J. Aust. Math. Soc. **93** (2012), 259–276
doi:10.1017/S1446788712000328

IRREDUCIBLE QUASIORDERS OF MONOUNARY ALGEBRAS

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(Received 7 September 2011; accepted 13 August 2012; first published online 1 March 2013)

Communicated by M. G. Jackson

Abstract

Rooted monounary algebras can be considered as an algebraic counterpart of directed rooted trees. We work towards a characterization of the lattice of compatible quasiorders by describing its join- and meet-irreducible elements. We introduce the limit $\mathcal{B}_\infty$ of all d-dimensional Boolean cubes $\mathbf{2}^d$ as a monounary algebra; then the natural order on $\mathbf{2}^d$ is meet-irreducible. Our main result is that any completely meet-irreducible quasiorder of a rooted algebra is a homomorphic preimage of the natural partial order (or its inverse) of a suitable subalgebra of $\mathcal{B}_\infty$. For a partial order, it is known that complete meet-irreducibility means that the corresponding partially ordered structure is subdirectly irreducible. For a rooted monounary algebra it is shown that this property implies that the unary operation has finitely many nontrivial kernel classes and its graph is a binary tree.

2010 *Mathematics subject classification*: primary 06F99; secondary 08A60, 06B99, 05C05.

Keywords and phrases: partial order, monounary algebra, quasiorder lattice, meet-irreducible quasiorder, critical pair, rooted tree.

1. Introduction

Rooted monounary algebras can be considered as an algebraic counterpart of directed rooted trees. Namely, a rooted tree T with vertex set A and root $\top \in A$ can be interpreted as the graph of a unary mapping $f : A \to A$, yielding a rooted monounary algebra $\mathcal{A} = (A, f)$. Clearly, the top element $\top$ then corresponds to the unique fixed point of f, and $f(a)$ is the father of $a \in A \setminus \{\top\}$ in the tree T.

Rooted monounary algebras also form a special class of acyclic monounary algebras, which have the remarkable property that each compatible partial order can

The first author was supported by Slovak VEGA grant 2/0194/10. The second and third author were supported by DFG grant 436 UNG 113/173/0-2. The research of the third author was carried out as part of the TAMOP-4.2.1.B-10/2/KONV-2010-0001 project, supported by the European Union, co-financed by the European Social Fund.

be extended to a compatible linear order (see Szigeti and Nagy [8]). For arbitrary monounary algebras and partial orders we refer to Foldes and Szigeti [3].

Because compatible partial orders do not form a lattice, their investigation led us naturally to the study of the lattice Quord $\mathcal{A}$ of compatible quasiorders, that is, reflexive and transitive relations, which constitute the least common generalization of congruences and compatible partial orders. In [6] several structure properties of Quord $\mathcal{A}$ and of its down-set of compatible partial orders are given (for example, Quord $\mathcal{A}$ is a glued sum of intervals in the lattice Quord(A) of all quasiorders on the set A).

It is well known that the 'building bricks' of each finite or, more generally, double-founded lattice, are its (completely) $\wedge$- and $\vee$-irreducible elements (see, for example, Ganter and Wille [4]). Aiming to characterize the lattice Quord $\mathcal{A}$, the present paper is devoted to the description of its (completely) $\wedge$- and $\vee$-irreducible elements. The completely $\vee$-irreducible quasiorders are easy to describe (see Proposition 3.1). However, $\wedge$-irreducible quasiorders not only need much more effort but are also of special importance: a compatible quasiorder q of an algebra $\mathcal{A}$ is completely $\wedge$-irreducible if and only if the corresponding canonical factor structure $\mathcal{A}/q_0$ (see Definition 2.2) considered as an ordered algebra is subdirectly irreducible in the sense of Czédli and Lenkehegyi [1, Section 3]; see Proposition 2.8 (and also Remark 5.7). Thus, if a partially ordered algebra (A, f, r) is represented as a subdirect product of subdirect irreducibles, then this means that it is represented as a subalgebra of a direct product of smaller algebraic structures with completely $\wedge$-irreducible partial order relations.

Therefore the main part of the paper aims to characterize compatible completely $\wedge$-irreducible quasiorders. The first step is a reduction to $\wedge$-irreducible partial orders (Proposition 2.6). These are characterized in Section 3 via so-called critical pairs. In Section 4, the d-dimensional Boolean cubes with their $\wedge$-irreducible partial orders (induced by the natural order of the Boolean cube) and their limit $\mathcal{B}_\infty$ are considered. It turns out in Section 5, that $\mathcal{B}_\infty$ – or for finite algebras, the Boolean cubes $\mathcal{B}_d$ – include all necessary information for the characterization of arbitrary completely $\wedge$-irreducible quasiorders. The main result is the characterization theorem for completely $\wedge$-irreducible partial orders (Theorem 5.2) together with Corollary 5.6 for completely $\wedge$-irreducible quasiorders.

2. Preliminaries

Definition 2.1. A *quasiorder* q on a set A is a reflexive and transitive relation $q \subseteq A \times A$. The quasiorders on A form (with respect to $\subseteq$) an algebraic, complemented, atomistic and dually atomistic lattice (see Erné and Reinhold [2]) denoted by Quord(A), which contains as a complete sublattice the lattice Equ(A) of all equivalence relations on A. For $q \in$ Quord(A) its *inverse* $q^{-1} := \{(y, x) \mid (x, y) \in q\}$ is also a quasiorder and the relation

$$q_0 := q \cap q^{-1}$$

https://doi.org/10.1017/S1446788712000328 Published online by Cambridge University Press

is an equivalence on A. Obviously, q is a partial order if and only if $q_0 = \Delta$ (where $\Delta := \{(x, x) \mid x \in A\}$). The q_0-equivalence class of an element $a \in A$ will be denoted by $[a]_{q_0}$. It is well known that $q \in \mathrm{Quord}(A)$ induces a natural partial order q/q_0 on the factor set $A/q_0 := \{[a]_{q_0} \mid a \in A\}$ given by

$$([a]_{q_0}, [b]_{q_0}) \in q/q_0 :\Longleftrightarrow \exists\, u \in [a]_{q_0}\ \exists\, v \in [b]_{q_0} : (u, v) \in q.$$

It is easy to verify that $([a]_{q_0}, [b]_{q_0}) \in q/q_0$ holds if and only if $(x, y) \in q$ for all $x \in [a]_{q_0}$ and $y \in [b]_{q_0}$.

For a unary mapping $f : A \to A$ we write for simplicity fx and $f^n x$ instead of $f(x)$ and $f^n(x)$ (where $f^0(x) := x$, $f^{n+1}(x) := f(f^n(x))$, $n \in \mathbb{N}$).

Definition 2.2. A unary mapping $f : A \to A$ *preserves* a binary relation $\varrho \subseteq A \times A$ (or f is an *endomorphism* of ϱ, or ϱ is *invariant for* or *compatible with* f; notation $f \triangleright \varrho$), if

$$\forall x, y \in A : (x, y) \in \varrho \implies (fx, fy) \in \varrho.$$

For a monounary algebra $\mathcal{A} = (A, f)$ we consider the following sets of compatible relations:

$$\begin{aligned} \mathrm{Quord}\,\mathcal{A} &:= \{\varrho \subseteq A^2 \mid \varrho \text{ quasiorder}, f \triangleright \varrho\}, \\ \mathrm{Con}\,\mathcal{A} &:= \{\varrho \subseteq A^2 \mid \varrho \text{ equivalence relation}, f \triangleright \varrho\}, \\ \mathrm{Pord}\,\mathcal{A} &:= \{\varrho \subseteq A^2 \mid \varrho \text{ partial order}, f \triangleright \varrho\}. \end{aligned}$$

It is known that $\mathrm{Quord}\,\mathcal{A}$ is a complete sublattice of $\mathrm{Quord}(A)$, and $\mathrm{Con}\,\mathcal{A}$ is a complete sublattice of $\mathrm{Equ}(A)$ (we sometimes write $r \le q$ and $r < q$ instead of $r \subseteq q$ and $r \subsetneq q$). This follows from the fact that the infimum of a system $\{\alpha_i \mid i \in I\}$ of elements in each of these lattices is equal to the set-theoretical intersection $\bigcap\{\alpha_i \mid i \in I\}$; and that the supremum is equal to $(\bigcup\{\alpha_i \mid i \in I\})^{\mathrm{tra}}$ where α^{tra} stands for the *transitive closure* of a binary relation $\alpha \subseteq A^2$. $\mathrm{Pord}\,\mathcal{A}$ is an order ideal of $\mathrm{Quord}\,\mathcal{A}$, since $r \in \mathrm{Pord}\,\mathcal{A}$ and $q \subseteq r$ imply that $q \in \mathrm{Pord}\,\mathcal{A}$ for any $q \in \mathrm{Quord}\,\mathcal{A}$.

If $q \in \mathrm{Quord}(A, f)$ then $q_0 \in \mathrm{Con}(A, f)$ and $\mathcal{A} = (A, f)$ has a natural homomorphic image $\mathcal{A}/q_0 := (A/q_0, \bar{f})$, where $\bar{f}$ is defined by $\bar{f}([x]_{q_0}) := [f(x)]_{q_0}$. The partial order q/q_0 is also compatible: $q/q_0 \in \mathrm{Pord}(A/q_0, \bar{f})$.

Definition 2.3. A monounary algebra $\mathcal{A} = (A, f)$ is *acyclic*, if $f^n a = a$ for some $a \in A$ and positive $n \in \mathbb{N}_+$ implies that $fa = a$ (that is, there are no cycles except loops). A monounary algebra $\mathcal{A}$ is called *connected* if for any $x, y \in A$ there exist some $n, m \in \mathbb{N}$ such that $f^n x = f^m x$ holds (see, for example, Jakubíková-Studenovská and Pócs [5]). From the definition it follows easily that for a monounary algebra $\mathcal{A} = (A, f)$ the following conditions are equivalent.

(A) $\forall x, y \in A\ \exists\, n \in \mathbb{N} : f^n x = f^n y$.

(B) $\mathcal{A}$ is acyclic, connected and has a fixed point.

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In particular, (A) implies that there is a unique fixed point (denoted by $\top_{\mathcal{A}}$ or just $\top$). We call an algebra satisfying (A) also a *rooted algebra* since the unique fixed point is the 'root' of the graph of f which is a rooted directed tree. In rooted algebras the following definition makes sense: the *depth* of an element $a \in A \setminus \{\top\}$ is the natural number $n \in \mathbb{N}$ such that $f^{n-1}(a) \neq f^n(a) = \top$, and will be denoted by $d(a)$. Moreover, we put $d(\top) = 0$. The *branching depth* of a rooted monounary algebra is defined by $\mathrm{bd}(\mathcal{A}) := \max\{d(a) \mid a \in A, |[a]_{\ker f}| \geq 2\}$ if this maximum exists, otherwise $\mathrm{bd}(\mathcal{A}) = \infty$.

Remark 2.4. (a) Let $\mathcal{A} = (A, f)$ be a monounary algebra. We denote by $q(a, b)$ the least compatible quasiorder of $\mathcal{A}$ containing the pair (a, b), that is, it is generated by (a, b). It is easy to verify that

$$q(a, b) = \Delta \cup \{(f^n a, f^n b) \mid n \in \mathbb{N}\}^{\mathrm{tra}},$$

and in particular that

$$q(a, b) = \Delta \cup \{(a, b)\} \text{ if } (a, b) \in \ker f.$$

(b) If $\mathcal{A}$ is acyclic, then $q(a, b)$ is known to be a partial order (see Radeleczki and Szigeti [7]). Further, if $\mathcal{A}$ is rooted with the fixed point $\top$, then for arbitrary $a, b \in A$,

$$q(a, \top) = \Delta \cup \{(f^n a, \top) \mid n \in \mathbb{N}\},$$
$$q(\top, b) = \Delta \cup \{(\top, f^n b) \mid n \in \mathbb{N}\}.$$

(c) Because $(f^k a, f^k b) \in \ker f \setminus \Delta$ for $k = \max\{d(a), d(b)\} - 1$, in view of (a) it immediately follows that a quasiorder $q \in \operatorname{Quord} \mathcal{A}$ of a rooted monounary algebra $\mathcal{A} = (A, f)$ is an atom in $\operatorname{Quord} \mathcal{A}$ if and only if $q = q(a, b)$ for some $(a, b) \in \ker f \setminus \Delta$.

Definition 2.5. Let A be a set. For $\theta \in \operatorname{Equ}(A)$, $\theta \subseteq \gamma \in \operatorname{Quord}(A)$ and $\gamma' \in \operatorname{Quord}(A/\theta)$ define

$$\iota_\theta(\gamma) := \gamma/\theta = \{([x]_\theta, [y]_\theta) \mid (x, y) \in \gamma\},$$
$$\lambda_\theta(\gamma') := \{(x, y) \in A^2 \mid ([x]_\theta, [y]_\theta) \in \gamma'\}.$$

From these definitions it easily follows that $\theta \subseteq \lambda_\theta(\gamma')$, $\lambda_\theta(\gamma/\theta) = \gamma$ and $\lambda_\theta(\gamma')/\theta = \gamma'$, that is, $\iota_\theta : [\theta\rangle_{\operatorname{Quord}(A)} \to \operatorname{Quord}(A/\theta)$ is a bijection, and λ_θ is the inverse. Moreover, these mappings are inclusion-preserving and therefore lattice isomorphisms.

Recall that an element r of a lattice $(L, \leq)$ is $\wedge$-*irreducible* if and only if $r = x_1 \wedge x_2$ implies $r = x_1$ or $r = x_2$ for $x_1, x_2 \in L$. In a complete lattice, r is *completely* $\wedge$-*irreducible* if and only if $r = \bigwedge X$ implies $r \in X$ for $X \subseteq L$ (this is equivalent to $r < \bigwedge\{x \in L \mid r < x\}$). The *(completely)* $\vee$-*irreducible elements* can be defined dually.

The following proposition will be a helpful tool for investigating $\wedge$-irreducible quasiorders of monounary algebras. Since compatibility of quasiorders depends only on the unary polynomial operations of an algebra, we formulate the proposition for arbitrary algebras but need to prove it only for unary ones.

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Proposition 2.6. *Let $\mathcal{A}$ be an arbitrary algebra.*

(a) *Let $\theta \in \operatorname{Con} \mathcal{A}$. Then $\iota_\theta : [\theta\rangle_{\operatorname{Quord} \mathcal{A}} \to \operatorname{Quord}(\mathcal{A}/\theta)$ is a lattice isomorphism.*

(b) *$q \in \operatorname{Quord} \mathcal{A}$ is (completely) $\wedge$-irreducible in $\operatorname{Quord} \mathcal{A}$ if and only if $q/q_0 \in \operatorname{Pord}(\mathcal{A}/q_0)$ is (completely) $\wedge$-irreducible in $\operatorname{Quord}(\mathcal{A}/q_0)$.*

Proof. Because of the properties mentioned after Definition 2.5, to verify (a), it remains to prove that the mappings ι_θ and λ_θ preserve compatibility. Let $f : A \to A$ be a (without loss of generality unary) operation of $\mathcal{A}$; the corresponding operation in the factor algebra $\mathcal{A}/\theta$ will be denoted here by $\bar{f}$. Let $\theta \subseteq q \in \operatorname{Quord} \mathcal{A}$ and $([x]_\theta, [y]_\theta) \in \iota_\theta(q) = q/\theta$. Then $(x, y) \in q$ and, by compatibility, $(fx, fy) \in q$, and therefore $(\bar{f}[x]_\theta, \bar{f}[y]_\theta) = ([fx]_\theta, [fy]_\theta) \in q/\theta$. Similarly, for $\gamma' \in \operatorname{Quord}(\mathcal{A}/\theta)$ and $(x, y) \in \lambda_\theta(\gamma')$ we get $([x]_\theta, [y]_\theta) \in \gamma'$, thus $([fx]_\theta, [fy]_\theta) = (\bar{f}[x]_\theta, \bar{f}[y]_\theta) \in \gamma'$ which gives $(fx, fy) \in \lambda_\theta(\gamma')$, showing that $\lambda_\theta(\gamma') \in \operatorname{Quord} \mathcal{A}$.

(b) immediately follows from (a) with $\theta := q_0 \in \operatorname{Con} \mathcal{A}$. □

Now we are also able to characterize the dual atoms.

Corollary 2.7. *Let $\mathcal{A}$ be a rooted monounary algebra with at least two elements. A quasiorder q is a dual atom in $\operatorname{Quord} \mathcal{A}$ if and only if $q \neq q^{-1}$ and $q_0 = q \cap q^{-1}$ has exactly two equivalence classes.*

Proof. By Proposition 2.6, a quasiorder q is a dual atom in $\operatorname{Quord} \mathcal{A}$ if and only if the corresponding partial order $r = q/q_0$ is a dual atom in $\operatorname{Quord}(\mathcal{B})$ for $\mathcal{B} = \mathcal{A}/q_0$. We shall see in Proposition 3.4 that an algebra must be of binary type if there exists a compatible partial order which is $\wedge$-irreducible in the quasiorder lattice. Since a dual atom r is $\wedge$-irreducible (in $\operatorname{Quord} \mathcal{B}$), it follows that $\mathcal{B}$ is of binary type. Thus the kernel class $[\top]_{\ker f}$ consists of exactly two elements and $\tilde{r} = r \cup \{(u, \top), (\top, u)\}$ is a larger nontrivial compatible quasiorder containing r. If $\mathcal{B}$ has more than two elements, then $\tilde{r}$ is nontrivial, a contradiction because r is a dual atom. Hence r is a linear order on a two-element rooted monounary algebra. □

We mention here an interesting connection with subdirectly irreducible partially ordered algebraic structures (ordered algebras in the sense of Czédli and Lenkehegyi [1]). It is a direct consequence of [1, Theorem 3.1] that, for an algebra $\mathcal{A} = (A, F^{(\mathcal{A})})$, a compatible partial order $r \in \operatorname{Pord} \mathcal{A}$ is completely $\wedge$-irreducible if and only if the partially ordered structure $(A, F^{(\mathcal{A})}, r)$ is subdirectly irreducible. Therefore, from Proposition 2.6(b) we immediately obtain the following.

Corollary 2.8. *Let $\mathcal{A} = (A, F^{(\mathcal{A})})$ be an algebra and let $q \in \operatorname{Quord} \mathcal{A}$. Then q is completely $\wedge$-irreducible if and only if the partially ordered factor algebra $(A/q_0, F^{(\mathcal{A}/q_0)}, q/q_0)$ is a subdirectly irreducible partially ordered structure.*

3. The irreducible quasiorders

As already mentioned in the introduction, completely $\vee$- and $\wedge$-irreducible quasiorders completely determine the quasiorder lattice $\operatorname{Quord} \mathcal{A}$. The $\vee$-irreducibles

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cause no problems, so we start with them; the remainder of the paper is devoted to the characterization of the $\wedge$-irreducible quasiorders. It is clear that each completely $\vee$-irreducible quasiorder in Quord $\mathcal{A}$ is of the form $q(a,b)$ (see the first formula in 2.4(a)) for some $a, b \in A$, $a \neq b$. The following proposition shows the converse.

PROPOSITION 3.1. *For a rooted monounary algebra $\mathcal{A} = (A, f)$, each $q(a,b)$ with $(a,b) \in A^2 \setminus \Delta$ is completely $\vee$-irreducible in the lattice* Quord $\mathcal{A}$. *Moreover, $\langle q(a,b)]_{\mathrm{Quord}\,\mathcal{A}}$ is finite.*

PROOF. Let $(a,b) \in A^2 \setminus \Delta$. Since the graph of f is a rooted tree (see Definition 2.3), there exist a least m and a least n such that $f^m a = f^n b$ (observe that then $f^{m'} a = f^{n'} b$ implies in particular that $n \leq n'$). Without loss of generality, we can assume that $m \geq n$ (otherwise consider $q(b,a) = (q(a,b))^{-1}$). Note that $m = 0$ is excluded, otherwise also $n = 0$ and $a = f^m a = f^n b = b$ in contradiction to $(a,b) \notin \Delta$.

Let $q^* := q(a,b) \setminus \{(a,b)\}$. Obviously, $p \leq q^*$ for each $p \in \mathrm{Quord}\,\mathcal{A}$ with $p < q(a,b)$ (since the latter implies that $(a,b) \notin p$). We shall prove that q^* is transitive, therefore it is a quasiorder and consequently $q(a,b)$ is completely $\vee$-irreducible.

First we show that

$$\begin{aligned} q(a,b) = \Delta &\cup \{(a,b)\} \cup \{(f^j a, f^j b) \mid 1 \leq j < n\} \\ &\cup \{(f^i a, f^{i+k(m-n)} a) \mid n \leq i \in \mathbb{N}, k \in \mathbb{N}_+\}. \end{aligned} \tag{3.1}$$

Denote the right-hand side of this equation by r. Obviously, since a is of finite depth, r contains only finitely many pairs, thus the last statement in Proposition 3.1 follows from (3.1). Now, it is easy to see that r is reflexive and compatible. Moreover, r is contained in $q(a,b)$: in fact, for $n \leq i$,

$$(f^i a, f^{i+(m-n)} a) = (f^i a, f^{i-n}(f^m a)) = (f^i a, f^{i-n}(f^n b)) = (f^i a, f^i b) \in q(a,b)$$

and hence inductively by transitivity of $q(a,b)$ we have $(f^i a, f^{i+k(m-n)} a) \in q(a,b)$ for all $k \in \mathbb{N}_+$. Note that r contains (a,b). Thus, to show the equality $r = q(a,b)$, it suffices to prove that r is also a quasiorder, that is, r is transitive. Assume that $(u,v), (v,w) \in r \setminus \Delta$. We shall prove that $(u,w) \in r$. Observe that $f^j b = v = f^i a$ is possible only if $j \geq n$. Therefore the only nontrivial case which must be considered is

$$(u,v) = (f^i a, f^{i+k(m-n)} a) \quad \text{and} \quad (v,w) = (f^j a, f^{j+k'(m-n)} a)$$

for some $i, j \geq n$. If $v = f^j a = \top$ then $w = f^{k(m-n)}(f^j a) = \top$ in contradiction to $(v,w) \notin \Delta$. Thus $f^{i+k(m-n)} a = v = f^j a \neq \top$ implies that $j = i + k(m-n)$, that is, $(u,w) = (f^i a, f^{i+k''(m-n)} a) \in r$ where $k'' = k + k'$. The transitivity of r is proved, and, as mentioned above, this proves (3.1).

In order to proceed further, notice that $a \neq \top$, because $a = \top$ would imply $m = 0$ (by minimality of m) which is excluded. Moreover, if $b = \top$ then from the first formula in 2.4(b) we immediately conclude that $q^* = q(a,\top) \setminus \{(a,\top)\} = q(fa,\top)$ is a quasiorder and, as mentioned above, $q(a,b)$ is completely $\vee$-irreducible. Thus we can also assume that $b \neq \top$.

Now we want to derive from (3.1) a description of $q^* = r \setminus \{(a, b)\}$. For this we notice first that (a, b) cannot belong to $\{(f^j a, f^j b) \mid 1 \le j < n\}$ (because $f^j b = b$ would imply $b = \top$). Next we look at which pairs of r of the form $(f^i a, f^{i+k(m-n)} a)$, with $n \le i \in \mathbb{N}$, $k \in \mathbb{N}_+$, may be equal to (a, b) (see Equation (3.1)).

Claim: $(f^i a, f^{i+k(m-n)} a) = (a, b) \iff (n, i, k) = (0, 0, 1)$.

If $(n, i, k) = (0, 0, 1)$, then clearly

$$(f^i a, f^{i+k(m-n)} a) = (f^0 a, f^m a) = (a, f^n b) = (a, b).$$

Conversely, let $(f^i a, f^{i+k(m-n)} a) = (a, b)$. Then $f^i a = a$, which implies that $i = 0$ (because $a \neq \top$), consequently $n = 0$ since $n \le i$. For the second component of the pairs we have $f^{i+k(m-n)} a = b$, therefore

$$b = f^{km} a = f^{(k-1)m}(f^m a) = f^{(k-1)m}(f^n b) = f^{(k-1)m} b.$$

Since $b \neq \top$ we conclude that $(k-1)m = 0$ and hence $k = 1$ (since $m \neq 0$), and the claim is proved.

Using this claim, from the description (3.1) of $q(a, b)$,

$$\begin{aligned} q^* = {} & \Delta \cup \{(f^i a, f^i b) \mid 1 \le i < n\} \\ & \cup \{(f^i a, f^{i+k(m-n)} a) \mid n \le i \in \mathbb{N}, k \in \mathbb{N}_+, (n, i, k) \neq (0, 0, 1)\}. \end{aligned}$$

The above proof of transitivity of r shows that $q^* = r \setminus \{(a, b)\}$ is also transitive: in fact, $(u, v), (v, w) \in q^*$ implies that $(u, w) = (f^i a, f^{i+k''(m-n)} a) \in q^*$ because $k'' = k + k' \ge 2$ and thus $(n, i, k'') \neq (0, 0, 1)$. Since transitivity of q^* is proved, we are done. □

Now we start to consider (completely) $\wedge$-irreducible quasiorders. Proposition 2.6(b) shows that every $\wedge$-irreducible quasiorder can be obtained from a $\wedge$-irreducible partial order (in a suitable factor algebra). Thus, from now on, we restrict mainly to partial orders. Note that throughout the paper $\wedge$-irreducibility will be meant always in the *quasiorder* lattice Quord $\mathcal{A}$. Proposition 3.4 below shows that $\wedge$-irreducible partial orders can exist only under very special conditions on the algebra. In preparation we need the following lemma.

Lemma 3.2. *Let $\mathcal{A} = (A, f)$ be a monounary algebra such that there exists a $\wedge$-irreducible partial order $r \in$ Pord $\mathcal{A}$. Then each congruence class of the kernel of f is linearly ordered by r.*

Proof. Assume by way of contradiction that $(a, b) \notin r$, $(b, a) \notin r$ for some $(a, b) \in \ker f$. Then $\gamma_1 := r \vee q(a, b)$ and $\gamma_2 := r \vee q(b, a)$ are quasiorders in Quord $\mathcal{A}$ strictly containing r. We will show that $r = \gamma_1 \cap \gamma_2$, which contradicts the $\wedge$-irreducibility of r and which therefore finishes the proof.

In fact, if $(x, y) \in \gamma_1 \cap \gamma_2$, then we have chains

$$\begin{aligned} x = u_0 \xrightarrow{r} u_1 \xrightarrow{q(a,b)} u_2 \xrightarrow{r} \cdots \xrightarrow{q(a,b)} u_{n-1} \xrightarrow{r} u_n = y, \\ x = v_0 \xrightarrow{r} v_1 \xrightarrow{q(b,a)} v_2 \xrightarrow{r} \cdots \xrightarrow{q(b,a)} v_{m-1} \xrightarrow{r} v_m = y \end{aligned}$$

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(here we use the notation $v \xrightarrow{s} w$ for $(v, w) \in s$ where s is any binary relation). If $q(a, b)$ or $q(b, a)$ is not involved, then we immediately get $(x, y) \in r$. Otherwise we may assume $n \geq 3$, $m \geq 3$. Since $(a, b) \in \ker f$, by the second formula in 2.4(a) we get $x = u_0 \xrightarrow{r} u_1 = a = v_{m-1} \xrightarrow{r} v_m = y$, that is, we also have $(x, y) \in r$ by transitivity of r; thus $\gamma_1 \cap \gamma_2 = r$. □

Definition 3.3. We say that a monounary algebra $\mathcal{A} = (A, f)$ is of *binary type* if $|[a]_{\ker f}| \leq 2$ for all $a \in A$.

Proposition 3.4. *Let $\mathcal{A}$ be a monounary algebra. If there exists a $\wedge$-irreducible compatible partial order for $\mathcal{A}$ then $\mathcal{A}$ is of binary type.*

Proof. Let $r \in \operatorname{Pord} \mathcal{A}$ be $\wedge$-irreducible. Assume to the contrary that there is a kernel class with at least three elements. By Lemma 3.2 we find a three-element chain $a \xrightarrow{r} b \xrightarrow{r} c$ in $[a]_{\ker f}$. Then $\gamma_1 := r \vee q(b, a)$ and $\gamma_2 := r \vee q(c, b)$ are quasiorders in $\operatorname{Quord} \mathcal{A}$ strictly containing r (since (b, a), (c, b) does not belong to r, note that r is a partial order).

We proceed as in the proof of Lemma 3.2, taking $(x, y) \in \gamma_1 \cap \gamma_2$, obtaining chains

$$x = u_0 \xrightarrow{r} u_1 \xrightarrow{q(b,a)} u_2 \xrightarrow{r} \cdots \xrightarrow{q(b,a)} u_{n-1} \xrightarrow{r} u_n = y,$$
$$x = v_0 \xrightarrow{r} v_1 \xrightarrow{q(c,b)} v_2 \xrightarrow{r} \cdots \xrightarrow{q(c,b)} v_{m-1} \xrightarrow{r} v_m = y,$$

where we can assume $n \geq 3$, $m \geq 3$, and thus getting

$$x = u_0 \xrightarrow{r} u_1 = b = v_{m-1} \xrightarrow{r} v_m = y,$$

that is, $(x, y) \in r$. This shows that $r = \gamma_1 \cap \gamma_2$ which contradicts the $\wedge$-irreducibility of r. □

Remark 3.5.

(i) The converse statement, namely that a monounary algebra $\mathcal{A} = (A, f)$ of binary type has a completely $\wedge$-irreducible compatible partial order, is not true in general. There exist even finite monounary algebras of binary type which admit no $\wedge$-irreducible partial order (for example, a three-element connected monounary algebra with a cycle of length two). However, in Proposition 5.5 we shall see that a *rooted* monounary algebra $\mathcal{A} = (A, f)$ of binary type has a completely $\wedge$-irreducible compatible partial order if and only if its branched depth (see Definition 2.3) is finite. In particular, this holds for all *finite* rooted monounary algebras.

(ii) In view of [1] (as mentioned before Corollary 2.8), from Proposition 3.4 it immediately follows that if, for a rooted monounary algebra $\mathcal{A} = (A, f)$ and $r \in \operatorname{Pord} \mathcal{A}$, the ordered algebra (A, f, r) is subdirectly irreducible, then $\mathcal{A}$ must be of binary type, consequently it has at most countable cardinality.

Lemma 3.6. *Let $\mathcal{A} = (A, f)$ be a rooted monounary algebra of binary type and let* $r \in \operatorname{Pord} \mathcal{A}$.

(a) *If r is $\wedge$-irreducible, then the fixed point $\top$ is either the greatest or the least element with respect to r.*

(b) *Let $\top$ be the greatest element, $(x, y) \in r$ and $u \in A \setminus \{\top\}$ such that $f(u) = \top$. Then $d(x) \geq d(y)$, and u is maximal (with respect to r) in $A \setminus \{\top\}$.*

(c) *If $\top$ is the greatest element, then $\{a \in A \mid d(a) \leq k\}$ is finite for each $k \in \mathbb{N}$; in particular, each filter $[m\rangle_r$ is finite ($m \in A$).*

Proof. To prove (a), we assume to the contrary that there exist $a, b \in A$ such that $(a, \top) \notin r$ and $(\top, b) \notin r$. Analogously to the previous proofs, consider $\gamma_1 := r \vee q(a, \top)$ and $\gamma_2 := r \vee (\top, b)$ which strictly contain r. We will show that $r = \gamma_1 \cap \gamma_2$, contradicting the $\wedge$-irreducibility of r and hence finishing the proof.

Thus let $(x, y) \in \gamma_1 \cap \gamma_2$. Consequently, we have chains

$$x = u_0 \xrightarrow{r} u_1 \xrightarrow{q(a,\top)} u_2 \xrightarrow{r} \cdots \xrightarrow{q(a,\top)} u_{n-1} \xrightarrow{r} u_n = y,$$
$$x = v_0 \xrightarrow{r} v_1 \xrightarrow{q(\top,b)} v_2 \xrightarrow{r} \cdots \xrightarrow{q(\top,b)} v_{m-1} \xrightarrow{r} v_m = y.$$

From the formulas in 2.4(b) we conclude that $x = v_0 \xrightarrow{r} v_1 = \top = u_{n-1} \xrightarrow{r} u_n = y$, that is, $(x, y) \in r$, and we are done.

We prove (b). For $(x, y) \in r$ we have $(f^{d(x)}x, f^{d(x)}y) = (\top, f^{d(x)}y) \in r$, and consequently $f^{d(x)}y = \top$ because $\top$ is the greatest element, that is, $d(y) \leq d(x)$. Now let $(u, u') \in r$. Then $(\top, fu') = (fu, fu') \in r$. By assumption $(fu', \top) \in r$, consequently $fu' = \top$. Since $\mathcal{A}$ is of binary type we get $u' = u$ or $u' = \top$.

Let us show (c). Because $\mathcal{A}$ is of binary type there are only finitely many elements of a fixed depth k (namely $\leq 2^{k-1}$). Thus, by (b), there are only finitely many elements of bounded depth. □

Definition 3.7. Let $r \in \operatorname{Pord}(A, f)$. A pair $(a, b) \in A^2$ is called *critical for r in (A, f)* (or *r-critical* for short) if:

(1) $(fa, fb) \in r$;

(2) a is a minimal element (with respect to r) in $A \setminus \langle b]_r$;

(3) b is a maximal element (with respect to r) in $A \setminus [a\rangle_r$.

Remark. The definition implies that $(a, b) \notin r$ because otherwise $(a, b) \in r$ would imply that $a \in \langle b]_r$. Thus, instead of (1), we can also write $(a, b) \in f^{-1}(r) \setminus r$. A pair $(a, b) \notin r$ satisfying (2) and (3) is called a *critical pair of* (A, r); see Trotter [9].

Lemma 3.8. *Let $r \in \operatorname{Pord} \mathcal{A}$, $q \in \operatorname{Quord} \mathcal{A}$ and $r \prec q$ in $\operatorname{Quord} \mathcal{A}$ for some monounary algebra $\mathcal{A} = (A, f)$. Then there exists an r-critical pair (a, b) such that $q = r \cup \{(a, b)\}$.*

Proof. By [6, Proposition 5.3] (the proof of this proposition is formulated for finite algebras in [6], but it does not require finiteness and therefore holds also for arbitrary rooted monounary algebras), $r \prec q$ in $\operatorname{Quord} \mathcal{A}$ implies $r \prec q$ also in $\operatorname{Quord}(A)$

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and $\widetilde{f}(q) \subseteq r \prec q \subseteq f^{-1}(\widetilde{f}(q))$ where

$$\widetilde{f}(q) := \Delta \cup \{(fx, fy) \mid (x, y) \in q\}^{\mathrm{tra}}.$$

Due to a result of Erné and Reinhold, [2, Lemma 1.4], we have $r \prec q$ in $\mathrm{Quord}(A)$ if and only if there exists $(a, b) \in A^2 \setminus r$ such that $q = r \cup \{(a, b)\}$ where a is minimal in $\langle b]_r$ and b is maximal in $A \setminus [a\rangle_r$. It remains to show that $(fa, fb) \in r$: clearly $(a, b) \in f^{-1}(\widetilde{f}(q))$ yields $(fa, fb) \in \widetilde{f}(q) \subseteq q$. Thus (a, b) is r-critical. □

Proposition 3.9. *Let $\mathcal{A} = (A, f)$ be a rooted monounary algebra and let $r \in \mathrm{Pord}(A, f)$. Then the following are equivalent.*

(1) *r is completely $\wedge$-irreducible in $\mathrm{Quord}(A, f)$.*

(2) *There exists an r-critical pair (a, b) with*

$$\forall v, w \in A : (v, w) \notin r, (fv, fw) \in r \implies (a, v) \in r, (w, b) \in r. \tag{C}$$

Moreover, if (2) is satisfied, then (a, b) is the unique r-critical pair.

Note that condition (C) can also be written in the more condensed form,

$$f^{-1}(r) \setminus r \subseteq [a\rangle_r \times \langle b]_r. \tag{C$'$}$$

Proof. We show that (1) implies (2). Since r is completely $\wedge$-irreducible, for

$$r^* := \bigwedge \{q \in \mathrm{Quord}\,\mathcal{A} \mid r < q\}$$

we have $r \prec r^*$. According to Lemma 3.8 there exists an r-critical pair (a, b) such that $r^* = r \cup \{(a, b)\}$. To prove the required properties, take $(v, w) \in f^{-1}(r) \setminus r$. Then $(fv, fw) \in r$ and it is easy to check that $q_r := r \cup \langle v]_r \times [w\rangle_r$ is a quasiorder. Moreover, q_r is compatible. In fact, let $(x, y) \in q_r$. If $(x, y) \in r$ then $(fx, fy) \in r \subseteq q_r$. Otherwise, if $(x, y) \in \langle v]_r \times [w\rangle_r$, then $x \xrightarrow{r} v$, $w \xrightarrow{r} y$ and hence $fx \xrightarrow{r} fv \xrightarrow{r} fw \xrightarrow{r} fy$, consequently $(fx, fy) \in r$. Thus $q_r \in \mathrm{Quord}\,\mathcal{A}$. Since $(v, w) \notin r$, it follows that $r < q_r$, thus $r^* \subseteq q_r$. Therefore, $(a, b) \in q_r \setminus r$ implies $(a, b) \in \langle v]_r \times [w\rangle_r$, that is, $(a, v), (w, b) \in r$.

Now we show that (2) implies (1). Let (a, b) be r-critical with (C$'$). Then $(a, b) \notin r$. We show that $(a, b) \in q$ for every $q \in \mathrm{Quord}\,\mathcal{A}$ with $r \subset q$, which implies that r is completely $\wedge$-irreducible. Thus let $r < q$. Then there exists $(x, y) \in q \setminus r$. We can assume that $(fx, fy) \in r$ (otherwise choose $(f^n x, f^n y) \in q \setminus r$ satisfying $(ff^n x, ff^n y) \in r$; such an n exists because of condition (A) of Definition 2.3). Consequently, by (C$'$), we have $(x, y) \in f^{-1}(r) \setminus r \subseteq [a\rangle_r \times \langle b]_r$, that is, $(a, x), (y, b) \in r \subseteq q$. Hence, by transitivity of q and $(x, y) \in q$, we have $(a, b) \in q$.

To prove the last statement in Proposition 3.9, namely that (a, b) is unique, let (c, d) be another r-critical pair, $(c, d) \neq (a, b)$. We can assume that $c \neq a$ (the case $d \neq b$ can be treated analogously using condition (3) of Definition 3.7). Now $(c, d) \in f^{-1}(r) \setminus r \subseteq [a\rangle_r \times \langle b]_r$ shows that $(a, c) \in r$ and $(d, b) \in r$. Since c is minimal in $A \setminus \langle d]_r$ (see condition (2) of Definition 3.7) and $a \neq c$, it follows that $a \in \langle d]_r$. Hence, $(a, d) \in r$, and by transitivity $(a, b) \in r$, a contradiction. □

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We mention that, if (a, b) is the unique r-critical pair, then, in view of Lemma 3.8, $r^* = r \cup \{(a, b)\}$ is the unique upper cover of r. Consequently, for *finite* algebras, r is (completely) $\wedge$-irreducible (that is, condition (1) of Proposition 3.9 is satisfied).

Theorem 3.10. *Let $\mathcal{A} = (A, f)$ be a rooted algebra of binary type and let $[\top]_{\ker f} = \{u, \top\}$. Then the following conditions are equivalent for a partial order $r \in \operatorname{Pord} \mathcal{A}$.*

(I) *r is completely $\wedge$-irreducible with $(u, \top) \in r$.*

(II) $\exists\, m \in A \setminus \langle u]_r : f^{-1}(r) \setminus r \subseteq [m\rangle_r \times \langle u]_r$.

Proof. We show that (I) implies (II). Since r is completely $\wedge$-irreducible, by Proposition 3.9 there exists an r-critical pair (a, b) satisfying condition (C). For any $x \in A \setminus \langle u]_r$ we have $(x, u) \notin r$ and $(fx, fu) = (fx, \top) \in r$. Therefore, by condition (C), we get $(a, x) \in r$ and $(u, b) \in r$. Thus $(fu, fb) \in r$ since r is compatible, and therefore $(\top, fb) = (fu, fb) \in r$, which implies $fb = \top$ and hence $b = \top$ or $b = u$. Since $(a, b) \notin r$ and $\top$ is the greatest element (by Lemma 3.6), the case $b = \top$ is excluded and therefore $b = u$ and, for $m := a$, $(m, u) = (a, b)$ is an r-critical pair. It satisfies condition (C) as assumed above; moreover, we have $m \in A \setminus \langle u]_r$ because of $(m, u) \notin r$. Thus (II) is proved.

Now we show that (II) implies (I). We have $(u, \top) \in r$ because, by (II), $(u, \top) \notin r$ together with $(fu, f\top) = (\top, \top) \in r$ would imply that $u \in [m\rangle_r$, contradicting $m \notin \langle u]_r$. We show further that $(a, \top) \in r$ for all $a \in A$. Assume that $(a, \top) \notin r$. Then there exists a least $i \in \mathbb{N}$ such that $(f^i a, f^i \top) = (f^i a, \top) \notin r$ but $(f^{i+1} a, \top) \in r$. By (II) this implies $\top \in \langle u]_r$ in contradiction to $(u, \top) \in r$.

We now prove that (m, u) is an r-critical pair. Then condition (C′) is satisfied for $(a, b) = (m, u)$ because of (II), and r has to be completely $\wedge$-irreducible by Proposition 3.9, and the proof is finished.

Thus it remains to check conditions(1)–(3) of Definition 3.7 for the pair (m, u). (1) is satisfied since, as proved above, $(fm, fu) = (fm, \top) \in r$. Concerning (2) we prove even more: m is the least element in $A \setminus \langle u]_r$. In fact, let $c \notin \langle u]_r$, that is, $(c, u) \notin r$. Because of $(fc, fu) = (fc, \top) \in r$ we have $(c, u) \in f^{-1}(r) \setminus r$ and conclude from (II) that $c \in [m\rangle_r$, that is, $(m, c) \in r$. Thus m is the least element in $A \setminus \langle u]_r$ and therefore $A = \langle u]_r \cup [m\rangle_r$ is a partition of A, whence $A \setminus [m\rangle_r = \langle u]_r$. But the latter implies that u is the greatest element in $A \setminus [m\rangle_r$ and therefore also (3) is fulfilled. □

Definition 3.11.

(i) Let $\mathcal{A} = (A, f)$ be a rooted algebra. A pair (C_0, C_1) of nonempty subsets of A is called a *$\star$-partition* of $\mathcal{A}$ if it is a partition of A (that is, $A = C_0 \cup C_1$ and $C_0 \cap C_1 = \emptyset$) such that the unique fixed point $\top$ of f belongs to C_1 and for all $a \in A$ we have $|[a]_{\ker f} \cap C_0| \leq 1$ and $|[a]_{\ker f} \cap C_1| \leq 1$, consequently $|[a]_{\ker f}| \leq 2$, that is, $\mathcal{A}$ is of binary type. Thus in each kernel class there is at most one element of each class C_1, C_2, in particular $\top \in C_1$, and therefore, for $[\top]_{\ker f} = \{\top, u\}$, we have $u \in C_0$ (provided that $\mathcal{A}$ has at least two elements).

(ii) Two algebras (A, f) and (A', f') with $\star$-partitions (C_0, C_1) and (C'_0, C'_1), respectively, are called *$\star$-isomorphic* if there exists an isomorphism $\varphi : (A, f) \to (A', f')$ which maps C_0 onto C'_0, and C_1 onto C'_1.

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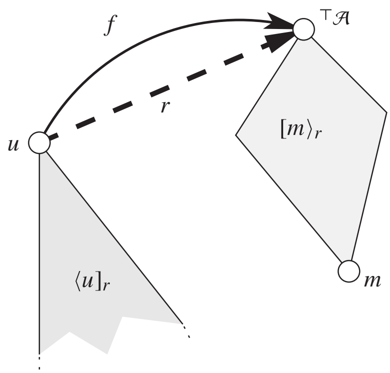

FIGURE 1. A $\wedge$-irreducible $r \in \operatorname{Pord}(A, f)$ with its r-critical pair (m, u) and corresponding $\star$-partition $(\langle u]_r, [m\rangle_r)$ of (A, f).

REMARK 3.12. Let $\mathcal{A} = (A, f)$ be rooted of binary type and let $r \in \operatorname{Pord} \mathcal{A}$ be completely $\wedge$-irreducible with $[\top]_{\ker f} = \{u, \top\}$, $(u, \top) \in r$ and m as in Theorem 3.10. From Proposition 3.9 and the proof of Theorem 3.10 we conclude the following.

(a) (u, m) is an r-critical pair, m is the least element in $A \setminus \langle u]_r$, u is the greatest element in $A \setminus [m\rangle_r$ and $A = \langle u]_r \cup [m\rangle_r$ is a disjoint union; see Figure 1.

(b) $(\langle u]_r, [m\rangle_r)$ is a $\star$-partition (in the sense of Definition 3.11), which we shall call the *$\star$-partition of $\mathcal{A}$ corresponding to r*.

In fact, clearly $\top \in [m\rangle_r$, and for $(a, b) \in \ker f \setminus \Delta$ we have either $(a, b) \notin r$ or $(b, a) \notin r$ (note that r is a partial order). If $(a, b) \notin r$ then by condition (II) of Theorem 3.10 we get $a \in [m\rangle_r$ and $b \in \langle u]_r$ since $(fa, fb) \in \Delta \subseteq r$. Analogously for $(b, a) \notin r$.

(c) We have $([m\rangle_r \times \langle u]_r) \cap r = \emptyset$ because $(m, u) \notin r$ (that is, there are no r-edges from the right part to the left part in Figure 1).

(d) The filter $[m\rangle_r$ is finite because of Lemma 3.6(c). If a kernel class $[a]_{\ker f}$ contains two elements then one of them must belong to $[m\rangle_r$ (due to the properties of a $\star$-partition; see Definition 3.11), thus for the branched depth (Definition 2.3) we get $\operatorname{bd}(\mathcal{A}) \le d_{\mathcal{A}}(m)$.

4. The d-dimensional Boolean cubes and their limit as monounary algebras

First we introduce several notions and notations. Let $\mathbf{2} := \{0, 1\}$ and consider the d-dimensional Boolean cube $\mathbf{2}^d := \{0, 1\}^d$ with $d \in \mathbb{N}_+$. We write the elements $w \in \mathbf{2}^d$ as words (of length d), and the components (letters) often will be denoted by w_i, that is, $w = w_1 \ldots w_d := (w_1, \ldots, w_d)$, $(w_i \in \{0, 1\}, i \in \{1, \ldots, d\})$. Further, consider the set $\mathbf{2}^{[\infty]}$ of all infinite strings $v = v_1 v_2 v_3 \ldots$ $(v_i \in \mathbf{2})$ with finitely many 0s. The concatenation of a word w and a word or a string v will be denoted as usual by wv. Further, w^n stands for $w \ldots w$ (n times, or for the empty word if $n = 0$).

Observe that all cubes $\mathbf{2}^d$ naturally embed into $\mathbf{2}^{[\infty]}$ via $w \mapsto w1^\infty$. Here 1^∞ denotes the string consisting of 1s only $(\forall i \in \mathbb{N}_+ : v_i = 1)$.

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We define the monounary algebra $\mathcal{B}_\infty := (\mathbf{2}^{[\infty]}, f_\infty)$ by

$$f_\infty(x_1x_2x_3\ldots) := x_2x_3\ldots$$

(this 'tail operation' just deletes the first component). The restriction of f_∞ to the d-dimensional Boolean cube gives the finite monounary algebra

$$\mathcal{B}_d := (\mathbf{2}^d, f_d), \quad \text{with } f_d(x_1x_2\ldots x_d) := x_2\ldots x_d\,1,$$

and the above natural embedding $\mathcal{B}_d \to \mathcal{B}_\infty$ is an injective homomorphism.

Clearly, $\mathcal{B}_\infty$ as well as $\mathcal{B}_d$ is a rooted algebra of binary tree form: for $x_1 \in \mathbf{2}$, $v \in \mathbf{2}^{[\infty]}$, we have $[x_1v]_{\ker f_\infty} = \{0v, 1v\}$, analogously $[x_1v]_{\ker f_d} = \{0v, 1v\}$ for $v \in \mathbf{2}^{d-1}$. Moreover, $\top = 1^\infty$ (respectively, $\top = 1^d$) is the unique fixed point of f_∞ (respectively, f_d) and the kernel class $[\top]_{\ker f_\infty}$ (respectively, $[\top]_{\ker f_d}$) is $\{\varepsilon, \top\}$ where $\varepsilon := 01^\infty$ (respectively, $\varepsilon := 01^{d-1}$). Let K_0 be the set of all strings $v \in \mathcal{B}_\infty$ (respectively, words in $\mathcal{B}_d$) with first letter $v_1 = 0$. Analogously K_1 denotes the set of all strings starting with $v_1 = 1$. Then (K_0, K_1) is a $\star$-partition (in the sense of Definition 3.11) of $\mathcal{B}_\infty$ as well as of $\mathcal{B}_d$.

Further, we consider the usual componentwise order $\leq$, denoted also by p_∞ on $\mathbf{2}^{[\infty]}$, and p_d on $\mathbf{2}^d$, respectively, given by

$$x_1x_2\ldots \leq y_1y_2\ldots :\iff \forall i \in \mathbb{N}_+ (\text{respectively, } \forall i \in \{1,\ldots,d\}) : x_i \leq y_i.$$

Note that $v \leq w$ implies $f_\infty v \leq f_\infty w$, thus p_∞ is compatible, that is, $p_\infty \in \operatorname{Pord}\mathcal{B}_\infty$ (and analogously $p_d \in \operatorname{Pord}\mathcal{B}_d$). For a subalgebra $\mathcal{B}$ of $\mathcal{B}_\infty$ (respectively, $\mathcal{B}_d$), the restriction of $\leq$ to the carrier set B of $\mathcal{B}$ will be denoted also by $p_B := \leq \cap (B \times B)$.

For any nonempty subalgebra $\mathcal{B}$ of $\mathcal{B}_\infty$ (respectively, $\mathcal{B}_d$) there is a canonical $\star$-partition, namely $(B \cap K_0, B \cap K_1)$, which we shall always use when subalgebras of $\mathcal{B}_\infty$ (respectively, $\mathcal{B}_d$) are considered. Note that $K_0 = \langle\varepsilon]_{p_\infty}$ and $K_1 = \bigcup_{d\in\mathbb{N}_+}[\mu_{(d)}\rangle_{p_\infty}$ with $\varepsilon = 01^\infty$ as above and $\mu_{(d)} := 10^{d-1}1^\infty$. For $d \in \mathbb{N}_+$, we use the notation $\varepsilon_d := 01^{d-1}$ and $\mu_d := 10^{d-1}$ (the indices may be omitted if d is clear from the context). These elements of $\mathcal{B}_d$ correspond to ε and $\mu_{(d)}$ in $\mathcal{B}_\infty$ via the natural embedding. Then $\mathcal{B}_d = K_0 \cup K_1 = \langle\varepsilon_d]_{p_d} \cup [\mu_d\rangle_{p_d}$.

As an example, $\mathcal{B}_3$ is shown in Figure 2 (with $\varepsilon := \varepsilon_3$, $\mu := \mu_3$). Part (a) emphasizes the usual cube order p_3 while part (b) reorders (a) with emphasis on the binary type (the graph of f_3 is drawn with bold arrows while the edges for the (Hasse) diagram of p_3 are given by dashed arrows; p_3 is the reflexive and transitive closure of these arrows).

Proposition 4.1. *Let $\mathcal{B} = (B, f_\infty\restriction_B)$ be a subalgebra of $\mathcal{B}_\infty$ such that $B \cap K_1$ has a least element μ with respect to p_∞. Then there exists exactly one completely $\wedge$-irreducible partial order $r \in \operatorname{Pord}\mathcal{B}$ with $\top$ as greatest element such that the corresponding $\star$-partition* (see Remark 3.12(b)) *equals $(B \cap K_0, B \cap K_1)$, namely $r = p_B = p_\infty \cap (B \times B)$. Moreover, (ε, μ) is a p_B-critical pair.*

Proof. In the proof we write f instead of f_∞, and, without loss of generality, we may assume that $|B| \geq 2$, thus $\varepsilon, \top \in B$. First we show that p_B is completely $\wedge$-irreducible.

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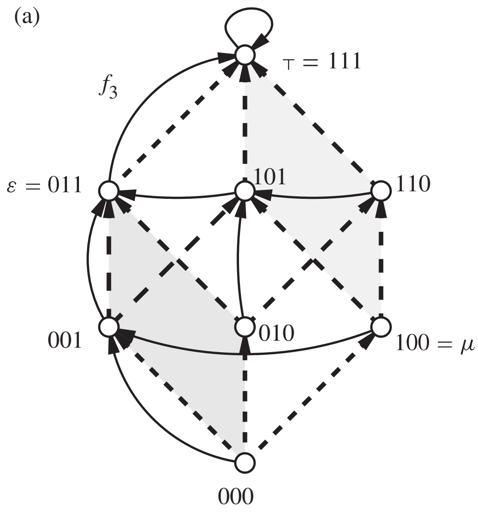

FIGURE 2. The Boolean cube $\mathcal{B}_3 = (\mathbf{2}^3, f_3)$ with $\wedge$-irreducible $p_3 \in \operatorname{Pord} \mathcal{B}_3$.

According to Theorem 3.10 it suffices to prove that condition (II) of Theorem 3.10 is satisfied for $u = \varepsilon = 01^\infty$ and $m = \mu$. Let $(a, b) \in f^{-1}(p_B) \setminus p_B$, that is, $a \nleq b$ but $fa \leq fb$ for $a, b \in \mathbf{2}^{[\infty]}$. By definition of f and $p_B = p_\infty \cap (B \times B)$, the second condition means that $a_i \leq b_i$ for $i \in \{2, 3, \ldots\}$ and therefore the first condition implies $a_1 > b_1$, that is, $a_1 = 1$ and $b_1 = 0$. Consequently, $a \in K_1$ and $b \in K_0$ and therefore $(a, b) \in [\mu\rangle_{p_B} \times \langle\varepsilon]_{p_B}$ which finishes the proof of condition (II) of Theorem 3.10. In view of Remark 3.12(a), (ε, μ) is a p_B-critical pair.

Now, let $r \in \operatorname{Pord} \mathcal{B}$ be completely $\wedge$-irreducible. According to Theorem 3.10 there exists an element $m \in B \setminus \langle\varepsilon]_r$ such that $f^{-1}(r) \setminus r \subseteq [m\rangle_r \times \langle\varepsilon]_r$. By our assumption for the $\star$-partition $(\langle\varepsilon]_r, [\mu\rangle_r)$ (see Remark 3.12(b)) we have $\langle\varepsilon]_r = B \cap K_0$, $[m\rangle_r = B \cap K_1$ and $\top$ is the greatest element of r. We are going to show that these conditions for r force r to be p_B.

First we show $r \subseteq p_B$. Assume that there exists a pair $(a, b) \in r \setminus p_B$. Recall that for $a = a_1 a_2 \ldots$ the first letter of $f^{i-1}a$ is a_i. Since $(a, b) \notin p_B$ there is some index $i \in \mathbb{N}_+$ with $1 = a_i > b_i = 0$, thus $(f^{i-1}a, f^{i-1}b) \in K_1 \times K_0$. On the other hand, f preserves r and therefore

$$(f^{i-1}a, f^{i-1}b) \in r \cap (K_1 \times K_0) = r \cap ((B \cap K_1) \times (B \cap K_0)) = r \cap ([m\rangle_r \times \langle\varepsilon]_r).$$

However, this is a contradiction because the latter intersection is empty according to Remark 3.12(c).

Now we show that $p_B \subseteq r$. Suppose that there exists a pair $(a, b) \in p_B \setminus r$. Without loss of generality we can assume that $(fa, fb) \in r$ (otherwise apply f several times until we get a pair with the required properties). Thus $(a, b) \in f^{-1}(r) \setminus r \subseteq [m\rangle_r \times \langle\varepsilon]_r$,

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that is, $a \in [m\rangle_r = B \cap K_1 = [\mu\rangle_{p_B}$ and $b \in \langle\varepsilon]_r = B \cap K_0 = \langle\varepsilon]_{p_B}$. Consequently, $(a, b) \in ([\mu\rangle_{p_B} \times \langle\varepsilon]_{p_B}) \cap p_B$. Again, this is a contradiction since this intersection is empty according to Remark 3.12(c) and the complete $\wedge$-irreducibility of p_B which was shown above. □

5. Characterization of completely ∧-irreducible quasiorders

From now on we explicitly assume that all algebras contain at least two elements. Moreover, we use all notations introduced in Section 4.

The next proposition shows how rooted algebras with a $\star$-partition (and therefore of binary type, see Definition 3.11) can be embedded into $\mathcal{B}_\infty$.

Proposition 5.1. *Let $\mathcal{A} = (A, f)$ be a rooted algebra and let (C_0, C_1) be a $\star$-partition. Then there exists a unique subalgebra $\mathcal{B} = (B, f_\infty \restriction_B)$ of $\mathcal{B}_\infty$ such that $\mathcal{B}$ with the canonical $\star$-partition $(B \cap K_0, B \cap K_1)$ is $\star$-isomorphic to $\mathcal{A}$.*

Proof. We recursively define a mapping $\varphi : \mathcal{A} \to \mathcal{B}_\infty$ as follows: $\varphi(\top_{\mathcal{A}}) := \top = 1^\infty$,

$$\varphi(a) := \begin{cases} 0\varphi(fa) & \text{if } a \in C_0, \\ 1\varphi(fa) & \text{if } a \in C_1. \end{cases}$$

Since for every $a \in A \setminus \{\top_{\mathcal{A}}\}$ there exists a least $n_a \in \mathbb{N}_+$ such that $f^{n_a} a = \top_{\mathcal{A}}$, the mapping φ is defined for every $a \in A$. By definition of f_∞ and φ we have $f_\infty(\varphi(a)) = \varphi(fa)$, that is, φ is a homomorphism and the image of φ is a subalgebra $\mathcal{B}$ of $\mathcal{B}_\infty$. Moreover, φ is injective: for $a \neq b$ there is a least $j \in \mathbb{N}_+$ with $f^j a = f^j b$. Therefore $f^{j-1}a$ and $f^{j-1}b$ belong to different classes of the $\star$-partition (C_0, C_1) and their φ-images differ in the first letter; consequently, by the above definition, the φ-images $\varphi(a)$ and $\varphi(b)$ must differ in the jth letter (from the left), that is, $\varphi(a) \neq \varphi(b)$. Finally, from the above definition of the mapping φ it follows that $a \in C_i \iff \varphi(a) \in K_i$ ($i \in \{0, 1\}$), that is, φ is a $\star$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

It remains to show that $\mathcal{B}$ is uniquely determined. In fact, every $\star$-isomorphism $\tilde{\varphi} : \mathcal{A} \to \tilde{\mathcal{B}}$ onto a subalgebra $\tilde{\mathcal{B}}$ of $\mathcal{B}_\infty$ with the canonical $\star$-partition $(\tilde{B} \cap K_0, \tilde{B} \cap K_1)$ must satisfy the same conditions as φ above since the $\star$-isomorphism condition ($a \in C_i \iff \tilde{\varphi}(a) \in K_i$; see Definition 3.11) implies that the first letter of the word $\tilde{\varphi}(a)$ must be i whenever $a \in C_i$ (for $i \in \{0, 1\}$). So there is no choice for another φ. □

Theorem 5.2 (Characterization of completely ∧-irreducible partial orders). *Let $\mathcal{A} = (A, f)$ be a rooted monounary algebra and $r \in \operatorname{Pord} \mathcal{A}$. Then the following are equivalent.*

(1) *r is completely $\wedge$-irreducible.*
(2) *$r = \varphi^{-1}(p_B)$ or $r = (\varphi^{-1}(p_B))^{-1}$ for some isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ onto a subalgebra $\mathcal{B} = (B, f_\infty \restriction_B)$ of $\mathcal{B}_\infty$ such that $B \cap K_1$ has a least element.*

Proof. First we show that (1) implies (2). According to Lemma 3.6(a), $\top_{\mathcal{A}}$ is either the least or the greatest element with respect to r. Without loss of generality, let $\top_{\mathcal{A}}$ be the

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greatest element. Let $(C_0, C_1) := (\langle u]_r, [m\rangle_r)$ be the $\star$-partition (see Remark 3.12(b)) corresponding to r. By Proposition 5.1 there exists a unique subalgebra $\mathcal{B}$ of $\mathcal{B}_\infty$ which (together with its canonical $\star$-partition) is $\star$-isomorphic to $\mathcal{A}$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be this unique isomorphism. Then the image $\varphi(r) = \{(\varphi(a), \varphi(b)) \mid (a, b) \in r\}$ is a completely $\wedge$-irreducible compatible partial order in Quord $\mathcal{B}$ with the $\star$-partition $(\varphi(C_0), \varphi(C_1)) = (B \cap K_0, B \cap K_1)$. By Proposition 4.1 we get $\varphi(r) = p_B$. Analogously we get $\varphi(r)^{-1} = \varphi(r^{-1}) = p_B$ if $\top_{\mathcal{A}}$ is the least element.

Now we show that (2) implies (1). By Proposition 4.1, p_B is completely $\wedge$-irreducible and therefore, since φ^{-1} is an isomorphism, $\varphi^{-1}(p_B)$ and $(\varphi^{-1}(p_B))^{-1}$ are also completely $\wedge$-irreducible. □

Remark 5.3. For concrete applications finite rooted algebras play a special role. In such situations, complete $\wedge$-irreducibility coincides with $\wedge$-irreducibility, and each such algebra $\mathcal{A}$ is of some finite depth d (Definition 2.3). Then any embedding $\mathcal{A} \to \mathcal{B}_\infty$ is in fact an embedding into the subalgebra $\{v \in \mathbf{2}^{[\infty]} \mid d(v) \le d\}$ which is isomorphic to $\mathcal{B}_d$ via

$$v_1 \ldots v_d 1^\infty \mapsto v_1 \ldots v_d.$$

Therefore, for finite $\mathcal{A}$ of maximal depth d, Theorem 5.2 can be rewritten by using $\mathcal{B}_d$ instead of $\mathcal{B}_\infty$. Note that p_d is (completely) $\wedge$-irreducible in Quord $\mathcal{B}_d$ because K_1 (as a subset of $\mathbf{2}^d$) has 10^{d-1} as the least element. It is not difficult to see that also p_∞ is $\wedge$-irreducible in Quord $\mathcal{B}_\infty$; however, p_∞ is not completely $\wedge$-irreducible because K_1 has no least element ($101^\infty > 1001^\infty > 10\,001^\infty > \cdots > 10^n1^\infty > \cdots$ is an infinite descending chain).

Example 5.4. Let $\mathcal{A} = (\{0, 1, 2, 3\}, f)$ where the graph of f is shown in the left-hand part of Figure 3. In order to find all $\wedge$-irreducible compatible partial orders, according to Remark 5.3 and Theorem 5.2 we must find all subalgebras of $\mathcal{B}_3$ isomorphic to $\mathcal{A}$. There are four such subalgebras $\mathcal{B}, \mathcal{B}', \mathcal{B}'', \mathcal{B}'''$, as given in Figure 3. The corresponding partial orders (restriction of p_3 to the base set of the subalgebras) are drawn by dashed arrows. The elements of $\mathcal{B}_3$ which belong to K_1 are marked as black filled circles. From Figure 3 it is clear that only three of the subalgebras satisfy the condition of Theorem 5.2 that $B \cap K_1$ has a least element, namely $\mathcal{B}, \mathcal{B}', \mathcal{B}''$. In $\mathcal{B}'''$ both 110 and 101 are minimal in $B''' \cap K_1$. Thus there are three $\wedge$-irreducible partial orders in Quord $\mathcal{A}$ with $\top$ as greatest element, that is, with their inverses, $\mathcal{A}$ has altogether six $\wedge$-irreducible partial orders.

According to Proposition 3.4, compatible completely $\wedge$-irreducible partial orders exist only if $\mathcal{A}$ is of binary type. We are now able to give a necessary and sufficient condition.

Proposition 5.5. *A rooted monounary algebra $\mathcal{A}$ has (at least one) compatible completely $\wedge$-irreducible partial order if and only if $\mathcal{A}$ is of binary type and of finite branched depth.*

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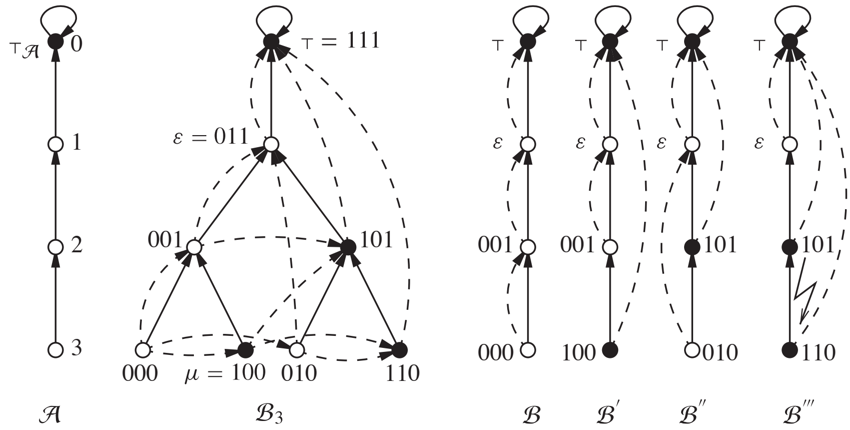

FIGURE 3. The $\wedge$-irreducible partial orders of $\mathcal{A}$.

PROOF. Necessity of the condition follows directly from Proposition 3.4 and Remark 3.12(d). To show sufficiency, let $\mathcal{A} = (A, f)$ be a rooted algebra of binary type and of branched depth d. If $d = 1$, then the graph of f is a directed path with a loop in the root (no branching), and $(C_0, C_1) := (A \setminus \{\top\}, \{\top\})$ is a $\star$-partition and there is a $\star$-isomorphism onto a subalgebra $\mathcal{B}$ of $(\{1^\infty, 01^\infty, 001^\infty, \dots\}, f_\infty) \le \mathcal{B}_\infty$, for which p_B obviously is a $\wedge$-irreducible compatible partial order since $\top$ is the least element of $\mathcal{B} \cap K_1 = \{\top\}$.

Thus we can assume now that $d \ge 2$. Then there exists $b \in A$ such that $|[b]_{\ker f}| = 2$, $f^{d-1}(b) \ne f^d(b) = \top$. We can choose a $\star$-partition (C_0, C_1) of $\mathcal{A}$ such that $b \in C_1$, $fb, \dots, f^{d-1}b \in C_0$ and $\{a \in A \mid |[a]_{\ker f}| = 1\} \subseteq C_0$ (clearly such a $\star$-partition exists since for each kernel class $[a]_{\ker f}$ with two elements one can put, arbitrarily, one element in C_0, and the other in C_1). Because of Proposition 5.1 there exists a unique subalgebra $\mathcal{B}$ of $\mathcal{B}_\infty$ with canonical $\star$-partition $(B \cap K_0, B \cap K_1)$ such that there is a $\star$-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$. Then, by definition of branched depth, all elements in $B \cap K_1$ have depth at most d. According to Definition 3.11 ($\star$-isomorphism) we have $c_0 := \varphi(b) \in K_1$, $c_i := \varphi(f^i b) = f_\infty^i(c_0) \in K_0$ for $i \in \{1, \dots, d-1\}$ and $\varphi(f^d b) = f_\infty^d(c_0) = \top$. There exists exactly one sequence in $\mathcal{B}_\infty$ with these properties, namely $(c_0, c_1, c_2 \dots, c_{d-1}) = (10^{d-1}1^\infty, 0^{d-1}1^\infty, 0^{d-2}1^\infty, \dots, 01^\infty)$. In particular, $c_0 = \mu_{(d)} = 10^{d-1}1^\infty$ is the least element among all elements of K_1 with depth at most d, and therefore it is also the least element in $B \cap K_1$. Consequently, p_B is completely $\wedge$-irreducible (see Proposition 4.1) and thus $\varphi^{-1}(p_B)$ is a compatible completely $\wedge$-irreducible partial order of $\mathcal{A}$ (see Theorem 5.2). □

Now we can collect all our results and give a full description of all completely $\wedge$-irreducible *quasi*orders.

COROLLARY 5.6 (Characterization of $\wedge$-irreducible quasiorders). *Let $\mathcal{A} = (A, f)$ be a monounary rooted algebra. The completely $\wedge$-irreducible compatible quasiorders q*

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of $\mathcal{A}$ are precisely those of the form $\psi^{-1}(p_B)$ or $(\psi^{-1}(p_B))^{-1}$ for some homomorphism $\psi : \mathcal{A} \to \mathcal{B}_\infty$, where B denotes the image of ψ, $|B| \geq 2$ and $B \cap K_1$ has a least element.

Proof. Every homomorphism $\psi : \mathcal{A} \to \mathcal{B}_\infty$ can be factorized according to $\mathcal{A} \to \mathcal{A}/\ker\psi \xrightarrow{\varphi} \mathcal{B}_\infty$, where φ is an isomorphism. Thus the result follows from Proposition 2.6(b) and Theorem 5.2 (observe that $\psi^{-1}(p_B) = \lambda_{\ker f}(\varphi^{-1}(p_B))$). □

Remark 5.7. By Corollary 2.8 and Theorem 5.2, the subdirectly irreducible partially ordered rooted monounary algebras (in the sense of [1]) are precisely those of the form $(A, f, \varphi^{-1}(p_B))$ with φ as in Theorem 5.2. This leads to the following problem.

Problem. Find *all* (not only the rooted) subdirectly irreducible partially ordered monounary algebras.

For instance, the partially ordered monounary algebras (A, f, Δ) and $(\{a, u, \top\}, g, r)$ are not rooted but are subdirectly irreducible, where (A, f) is a cycle of prime power length, $g(u) = g(\top) = \top$, $g(a) = a$, and r is the linear order given by $a < u < \top$.

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