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{r-c\frac{1+(-1)^a\epsilon}{2}}\binom{s+t-1}{n+bc-r}.\end{aligned}$$

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For $b=0$, applying [BK11, Lemma A.1(ii)], the sum over c becomes

$$
\begin{aligned}
&(-1)^n \sum_{r=0}^{n} \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t-1+\frac{1-(-1)^a\epsilon}{2}-\nu)/2}{r} \binom{-(s+t-1+\frac{1+(-1)^a\epsilon}{2}+\nu)/2}{r} \binom{s+t-1}{n-r} \\
&\quad = \frac{4^n n!^2}{(2n)!} \binom{(s+2n-2+\frac{1-(-1)^a\epsilon}{2}-\nu)/2}{n} \binom{(s+2n-2+\frac{1+(-1)^a\epsilon}{2}+\nu)/2}{n} \\
&\quad = \frac{(-2\pi)^{2n}}{(2n)!} \frac{\Gamma_{\mathbb{R}}(1-s+(1+(-1)^a\epsilon)/2+\nu)}{\Gamma_{\mathbb{R}}(1-s-2n+(1+(-1)^a\epsilon)/2+\nu)} \frac{\Gamma_{\mathbb{R}}(1-s+(1-(-1)^a\epsilon)/2-\nu)}{\Gamma_{\mathbb{R}}(1-s-2n+(1-(-1)^a\epsilon)/2-\nu)} \\
&\quad = \frac{(-2\pi)^t}{t!} \frac{\gamma_f^{(-)^a}(1-s)}{\gamma_f^{(-)^a}(1-s-2n)} = \frac{(-2\pi)^t}{t!} \frac{\gamma_f^{(-)^{a+t}}(1-s)}{\gamma_f^{(-)^{a+t}}(1-s-2\lfloor t/2 \rfloor)}.
\end{aligned}
$$

For $b=1$ and $c=0$, the inner sum is

$$
(-1)^n \sum_{r=0}^{n} \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{r} \binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{r} \binom{s+t-1}{n-r}.
$$

Writing $\binom{s+t-1}{n-r} = \binom{s+t}{n-r+1} - \binom{s+t-1}{n-r+1}$ and applying [BK11, Lemma A.1(ii)], we get

$$
\begin{aligned}
&(-1)^n \bigg[\frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!} \binom{(s+t-\frac{1+(-1)^a\epsilon}{2}+\nu)/2}{n+1} \binom{(s+t-\frac{1-(-1)^a\epsilon}{2}-\nu)/2}{n+1} \\
&\qquad - \frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!} \binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{n+1} \binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{n+1} \bigg] \\
&\qquad + (-1)^{n+1} \bigg[\sum_{r=0}^{n+1} \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{r} \\
&\qquad \times \binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{r} \binom{s+t-1}{n-r+1} \\
&\qquad - \frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!} \binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{n+1} \binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{n+1} \bigg] \\
&\quad = (-1)^n \frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!} \binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1} \binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1} \\
&\qquad + (-1)^{n+1} \sum_{r=0}^{n+1} \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t-(-1)^a\epsilon\nu+1)/2}{r} \binom{-(s+t+(-1)^a\epsilon\nu)/2}{r} \binom{s+t-1}{n-r+1}.
\end{aligned}
$$

For $b=1$ and $c=1$ the inner sum is

$$
(-1)^{n+1} \sum_{r=1}^{n+1} \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t-(-1)^a\epsilon\nu+1)/2}{r-1} \binom{-(s+t+(-1)^a\epsilon\nu)/2}{r} \binom{s+t-1}{n+1-r},
$$

and, adding this to the contribution from $c=0$, for $b=1$ we obtain

$$
\begin{aligned}
&(-1)^n \frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!} \binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1} \binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1} + (-1)^{n+1} \binom{s+t-1}{n+1} \\
&\quad + (-1)^{n+1} \sum_{r=1}^{n+1} \frac{(-4)^r r!^2}{(2r)!} \binom{1-(s+t-(-1)^a\epsilon\nu+1)/2}{r} \binom{-(s+t+(-1)^a\epsilon\nu)/2}{r} \binom{s+t-1}{n-r+1}
\end{aligned}
$$

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$$= (-1)^n \frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!} \binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1} \binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1}$$
$$+ (-1)^{n+1} \sum_{r=0}^{n+1} \frac{(-4)^r r!^2}{(2r)!} \binom{1-(s+t-(-1)^a\epsilon\nu+1)/2}{r} \binom{-(s+t+(-1)^a\epsilon\nu)/2}{r} \binom{s+t-1}{n-r+1}.$$

Applying [BK11, Lemma A.1(ii)], this is

$$-\frac{4^{n+1}(n+1)!^2}{(2n+2)!} \binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1}$$
$$\times \left[\binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1} - \binom{(s+t-(-1)^a\epsilon\nu)/2-1}{n+1} \right]$$
$$= -\frac{4^{n+1}(n+1)!^2}{(2n+2)!} \frac{(s+(-1)^a\epsilon\nu+2n)/2}{n+1}$$
$$\cdot \binom{(s+2n-2+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{n} \binom{(s+2n-2+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{n}$$
$$= \frac{s+2\lfloor t/2 \rfloor-(-1)^{a+t}\epsilon\nu}{2\pi} \frac{(-2\pi)^t}{t!} \frac{\gamma_f^{(-)^{a+t}}(1-s)}{\gamma_f^{(-)^{a+t}}(1-s-2\lfloor t/2 \rfloor)}.$$

In all cases, the result matches the formula for $P_f(s; a+t, t)$. Taking $l_0 = \lceil T/2 \rceil$, $\sigma = 1$ and applying Mellin inversion, we get (20), with $T+1$ in place of T when T is odd. In that case, we estimate the final term of the sum by shifting the contour to $\Re(s) = \frac{3}{2} - T$, which yields $O(y^{T-1})$. □

LEMMA 2.5. *Assume that $\Lambda_f(s)$ has at most finitely many simple zeros, and let $\alpha \in \mathbb{Q}^\times$ and $z = \alpha + iy$ for some $y \in (0, |\alpha|/4]$. Then there are numbers $a_j(\alpha), b_j(\alpha) \in \mathbb{C}$ such that, for any integer $M \geqslant 0$, we have*

$$B(\alpha+iy) = O_{\alpha,f,M}(y^M) + \sum_{j=0}^{M-1} y^{j+1/2} \begin{cases} a_j(\alpha) + b_j(\alpha)\log y & \text{if } \nu = k = 0, \\ a_j(\alpha)y^\nu + b_j(\alpha)y^{-\nu} & \text{otherwise.} \end{cases} \tag{21}$$

Proof. Let $s \in \mathbb{C}$ with $\Re(s) \in (0,1)$, and set $\omega = \alpha/y$. We will show that there are numbers $a_j(\alpha,s), b_j(\alpha,s) \in \mathbb{C}$ satisfying

$$H_f(s,\omega)y^{1/2-s} = \sum_{j=0}^{\infty} y^{j+1/2} \begin{cases} a_j(\alpha,s) + b_j(\alpha,s)\log y & \text{if } \nu = k = 0, \\ a_j(\alpha,s)y^\nu + b_j(\alpha,s)y^{-\nu} & \text{otherwise,} \end{cases} \tag{22}$$

and

$$a_j(\alpha,s), b_j(\alpha,s) \ll_{f,\alpha,\varepsilon} (2e^{\pi/2})^{(1+\varepsilon)|s|} |2/\alpha|^{j+1/2} \sqrt{j+1} \quad \text{for all } \varepsilon > 0. \tag{23}$$

Let us assume this for now. Then, since $y \leqslant |\alpha|/4$, we have

$$\sum_{j=M}^{\infty} \left(\frac{2y}{|\alpha|}\right)^{j+1/2} \sqrt{j+1} \ll_{\alpha,M} y^{M+1/2},$$

so that (by the trivial estimate $|\Re(\nu)| < \frac{1}{2}$),

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$$H_f(s,\omega)y^{1/2-s} = O_{f,\alpha,M,\varepsilon}((2e^{\pi/2})^{(1+\varepsilon)|s|}y^M) + \sum_{j=0}^{M-1} y^{j+1/2}\begin{cases} a_j(\alpha,s)+b_j(\alpha,s)\log y & \text{if } \nu=k=0,\\ a_j(\alpha,s)y^{\nu}+b_j(\alpha,s)y^{-\nu} & \text{otherwise.}\end{cases} \tag{24}$$

We substitute this expansion into (18). By hypothesis, $\Lambda_f(s)$ has at most finitely many simple zeros, so the sum over ρ in (18) is a finite linear combination of the series (24) with $s=\rho$, which yields an expansion of the shape (21). As for the integral term in (18), by the convexity bound and Stirling's formula, we have

$$X_f(s)\Lambda_f(s) \ll_{f,\varepsilon} e^{-(3\pi/2-\varepsilon)|s|} \quad \text{for } \Re(s)=\tfrac{1}{2},\ \varepsilon>0.$$

Since $2<e^{\pi}$, the integral converges absolutely and again yields something of the shape (21).

It remains to show (22) and (23). First suppose that $k=0$. Then, by (15), we have

$$H_f(s,\omega)y^{1/2-s} = |\alpha/\omega|^{1/2-s}(2\pi i\omega)^{(1-\epsilon)/2}\ {}_2\mathrm{F}_1\left(\frac{s+(1-\epsilon)/2+\nu}{2},\frac{s+(1-\epsilon)/2-\nu}{2};1-\frac{\epsilon}{2};-\omega^2\right).$$

Applying the hypergeometric transformation [GR15, 9.132(2)] and the defining series (12), this is

$$\begin{aligned}
&(\pi i\,\mathrm{sgn}(\alpha))^{(1-\epsilon)/2}|\alpha|^{1/2-s}\pi^{1/2}\sum_{\pm}\frac{|y/\alpha|^{1/2\pm\nu}\Gamma(\mp\nu)}{\Gamma(1-(s+(1+\epsilon)/2\pm\nu)/2)\Gamma((s+(1-\epsilon)/2\mp\nu)/2)}\\
&\quad\cdot\ {}_2\mathrm{F}_1\left(\frac{s+(1-\epsilon)/2\pm\nu}{2},\frac{s+(1+\epsilon)/2\pm\nu}{2};1\pm\nu;-\left(\frac{y}{\alpha}\right)^2\right)\\
&\quad= (\pi i\,\mathrm{sgn}(\alpha))^{(1-\epsilon)/2}|\alpha|^{1/2-s}\pi^{1/2}\sum_{j=0}^{\infty}\sum_{\pm}\frac{\Gamma(\mp\nu)}{\Gamma(1-(s+(1+\epsilon)/2\pm\nu)/2)\Gamma((s+(1-\epsilon)/2\mp\nu)/2)}\\
&\quad\cdot\frac{\binom{-\frac{s+(1-\epsilon)/2\pm\nu}{2}}{j}\binom{-\frac{s+(1+\epsilon)/2\pm\nu}{2}}{j}}{\binom{-1\mp\nu}{j}}\left|\frac{y}{\alpha}\right|^{2j+1/2\pm\nu}.
\end{aligned}\tag{25}$$

To pass from this to (22), we replace $2j$ by j and set $a_j=b_j=0$ when j is odd.

When $\nu\neq 0$ we use the estimates

$$\left|\binom{-\frac{s+a\pm\nu}{2}}{j}\right| = \left|\binom{\frac{s+a\pm\nu}{2}+j-1}{j}\right| \leqslant 2^{|s+a\pm\nu|/2+j} \ll_f 2^{|s|/2+j} \quad \text{for } a\in\{0,1\},$$

$$\left|\binom{-1\mp\nu}{j}\right| = \prod_{l=1}^{j}\left|1\pm\frac{\nu}{l}\right| \geqslant \prod_{l=1}^{j}\left|1-\frac{1}{2l}\right| = \left|\binom{-\frac{1}{2}}{j}\right| \gg \frac{1}{\sqrt{2j+1}}$$

and

$$(\pi i\,\mathrm{sgn}(\alpha))^{(1-\epsilon)/2}|\alpha|^{1/2-s}\pi^{1/2}\frac{\Gamma(\mp\nu)}{\Gamma(1-(s+(1+\epsilon)/2\pm\nu)/2)\Gamma((s+(1-\epsilon)/2\mp\nu)/2)} \ll_{f,\varepsilon} e^{(\pi/2+\varepsilon)|s|} \quad \text{for all } \varepsilon>0$$

to obtain (23).

When $\nu=0$, (25) has a singularity arising from the $\Gamma(\pm\nu)$ factors, but we can still understand the formula by analytic continuation. To remove the singularity, we replace $y^{\pm\nu}$ by $(y^{\pm\nu}-1)+1$. Since

$$\lim_{\nu\to 0}\Gamma(\pm\nu)(y^{\pm\nu}-1) = \log y,$$

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in the terms with $y^{\pm\nu} - 1$ we can simply take the limit and estimate the remaining factors as before; this gives the b_j terms in (22) and (23). The terms with 1 can be written in the form $y^{2j+1/2}(h_j(\nu) + h_j(-\nu))$, where h_j is meromorphic with a simple pole at $\nu = 0$, and independent of y. Then $h_j(\nu) + h_j(-\nu)$ is even, so it has a removable singularity at $\nu = 0$. By the Cauchy integral formula, we have

$$\lim_{\nu\to 0}(h_j(\nu) + h_j(-\nu)) = \frac{1}{2\pi i}\int_{|\nu|=1/2} \frac{h_j(\nu) + h_j(-\nu)}{\nu}\, d\nu.$$

Since the above estimates hold uniformly for $\nu \in \mathbb{C}$ with $|\nu| = \frac{1}{2}$, they also hold for $\lim_{\nu\to 0}(h_j(\nu) + h_j(-\nu))$. This concludes the proof of (22) and (23) when $k = 0$.

Turning to $k = 1$, by (15) we have

$$\begin{aligned}
&H_f(s,\omega)y^{1/2-s}\\
&\quad = \sum_{\delta\in\{0,1\}} \left|\frac{\alpha}{\omega}\right|^{1/2-s} (i\omega(s-\epsilon\nu))^{\delta}\\
&\qquad \cdot\ {}_2\mathrm{F}_1\left(\frac{s+(-1)^{\delta}((1+\epsilon)/2)+\nu}{2}+\delta, \frac{s+(-1)^{\delta}((1-\epsilon)/2)-\nu}{2}+\delta; \frac{1}{2}+\delta; -\omega^2\right),
\end{aligned}$$

and applying [GR15, 9.132(2)], this becomes

$$\begin{aligned}
&\pi^{1/2}|\alpha|^{1/2-s} \sum_{\delta\in\{0,1\}} \left(\frac{i\,\mathrm{sgn}(\alpha)(s-\epsilon\nu)}{2}\right)^{\delta} \sum_{\pm} \left|\frac{y}{\alpha}\right|^{1/2+(1\pm(-1)^{\delta}\epsilon)/2\pm\nu}\\
&\quad \times \frac{\Gamma(\mp(\nu+(-1)^{\delta}(\epsilon/2)))}{\Gamma((s+(-1)^{\delta}((1\mp\epsilon)/2)\mp\nu)/2+\delta)\Gamma(1/2-(s+(-1)^{\delta}((1\pm\epsilon)/2)\pm\nu)/2)}\\
&\quad \cdot\ {}_2\mathrm{F}_1\Bigg(\frac{s+(-1)^{\delta}((1\pm\epsilon)/2)\pm\nu}{2}+\delta, \frac{s+(-1)^{\delta}((1\pm\epsilon)/2)\pm\nu}{2}+\frac{1}{2};\\
&\qquad 1\pm\left(\nu+(-1)^{\delta}\frac{\epsilon}{2}\right); -\left(\frac{y}{\alpha}\right)^2\Bigg).
\end{aligned}$$

In this case no singularity arises from the Γ-factor in the numerator, so expanding the final ${}_2\mathrm{F}_1$ as a series and applying a similar analysis to the above, we arrive at (22) and (23). $\square$

With the lemmas in place, we can now complete the proof of Proposition 1.2. Let

$$\chi_{(0,|\alpha|/4]}(y) = \begin{cases} 1 & \text{if } y \leqslant \dfrac{|\alpha|}{4},\\[2mm] 0 & \text{if } y > \dfrac{|\alpha|}{4}, \end{cases}$$

and define

$$\begin{aligned}
g(y) = F(\alpha+iy) + A(\alpha+iy) - \chi_{(0,|\alpha|/4]}(y)\sum_{j=0}^{M-1} y^{j+1/2} &\begin{cases} a_j(\alpha) + b_j(\alpha)\log y & \text{if } \nu = k = 0,\\ a_j(\alpha)y^{\nu} + b_j(\alpha)y^{-\nu} & \text{otherwise,}\end{cases}\\
- \eta(i\,\mathrm{sgn}(\alpha))^k \sum_{t=0}^{M-1}\frac{(2\pi i N\alpha)^t}{t!}\sum_{a\in\{0,1\}}\frac{i^{-a}}{2\pi i}&\\
\cdot \int_{\Re(s)=2} P_f(s; a+t, t)\Delta_{\bar{f}}\left(s+t, -\frac{1}{N\alpha}, \cos^{(a)}\right)\left(\frac{y}{N\alpha^2}\right)^{1/2-s} ds.&
\end{aligned}$$

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By Lemmas 2.1, 2.4 and 2.5, we have $g(y)=O_{\alpha,M}(y^{M-1})$ for $y\leqslant|\alpha|/4$. On the other hand, shifting the contour of the above to the right, we see that g decays rapidly as $y\to\infty$. Hence, $\int_0^\infty g(y)y^{s-1/2}(dy/y)$ converges absolutely and defines a holomorphic function for $\Re(s)>\frac{5}{2}-M$.

We have

$$\int_0^\infty F(\alpha+iy)y^{s-1/2}\frac{dy}{y}=\sum_{a\in\{0,1\}}i^{-a}\Delta_f\big(s,\alpha,\cos^{(a)}\big)\begin{cases}1 & \text{if } k=1 \text{ or } (-1)^a=\epsilon,\\ 0 & \text{otherwise.}\end{cases}$$

By Lemma 2.2, $\int_0^\infty A(\alpha+iy)y^{s-1/2}(dy/y)$ continues to a holomorphic function on Ω. Similarly,

$$\begin{aligned}\int_0^\infty & y^{s-1/2}\frac{dy}{y}\cdot\chi_{(0,|\alpha|/4]}(y)\sum_{j=0}^{M-1}y^{j+1/2}\begin{cases}a_j(\alpha)+b_j(\alpha)\log y & \text{if } \nu=k=0,\\ a_j(\alpha)y^\nu+b_j(\alpha)y^{-\nu} & \text{otherwise,}\end{cases}\\ &=\sum_{j=0}^{M-1}\begin{cases}\dfrac{|\alpha/4|^{s+j}}{s+j}\left[a_j(\alpha)+b_j(\alpha)\left(\log|\alpha/4|-\dfrac{1}{s+j}\right)\right] & \text{if } \nu=k=0,\\ a_j(\alpha)\dfrac{|\alpha/4|^{s+j+\nu}}{s+j+\nu}+b_j(\alpha)\dfrac{|\alpha/4|^{s+j-\nu}}{s+j-\nu} & \text{otherwise,}\end{cases}\end{aligned}$$

is holomorphic on Ω. Hence, by Mellin inversion,

$$\begin{aligned}&\sum_{a\in\{0,1\}}i^{-a}\Delta_f(s,\alpha,\cos^{(a)})\begin{cases}1 & \text{if } k=1 \text{ or } (-1)^a=\epsilon,\\ 0 & \text{otherwise,}\end{cases}\\ &-\eta(i\operatorname{sgn}(\alpha))^k(N\alpha^2)^{s-1/2}\sum_{t=0}^{M-1}\frac{(2\pi iN\alpha)^t}{t!}\sum_{a\in\{0,1\}}i^{-a}P_f(s;a+t,t)\Delta_{\bar f}\left(s+t,-\frac{1}{N\alpha},\cos^{(a)}\right)\end{aligned}\tag{26}$$

is holomorphic on $\{s\in\Omega:\Re(s)>\frac{5}{2}-M\}$.

Denoting (26) by $h(\alpha)$, we consider the combination $\frac{1}{2}(i^{k+a_0}h(\alpha)+i^{-k-a_0}h(-\alpha))$ for some $a_0\in\{0,1\}$. This picks out the term with $a\equiv k+a_0 \pmod 2$ in the first sum over a, and $a\equiv t+a_0 \pmod 2$ in the second. Therefore, since

$$P_f(s;a_0,0)=\begin{cases}1 & \text{if } k=1 \text{ or } (-1)^{a_0}=\epsilon,\\ 0 & \text{otherwise,}\end{cases}$$

we find that

$$\begin{aligned}&P_f(s;a_0,0)\Delta_f(s,\alpha,\cos^{(k+a_0)})\\ &\quad-\eta(-\operatorname{sgn}(\alpha))^k(N\alpha^2)^{s-1/2}\sum_{t=0}^{M-1}\frac{(2\pi N\alpha)^t}{t!}P_f(s;a_0,t)\Delta_{\bar f}\left(s+t,-\frac{1}{N\alpha},\cos^{(t+a_0)}\right)\end{aligned}\tag{27}$$

is holomorphic on $\{s\in\Omega:\Re(s)>\frac{5}{2}-M\}$. Finally, replacing M by $M+1$ and discarding the final term of the sum, we see that (27) is holomorphic on $\{s\in\Omega:\Re(s)>\frac{3}{2}-M\}$, as required.

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