Glasgow Math. J. **57** (2015) 1–5. © Glasgow Mathematical Journal Trust 2014.
doi:10.1017/S0017089514000093.

SAW*-ALGEBRAS ARE ESSENTIALLY NON-FACTORIZABLE

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(Received 17 July 2013; revised 16 August 2013; accepted 7 September 2013; first published online 26 August 2014)

Abstract. In this paper, we solve a question of Simon Wassermann, whether the Calkin algebra can be written as a C*-tensor product of two infinite dimensional C*-algebras. More generally, we show that there is no surjective *-homomorphism from a SAW^*-algebra onto C*-tensor product of two infinite dimensional C*-algebras.

2013 *Mathematics Subject Classification.* 46L05, 46L06, 46L85, 03E05.

1. Introduction. It was shown by Ge [**7**], using free entropy, that if the group von Neumann algebra of $\mathbb{F}_2$, $L(\mathbb{F}_2)$ is written as the von Neumann tensor product of two von Neumann algebras M and N, then either M or N has to be isomorphic to the algebra of $n \times n$ matrices $\mathbb{M}_n(\mathbb{C})$ for some n. For two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ the C*-algebra tensor product is not unique and for a C*-norm $\|.\|_\nu$ on the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ the completion is usually denoted by $\mathcal{A} \otimes_\nu \mathcal{B}$ (see [**1**]). A C*-algebra $\mathcal{A}$ is called *essentially non-factorizable* if it cannot be written as $\mathcal{B} \otimes_\nu \mathcal{C}$, where both $\mathcal{B}$ and $\mathcal{C}$ are infinite dimensional for any C*-algebra norm ν. In a presentation at the London Mathematical Society meeting held in Nottingham in 2010, Simon Wassermann demonstrated that the reduced C*-algebra of $\mathbb{F}_2$, $C_r^*(\mathbb{F}_2)$ is essentially non-factorizable. In fact, if $C_r^*(\mathbb{F}_2) = \mathcal{B} \otimes_\nu \mathcal{C}$ for some C*-norm ν and infinite dimensional C*-algebra $\mathcal{B}$ then $\mathcal{C} = \mathbb{M}_n(\mathbb{C})$ with $n = 1$. At the same meeting, it was asked by Simon Wassermann whether the Calkin algebra is essentially non-factorizable. We prove that the answer to this question is positive by showing that all SAW^*-algebras, of which the Calkin algebra is an example, are essentially non-factorizable.

It is well known that C*-algebras can be viewed as non-commutative topological spaces and the correspondence $X \leftrightarrow C(X)$ is a contravariant category equivalence between the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C*-algebras and unital *-homomorphisms. Each property of a locally compact Hausdorff space can be reformulated in terms of the function algebra $C_0(X)$, so it usually makes sense to ask about these properties for non-commutative C*-algebras. SAW^*-algebras were introduced by Pedersen [**9**] as non-commutative analogues of sub-Stonean spaces (also known as F-spaces) in topology, which are the locally compact Hausdorff spaces in which disjoint σ-compact open subspaces have disjoint compact closure. Analogously a C*-algebra $\mathcal{A}$ is called an SAW^*-algebra if for every two orthogonal elements x and y in $\mathcal{A}_+$ there is an element e in $\mathcal{A}_+$ such that $ex = x$ and $ey = 0$. It is not hard to see that an abelian C*-algebra $C_0(X)$ is a SAW^*-algebra if and only if X is a sub-Stonean space.

Some of the properties of sub-Stonean spaces are generalised to SAW^*-algebras in [**8**, **9**]. It is proved (cf. [**9**]) that the corona algebra of any σ-unital C*-algebra is a SAW^*-algebra. In particular for a separable Hilbert space, the Calkin algebra is a SAW^*-algebra. Another class of C*-algebras, called countably degree-1 saturated C*-algebras, are introduced in [**6**] using model theoretic notions. It was shown that countably degree-1 saturated C*-algebras contain all coronas of σ-unital C*-algebra, ultrapowers of C*-algebras and relative commutants of separable subalgebras of a countably degree-1 saturated C*-algebra, and these are all SAW^*-algebras. In this paper, we will use another property of sub-Stonean spaces to show that SAW^*-algebras are essentially non-factorizable. This will show that the ultrapowers of C*-algebras and relative commutants of separable subalgebras of a countably degree-1 saturated C*-algebra are also essentially non-factorizable. A similar result for ultrapowers of type II_1-factors with respect to a free ultrafilter is proved in [**2**].

In this paper we don't require any knowledge about sub-Stonean spaces and it's enough to know that $\beta\mathbb{N}$, the Stone–Čech compactification of $\mathbb{N}$, is a sub-Stonean space.

2. SAW^*-algebras are essentially non-factorizable. We adopt standard notations from the Ramsey theory and write $[\mathbb{N}]^2$ to denote the set of all $(m, n) \in \mathbb{N}^2$ such that $m < n$ and $\Delta^2\mathbb{N}$ to denote the diagonal of $\mathbb{N}^2$. For spaces X and Y, a rectangle is a subset of $X \times Y$ of the form $A \times B$ for $A \subset X$ and $B \subset Y$. We say a map f on $A \times B$ depends only on the first coordinate if $f(x, y) = f(x, z)$ for every (x, y) and (x, z) in $A \times B$. In [**10**, Lemma 5.1] Van Douwen proved that for any continuous map $f : \beta\mathbb{N}^2 \to \beta\mathbb{N}$ there is a clopen $U \subset \beta\mathbb{N}$ such that $f \restriction U^2$ depends on at most one coordinate and conjecture [**10**, Conjecture 8.4] that there is a disjoint open cover of $\beta\mathbb{N}^2$ into such sets. Farah in [**4**, Theorem 3] showed that for a sub-Stonean space Z, compact spaces X and Y, every continuous map $f : X \times Y \to Z$ is of a *very simple* form, which will be clear from Theorem 2.2 (in fact, the theorem is proved for a larger class of spaces, the so-called $\beta\mathbb{N}$- spaces, in the range and arbitrary powers of a compact space in the domain. However, the theorem remains true if products of arbitrary compact spaces are replaced in the domain of the map). We sketch the proof of this theorem for the convenience of the reader. Before this we need the following lemma.

LEMMA 2.1. *Suppose X, Y and Z are arbitrary sets, $\rho : X \times Y \to Z$ a map, then exactly one of the following holds:*

(1) *$X \times Y$ can be covered by finitely many mutually disjoint rectangles such that ρ depends on at most one coordinate on each of them.*
(2) *There are sequences $x_i \in X$, $y_i \in Y$ such that for all i and all $j < k$ we have $\rho(x_i, y_i) \neq \rho(x_j, y_k)$.*

Moreover if X, Y and Z are topological spaces and ρ is a continuous map, we can assume that the rectangles in (1) *are clopen.*

Proof. For any map from X^2 into X this is an immediate consequence of [**3**, Theorem 3]. One can check the proof of this theorem to see that a small adjustment in definitions would give the same result for any map from $X \times Y$ into Z. To see the second part, note that the closures of this rectangles are still rectangles, and since ρ is continuous, it depends on at most one coordinate on each of this closures. By [**4**, Theorem 8.2] we can assume that these rectangles are clopen. □

THEOREM 2.2. *If ρ is a continuous map from $X \times Y$ into Z, where X and Y are compact topological spaces and Z is a sub-Stonean space, then $X \times Y$ can be covered by finitely many mutually disjoint clopen rectangles such that ρ depends on at most one coordinate on each of them.*

Proof. We just need to show that case (2) of Lemma 2.1 does not happen. Suppose $\{x_i\}$ and $\{y_i\}$ are sequences guaranteed by (2). Define the map $g : \mathbb{N}^2 \longrightarrow X \times Y$ by $g(m, n) = (x_m, y_n)$. Then g continuously extends to a map $\beta g : \beta\mathbb{N}^2 \longrightarrow X \times Y$ and the continuous map $h : \beta\mathbb{N}^2 \longrightarrow Z$ defined by $h = \rho \circ \beta g$ has the property that $h(l, l) \neq h(m, n)$ for all l and all m, n such that $m < n$. This contradicts Corollary 7.6 in [**4**] which states that if $h : \beta\mathbb{N}^2 \longrightarrow Z$ is a continuous map and Z is a sub-Stonean space then the sets $h([\mathbb{N}]^2)$ and $h(\Delta^2\mathbb{N})$ have non-empty intersection. □

As a corollary to this, if X and Y are infinite, then any such ρ is not injective. For C*-algebras, the product of non-commutative spaces corresponds to the tensor product of algebras. By the Gelfand transform we can restate Farah's theorem in terms of commutative C*-algebras.

THEOREM 2.3. *Suppose $f : \mathcal{A} \to \mathcal{B} \otimes \mathcal{C}$ is a unital *-homomorphism, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are unital commutative C*-algebras and $\mathcal{A}$ is a SAW^*-algebra. Then there are finitely many projections $p_1, \dots p_s$ in $\mathcal{B}$ and $q_1, \dots q_t$ projections in $\mathcal{C}$ such that $\sum_{i=1}^{s} p_i = 1_{\mathcal{B}}$ and $\sum_{i=1}^{t} q_i = 1_{\mathcal{C}}$ and for every $1 \leq i \leq s$ and $1 \leq j \leq t$ and either for every $a \in \mathcal{A}$ we have $(p_i \otimes q_j)f(a) \in (p_i\mathcal{B}p_i) \otimes q_j$ or for every $a \in \mathcal{A}$ we have $(p_i \otimes q_j)f(a) \in p_i \otimes (q_j\mathcal{C}q_j)$.*

Note that in particular every element in the image of f is a finite sum of elementary tensor products and if $\mathcal{A}$ is a commutative SAW^*-algebra with no projections (e.g. $A = C(X)$, where X is a connected sub-Stonean space like $\beta\mathbb{R} \setminus \mathbb{R}$), the image of f can be identified with a C*-subalgebra of $\mathcal{B}$ or $\mathcal{C}$.

LEMMA 2.4. *If $\mathcal{B}$ is an infinite-dimensional, unital C*-algebra, we can find an orthogonal sequence $\{a_1, a_2, \dots\}$ in $\mathcal{B}$ such that $0 \leq a_i \leq 1_{\mathcal{B}}$ for all i and a sequence of states on $\mathcal{B}$, $\{\phi_n\}$, such that $\phi_n(a_n) = 1$ and $\phi_n(a_m) = 0$ if $m \neq n$.*

Proof. It is well known that any maximal abelian subalgebra (MASA) of an infinite-dimensional C*-algebra $\mathcal{B}$ is also infinite-dimensional. If not, then there are orthogonal 1-dimensional projections $\{p_1, p_2, \dots, p_n\}$ in MASA such that $\sum_{i=1}^{n} p_i = 1_{\mathcal{B}}$. Since $\mathcal{B} = \sum_{i,j=1}^{n} p_i\mathcal{B}p_j$ and for each pair i, j, we have $p_i\mathcal{B}p_j$ as either $\{0\}$ or 1-dimensional, and $\mathcal{B}$ is finite-dimensional. Fix such a MASA, and by the Gelfand–Naimark theorem identify it with $C(X)$ for some compact Hausdorff space X. We can also identify the set of pure states of $C(X)$ with X. Since X is an infinite normal space, we can choose a discrete sequence of pure states $\{\phi_n\}$ in X and find a pairwise disjoint sequence $\{U_n\}$ of open neighbourhoods of $\{\phi_n\}$. By Uryshon's lemma we get an orthogonal sequence $0 \leq a_n \leq 1_{\mathcal{B}}$ in $C(X)$ such that $\phi_n(a_n) = a_n(\phi_n) = 1$ and a_n vanishes outside U_n. So $\phi_n(a_m) = a_m(\phi_n) = 0$ if $m \neq n$. Now by the Hahn–Banach extension theorem extend ϕ_n to a functional on $\mathcal{B}$ of norm 1. Since $\phi_n(1_{\mathcal{B}}) = 1$, this extension is a state. □

Note that if ϕ is a state on a C*-algebra $\mathcal{A}$ and $\phi(a) = 1$ for $0 \leq a \leq 1_{\mathcal{A}}$, as a consequence of the Cauchy–Schwartz inequality for states we have $\phi(b) = \phi(aba)$ for any $b \in \mathcal{A}$ (cf. [**5**, Lemma 4.8]).

LEMMA 2.5. *Let $\{\phi_n\}$ be a sequence of states on a SAW^*-algebra $\mathcal{A}$. If there exists a sequence $\{a_n\}$ of mutually orthogonal positive elements in $\mathcal{A}$ such that $\|a_n\| = \phi_n(a_n) = 1$ and $\phi_n(a_m) = 0$ if $m \neq n$, then the weak*-closure of $\{\phi_n\}$ is homeomorphic to $\beta\mathbb{N}$.*

Proof. Let D be a subset of $\mathbb{N}$. We show that $\overline{\{\phi_n : n \in D\}} \cap \overline{\{\phi_n : n \in D^c\}} = \emptyset$. Take $\psi \in \overline{\{\phi_n : n \in D\}}$. Let $a = \sum_{i \in D} 2^{-i}a_i$ and $b = \sum_{i \in D^c} 2^{-i}a_i$. Since $\mathcal{A}$ is a SAW^*-algebra, there exists a positive $e \in \mathcal{A}$ such that $ea = a$ and $eb = 0$. Then $ea_n = a_n$ for $n \in D$ and $ea_n = 0$ for every $n \in D^c$. For $n \in D$ we have $\phi_n(e) = \phi_n(ea_n) = \phi_n(a_n) = 1$

https://doi.org/10.1017/S0017089514000093 Published online by Cambridge University Press

and for $n \in D^c$ we have $\phi_n(e) = \phi_n(ea_n) = \phi_n(0) = 0$. Hence, $\psi(e) = 1$ and ψ is not in $\overline{\{\phi_n : n \in D^c\}}$.

Now let $F : \beta\mathbb{N} \longrightarrow \overline{\{\phi_n : n \in \mathbb{N}\}}$ be the continuous map such that $F(n) = \phi_n$. Let $\mathcal{U}$ and $\mathcal{V}$ be two distinct ultra filters in $\beta\mathbb{N}$ and pick $X \subseteq \mathbb{N}$ such that $X \in \mathcal{U}$ but X is not in $\mathcal{V}$. Therefore,

$$F(\mathcal{U}) \in F(\overline{X}) \subseteq \overline{F(X)} = \overline{\{\phi_n : n \in X\}}.$$

Similarly, $F(\mathcal{V}) \in \overline{\{\phi_n : n \in X^c\}}$. So F is injective and clearly surjective. Since $\beta\mathbb{N}$ is compact and $\overline{\{\phi_n\}}$ is Hausdorff, it follows that F is a homeomorphism. □

THEOREM 2.6. *Any SAW^* algebra is essentially non-factorizable.*

Proof. Let $\mathcal{A}$ be a SAW^*-algebra. Suppose that $\mathcal{A} = \mathcal{B} \otimes_\nu \mathcal{C}$ for the C*-completion of the algebraic tensor product $\mathcal{B} \odot \mathcal{C}$ of infinite dimensional C*-algebras $\mathcal{B}$ and $\mathcal{C}$ with respect to some C*-norm $\|.\|_\nu$. By Lemma 2.4 there are orthogonal sequences of positive contractions $\{b_n\} \subseteq \mathcal{B}$ and $\{c_n\} \subseteq \mathcal{C}$ and sequences $\{\phi_n\} \subseteq \mathcal{B}^*$ and $\{\psi_n\} \subseteq \mathcal{C}^*$ such that $\phi_n(b_n) = \psi_n(c_n) = 1$ and $\phi_n(b_m) = \psi_n(c_m) = 0$ for $m \neq n$. Identifying $\mathcal{A}$ with $\mathcal{B} \otimes_\nu \mathcal{C}$ and letting $a_{m,n} = b_m \otimes c_n$ and $\gamma_{m,n} = \phi_m \otimes \psi_n$, it is immediate that

$$\gamma_{m,n}(a_{m',n'}) = \begin{cases} 1 & (m, n) = (m', n') \\ 0 & (m, n) \neq (m', n') \end{cases}.$$

Let $X = \overline{\{\phi_n : n \in \mathbb{N}\}}^{w^*}$ and $Y = \overline{\{\psi_n : n \in \mathbb{N}\}}^{w^*}$. Then X and Y are compact subsets of $\mathcal{B}^*$ and $\mathcal{C}^*$, respectively, $\{\phi_n : n \in \mathbb{N}\} \times \{\psi_n : n \in \mathbb{N}\}$ is a dense subset of $\overline{\{\gamma_{m,n}\}}^{w^*}$ and by compactness $X \times Y$ is homeomorphic to $\overline{\{\gamma_{m,n}\}}^{w^*}$, which is homeomorphic to $\beta\mathbb{N}$ by Lemma 2.5. This contradicts the remark following the proof of Theorem 2.2. Hence, there is no *-isomorphism $\mathcal{A} \cong \mathcal{B} \otimes \mathcal{C}$ with $\mathcal{B}$ and $\mathcal{C}$ infinite dimensional. □

COROLLARY 2.7. *The Calkin algebra is essentially non-factorizable.*

We do not know whether Theorem 2.3 is true for non-commutative C*-algebras. But an analogous theorem for non-commutative C*-algebras would provide us with a strong tool to study the automorphisms between tensorial powers of the Calkin algebra or other SAW^*-algebras such as ultrapowers of C*-algebras.

ACKNOWLEDGEMENT. The author is indebted to Ilijas Farah for several suggestions and useful conversations. He would also like to thank the referee for suggesting several improvements.

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