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φ-SYMMETRIC CONTACT METRIC SPACES

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Abstract. We give the first examples of contact metric spaces which are weakly locally φ-symmetric (that is, $\nabla R = 0$ on horizontal vectors), but not strongly (that is, not all reflections with respect to the characteristic lines are local isometries). These examples are three-dimensional non-unimodular Lie groups with a left-invariant contact metric structure. We exhibit additional symmetries on these spaces.

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1. Introduction. Within the class of K-contact or Sasakian manifolds, local symmetry is a very strong requirement: a locally symmetric K-contact manifold necessarily has constant curvature equal to 1. See [**13**] and [**16**]. For this reason, T. Takahashi introduced the weaker notion of a (*locally*) *φ-symmetric space* in the context of Sasakian geometry [**15**]: a Sasakian space $(M, \xi, \eta, \varphi, g)$ is locally φ-symmetric if it satisfies the curvature condition

$$g((\nabla_X R)(Y, Z)V, W) = 0, \tag{1}$$

for all vector fields X, Y, Z, V and W which are horizontal; that is, orthogonal to the characteristic vector field ξ. Here, ∇ is the Levi Civita connection and R the associated Riemann curvature tensor. T. Takahashi proves that this condition is equivalent to having characteristic reflections (that is, reflections with respect to the integral curves of ξ) which are local automorphisms of the Sasakian structure. In [**4**], it is shown that the isometry property of the reflections is already sufficient. (For a slightly more general result, see [**8**].)

At least two different generalizations of the notion of a locally φ-symmetric space to the broader class of contact metric manifolds have appeared in the literature. In [**3**], a contact metric space is called *locally φ-symmetric* if it satisfies the curvature condition (1). This is a very workable definition technically, but without immediate geometric content. A second definition, by the first and the third author, starts from a geometric reality: a contact metric space is called *locally φ-symmetric* if its characteristic reflections are local isometries [**7**]. This definition leads to an infinite number of curvature conditions; (see Proposition 2). To distinguish between the two notions, we shall speak about *weak local φ-symmetry* (for the first one) and *strong local φ-symmetry* (for the second one).

In [**7**], the first and the third author show that the unit tangent sphere bundle of a Riemannian manifold (M, g), equipped with its natural contact metric structure, is

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strongly locally φ-symmetric if and only if (M, g) has constant curvature. When the constant curvature differs from 1, the associated unit tangent sphere bundle is not Sasakian [**17**]. This gave the first class of examples of strongly locally φ-symmetric contact metric spaces which are not Sasakian. In [**5**], this class was extended to the class of (non-Sasakian) contact metric manifolds whose characteristic vector field ξ belongs to the (k, μ)-nullity distribution, for some real numbers k and μ. This means that the curvature tensor R satisfies

$$R(X, Y)\xi = k\left(\eta(Y)X - \eta(X)Y\right) + \mu\left(\eta(Y)hX - \eta(X)hY\right), \tag{2}$$

for all vector fields X and Y, where h denotes, up to a scaling factor, the Lie derivative of the structure tensor φ in the direction of ξ. The unit tangent sphere bundles of spaces of constant curvature c satisfy this condition with $k = c(2 - c)$ and $\mu = -2c$. Moreover, within the class of unit tangent sphere bundles, they are the only ones with this property [**2**]. Further, all three-dimensional unimodular Lie groups, except the commutative one, can be equipped with a left-invariant contact metric structure satisfying (2). (See [**2**].)

Recently, D. Perrone made an in-depth study of three-dimensional homogeneous contact metric manifolds [**14**]. He showed that all these spaces are locally isometric to Lie groups with a left-invariant contact metric structure. For the unimodular Lie groups, he proved that these structures are weakly locally φ-symmetric. He does not undertake a similar study for the non-unimodular ones, however, and neither does he consider strong local φ-symmetry.

In this note, we clarify the questions regarding local φ-symmetry. All unimodular Lie groups in Perrone's classification turn out to be strongly locally φ symmetric. The situation is more complicated for the non-unimodular ones. In this case, we find the first examples of contact metric spaces which are weakly locally φ-symmetric, but not strongly (Theorem 5). Hence, the two notions of local φ-symmetry do not agree. The geometry of the non-unimodular spaces has an additional interesting feature relating to isometric reflections, which we discuss in the last section of this note.

2. Preliminaries. We first collect some basic facts about contact metric manifolds. We refer to [**1**] for a detailed treatment. All manifolds are assumed to be connected and smooth.

A manifold M^{2n+1} has an *almost contact structure* if it admits a vector field ξ, a one-form η and a (1, 1)-tensor field φ satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \varphi^2 = -\operatorname{id} + \eta \otimes \xi.$$

Then one can always find a Riemannian metric g which is compatible; that is, such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y. (ξ, η, φ, g) is called an *almost contact metric structure* and $(M, \xi, \eta, \varphi, g)$ an *almost contact metric manifold*. If in addition the equation

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$d\eta(X, Y) = g(X, \varphi Y)$ holds, then $(M, \xi, \eta, \varphi, g)$ is called a *contact metric manifold*. As a consequence, the integral curves of the characteristic vector field ξ are geodesics.

On a contact metric manifold M, one defines the $(1, 1)$-tensor field h by $hX = (1/2)(\mathcal{L}_\xi \varphi)(X)$ where $\mathcal{L}_\xi$ denotes Lie differentiation in the direction of ξ. If $h = 0$, then ξ is a Killing vector field and $(M, \xi, \eta, \varphi, g)$ is called a *K-contact manifold*. Finally, if the Riemann curvature tensor satisfies

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]}\xi = \eta(Y)X - \eta(X)Y, \tag{3}$$

for all vector fields X and Y, then the contact metric manifold is *Sasakian*. In that case, ξ is a Killing vector field and hence every Sasakian manifold is K-contact. The converse is true in dimension three, but not in general.

As mentioned in the introduction, we have two different notions of local φ-symmetry for contact metric manifolds.

Definition 1. A contact metric space $(M, \xi, \eta, \varphi, g)$ is said to be *weakly locally φ-symmetric* if its Riemann curvature tensor R satisfies $g((\nabla_X R)(Y, Z)V, W) = 0$, for all horizontal vector fields X, Y, Z, V and W. It is *strongly locally φ-symmetric* if the characteristic reflections are local isometries.

The first requirement for the local reflection with respect to a curve to be a local isometry is that the curve is a geodesic, which is the case for the integral curves of ξ on a contact metric manifold. From [**10**], we have the following analytic criterion.

Proposition 2. *Let $(M, \xi, \eta, \varphi, g)$ be a contact metric manifold. If it is a strongly locally φ-symmetric space, then the following curvature conditions hold:*

$$g((\nabla^{2k}_{X\cdots X} R)(X, Y)X, \xi) = 0, \tag{4}$$

$$g((\nabla^{2k+1}_{X\cdots X} R)(X, Y)X, Z) = 0, \tag{5}$$

$$((\nabla^{2k+1}_{X\cdots X} R)(X, \xi)X, \xi) = 0, \tag{6}$$

for all horizontal vectors X, Y and Z, and all $k \in \mathbb{N}$. Moreover, if (M, g) is analytic, these conditions are also sufficient for the contact metric manifold to be a strongly locally φ-symmetric space.

Condition (5) with $k = 0$ is equivalent to the condition (1) for weak φ-symmetry. Hence, *every strongly locally φ-symmetric space is weakly φ-symmetric*. Moreover, as an immediate consequence of condition (4) with $k = 0$, we have the following result

Proposition 3. *Let $(M, \xi, \eta, \varphi, g)$ be a strongly locally φ-symmetric space. Then ξ is an eigenvector of the Ricci operator.*

3. Local φ-symmetry on three-dimensional homogeneous contact metric spaces. In [**14**], D. Perrone studies three-dimensional manifolds with a homogeneous contact metric structure. He shows that these manifolds are locally isometric to a Lie group G with a left-invariant contact metric structure. We now consider these spaces with regard to local φ-symmetry. Following [**14**], we distinguish between the unimodular and the non-unimodular case.

3.1. The unimodular case. Let G be a unimodular Lie group with a left-invariant contact metric structure (ξ, η, φ, g). Because of the invariance under left translations, the structure is completely determined if we know how it acts on the associated Lie algebra $\mathfrak{g}$. It is shown in [**14**] that an orthonormal basis $\{e_1, e_2 = \varphi e_1, \xi\}$ for $\mathfrak{g}$ can be chosen such that the bracket relations are given by

$$[e_1, e_2] = 2\xi, \qquad [e_2, \xi] = c_1 e_1, \qquad [\xi, e_1] = c_2 e_2. \tag{7}$$

REMARK. From (7), we see immediately that the commutative Lie group $\mathbb{R}^3$ does not admit any left-invariant contact metric structure. This is the only three-dimensional unimodular Lie group in which this occurs. Indeed, for all others, one can construct such a structure explicitly (see, for example [**2**], [**14**]). Note also that there exists a *flat* left-invariant contact metric structure on the (non-commutative) Lie group $E(2)$.

We return to the metric Lie group G with Lie algebra $\mathfrak{g}$ determined by (7). In [**2**], it is proved that the characteristic vector field ξ of the contact metric structure belongs to the (k, μ)-nullity distribution for $k = 1 - (c_1 - c_2)^2/4$ and $\mu = 2 - (c_1 + c_2)$. When $c_1 \neq c_2$, $k \neq 1$ and the contact metric manifold is not Sasakian. By [**5**, Theorem 1], it is strongly locally φ-symmetric. (For a direct proof, see [**9**].) In the Sasakian case ($c_1 = c_2$), the constancy of the scalar curvature already implies that the manifold is (strongly) locally φ-symmetric ([**19**]). Summarizing, we have the following result.

THEOREM 4. *Every three-dimensional unimodular Lie group, with the exception of the commutative one, admits a left-invariant contact metric structure. For all of these structures, the characteristic vector field ξ belongs to the (k, μ)-nullity distribution and the structure is strongly locally φ-symmetric.*

3.2. The non-unimodular case. Let G be a non-unimodular Lie group with a left-invariant contact metric structure. Again, it suffices to describe this structure on the associated Lie algebra $\mathfrak{g}$. D. Perrone [**14**] shows that one can find an orthonormal basis $\{e_1, e_2 = \varphi e_1, \xi\}$ for $\mathfrak{g}$ such that the bracket relations are given by

$$[e_1, e_2] = \alpha e_2 + 2\xi, \qquad [e_2, \xi] = 0, \qquad [e_1, \xi] = \gamma e_2 \tag{8}$$

with $\alpha \neq 0$. There is little freedom in the choice of this basis: the *unimodular kernel* $\mathfrak{a} = \{X \in \mathfrak{g} \mid \operatorname{tr}\operatorname{ad}_X = 0\}$ is two-dimensional [**12**] and contains ξ [**14**]. One takes e_1 as the unique unit vector orthogonal to $\mathfrak{a}$ (up to sign), and $e_2 = \varphi e_1$. Note that Milnor's isomorphism invariant D [**12**], given in this case by $D = -8\gamma/\alpha^2$, can be assigned any real value by varying α and γ.

REMARK. It is known (see [**12**]) that D is a complete isomorphism invariant for the non-unimodular Lie algebras if we exclude the exceptional algebra determined by the bracket

$$[e_1, e_2] = \lambda\, e_2, \qquad [e_1, e_3] = \lambda\, e_3, \qquad [e_2, e_3] = 0 \tag{9}$$

where $\lambda \neq 0$; that is, the case in which ad_{e_1} is a non-zero multiple of the identity on $\mathfrak{a}$. From the above, we see that the Lie group associated to this specific Lie algebra *does not admit any left-invariant contact metric structure,* the only three-dimensional non-unimodular Lie group where this occurs (as D takes all real values). We note that the metric Lie group with Lie algebra (9) has constant negative curvature, and that this real space form does not admit any other Lie group structure.

We return to the non-unimodular Lie group G with the Lie algebra (8). Using the Koszul formula, we calculate the covariant derivative:

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_2} e_1 = -\alpha\, e_2 - \dfrac{2+\gamma}{2}\,\xi, & \nabla_{\xi} e_1 = -\dfrac{2+\gamma}{2}\, e_2, \\ \nabla_{e_1} e_2 = \dfrac{2-\gamma}{2}\,\xi, & \nabla_{e_2} e_2 = \alpha\, e_1, & \nabla_{\xi} e_2 = \dfrac{2+\gamma}{2}\, e_1, \\ \nabla_{e_1} \xi = -\dfrac{2-\gamma}{2}\, e_2, & \nabla_{e_2} \xi = \dfrac{2+\gamma}{2}\, e_1, & \nabla_{\xi} \xi = 0. \end{array} \tag{10}$$

From this it follows at once that ξ is a Killing vector field (and hence the structure is K-contact or Sasakian) if and only if $\gamma = 0$. In that case, it follows again from the constancy of the scalar curvature that the manifold is (strongly) locally φ-symmetric [**19**].

The Riemann curvature tensor is given by

$$\begin{aligned} R(e_1, e_2)e_1 &= (\alpha^2 + \gamma + 3 - \gamma^2/4)\, e_2 + \alpha\gamma\,\xi, \\ R(e_1, e_2)e_2 &= -(\alpha^2 + \gamma + 3 - \gamma^2/4)\, e_1, \\ R(e_1, e_2)\xi &= -\alpha\gamma\, e_1, \\ R(e_1, \xi)e_1 &= \alpha\gamma\, e_2 + (3\gamma^2/4 + \gamma + 1)\,\xi, \\ R(e_1, \xi)e_2 &= -\alpha\gamma\, e_1, \\ R(e_1, \xi)\xi &= -(3\gamma^2/4 + \gamma + 1)\, e_1, \\ R(e_2, \xi)e_1 &= 0, \\ R(e_2, \xi)e_2 &= -(2+\gamma)^2/4\,\xi, \\ R(e_2, \xi)\xi &= (2+\gamma)^2/4\, e_2. \end{aligned} \tag{11}$$

In particular, as $R(e_1, e_2)\xi = -\alpha\gamma\, e_1$, we see that the condition (4) is not satisfied for $k = 0$ if $\gamma \neq 0$. Hence, the associated Lie group is not strongly locally φ-symmetric unless it is a Sasakian manifold.

As concerns weak local φ-symmetry, we compute

$$\begin{aligned} g((\nabla_{e_1} R)(e_1, e_2)e_1, e_2) &= \alpha\gamma(\gamma - 2), \\ g((\nabla_{e_2} R)(e_1, e_2)e_1, e_2) &= 0. \end{aligned}$$

Hence, if $\gamma = 2$, the associated Lie group is weakly locally φ-symmetric, but not strongly. In this case, the isomorphism invariant $D = -16/\alpha^2$ is strictly negative. The condition $\gamma = 2$ is equivalent to the geometric property that the basis $\{e_1, e_2, \xi\}$ is parallel along the integral curves of the vector field e_1.

We have established the following result.

THEOREM 5. *Every three-dimensional non-unimodular Lie group, with the exception of the one associated with the exceptional Lie algebra* (9), *admits a left-invariant contact metric structure. Furthermore, depending on the value of the Milnor isomorphism invariant D, the following occur:*

1. *if* $D > 0$, *none of the contact metric structures is locally φ-symmetric (in whatever sense);*
2. *if* $D = 0$, *the associated structures are Sasakian and locally φ-symmetric in both senses;*
3. *if* $D < 0$, *none of the contact metric structures is strongly φ-symmetric, but there exists one which is weakly φ-symmetric.*

From the curvature formulas (11), it follows at once that

$$\rho(e_1, \xi) = 0, \qquad \rho(e_2, \xi) = -\alpha\gamma,$$

where ρ is the Ricci curvature. Hence, ξ is an eigenvector of the Ricci operator if and only if $\gamma = 0$. Hence, in the homogeneous case, we find the following partial converse to Proposition 3.

PROPOSITION 6. *A three-dimensional homogeneous contact metric structure is strongly locally φ-symmetric if and only if ξ is an eigenvector of the Ricci operator.*

(See also [**11**] and [**9**] for related results.)

REMARK. The simply connected Lie group associated to the Lie algebra (8) can be realized explicitly on $\mathbb{R}^3$ by the orthonormal coframe

$$\begin{aligned}\omega^1 &= \frac{1}{2}\,dx,\\ \omega^2 &= \frac{1}{2}(\alpha\,y + \gamma\,z)\,dx + dy,\\ \eta &= y\,dx + dz.\end{aligned}$$

4. Symmetries on three-dimensional non-unimodular Lie groups. Consider again the non-unimodular Lie groups associated to the Lie algebra (8). As already mentioned before, the vector field e_1 has a specific geometric meaning: it is the unique unit vector field (up to sign) that is orthogonal to the unimodular kernel $\mathfrak{a}$. From (10) it follows that its integral curves are geodesics. We show now that the local reflections with respect to this family of geodesics are local isometries, regardless of the value of the invariant D. As a Lie group is analytic, it suffices to check that the curvature conditions (4)–(6) are satisfied when we replace ξ by e_1.

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Let $\{\omega^1, \omega^2, \eta\}$ be the dual orthonormal coframe of the basis $\{e_1, e_2, \xi\}$. It follows from the expressions (10) that

$$\begin{aligned}
\nabla\omega^1 &= -\alpha\,\omega^2 \otimes \omega^2 - \frac{2-\gamma}{2}\,(\omega^2 \otimes \eta + \eta \otimes \omega^2),\\
\nabla\omega^2 &= \frac{2-\gamma}{2}\,\omega^1 \otimes \eta + \alpha\,\omega^2 \otimes \omega^1 + \frac{2+\gamma}{2}\,\eta \otimes \omega^1,\\
\nabla\eta &= -\frac{2-\gamma}{2}\,\omega^1 \otimes \omega^2 + \frac{2+\gamma}{2}\,\omega^2 \otimes \omega^1.
\end{aligned}$$

In particular, we note that an *even* number of ω^1 becomes an *odd* number of ω^1 after covariant differentiation and vice versa. Further, the curvature tensor R in its $(0, 4)$-form is given by

$$\begin{aligned}
R = 4\Big(&(\alpha^2 + \gamma + 3 - \gamma^2/4)\,\omega^1 \wedge \omega^2 \otimes \omega^1 \wedge \omega^2\\
&+ \alpha\gamma\,(\omega^1 \wedge \omega^2 \otimes \omega^1 \wedge \eta + \omega^1 \wedge \eta \otimes \omega^1 \wedge \omega^2)\\
&+ (3\gamma^2/4 + \gamma - 1)\,\omega^1 \wedge \eta \otimes \omega^1 \wedge \eta\\
&- (2+\gamma)^2/4\,\omega^2 \wedge \eta \otimes \omega^2 \wedge \eta\Big).
\end{aligned}$$

We see that each term has an even number of ω^1. By induction, we derive from this and the observation above the following result.

Lemma 7. *The curvature expression $\nabla^\ell R$ consists of terms involving an even number of ω^1 if ℓ is even, and an odd number of ω^1 if ℓ is odd.*

The verification of the conditions (4)–(6) with e_1 in the role of ξ is now immediate. These conditions also hold trivially for the exceptional case (9), since the sectional curvature is constant, and hence the local reflection with respect to *any* geodesic is a local isometry [**18**]. We have proved the following result.

Proposition 8. *For any non-unimodular Lie group, the local reflections with respect to the integral curves of e_1 are local isometries.*

Remark. This is a special case of a more general result concerning isometric reflections on special semi-direct product Lie groups; (see the forthcoming paper [**6**] by the same authors).

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