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MODIFIED CAUCHY KERNELS AND FUNCTIONAL CALCULUS FOR OPERATORS ON BANACH SPACE

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Abstract

In Banach space operators with a bounded H^∞ functional calculus, Cowling et al. provide some necessary and sufficient conditions for a type-ω operator to have a bounded H^∞ functional calculus. We provide an alternate development of some of their ideas using a modified Cauchy kernel which is L^1 with respect to the measure $|dz|/|z|$. The method is direct and has the advantage that no transforms of the functions are necessary.

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1. Introduction

In [1] Cowling et al. show that type-ω operators (defined below) on Banach spaces which satisfy 'weak quadratic estimates' possess a bounded functional calculus for holomorphic functions. In Section 2 of this paper we obtain similar results using a modified Cauchy kernel applied to the Riesz-Dunford formula for functional calculi. In Section 3 we derive some 'strong quadratic estimates', a special case of those in [1], which are sufficient for a type-ω operator on $L^p(\Omega)$ to have a bounded H^∞ functional calculus. The derivation is almost immediate from the results in Section 2 and allows us in Section 4 to show that type-ω operators on Hilbert spaces which possess bounded functional calculi for functions also admit uniformly bounded functional calculi for matrices. This implies using the results of Paulsen [9], that such operators are similar to operators with functional calculus constant 1. This result has been obtained independently by Christian Le Merdy [8].

We begin with some notation, definitions, and assumptions. Throughout, X and

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$\mathcal{H}$ denote complex Banach spaces and complex Hilbert spaces respectively. All operators T acting on X or $\mathcal{H}$ will be assumed to be closed, one–one and have dense domains and dense ranges. For $0 \leq \mu < \pi$, let S_μ denote the open sector of angle μ, that is, $S_\mu = \{z \in \mathbb{C} : |\arg z| < \mu\}$. Let $H^\infty(S_\mu)$ denote the space of functions which are bounded and holomorphic on S_μ. For $h \in H^\infty(S_\mu)$ set $\|h\|_\infty = \sup_{z \in S_\mu} |h(z)|$.

We are interested in operators which satisfy the following condition on their resolvents.

DEFINITION 1.1. An operator T in X is said to be type-ω, where $0 \leq \omega < \pi$, if T is closed, $\sigma(T) \subseteq S_\omega \cup \partial S_\omega$, and for each μ in (ω, π) and z in $\mathbb{C} \backslash S_\mu$,

$$\|(T - z)^{-1}\| \leq c|z|^{-1}.$$

One-one type-ω operators always possess an H^∞ functional calculus. That is, for $\mu > \omega$ there exists a unique algebra homomorphism from $H^\infty(S_\mu)$ into the space of closed operators on X which takes $(\lambda - z)^{-1}$ to $(\lambda - T)^{-1}$. However it may happen that for some $h \in H^\infty(S_\mu)$ with $\|h\|_\infty = 1$ we have $\|h(T)\| = \infty$; see [6, 7]. (An operator with this property may also be obtained by taking the Cayley transform of Foguel's 1964 counterexample [2, 5].) To show that the conditions we derive in Sections 2 and 3 guarantee a bounded functional calculus, we shall need McIntosh's result for approximating operators, namely the Convergence Lemma.

LEMMA 1.2 (Convergence Lemma). *Suppose T is an operator of type-ω which is one-to-one with dense domain and dense range in X, and that $\mu > \omega$. Let $\{f_\alpha\}$ be a uniformly bounded net of functions in $H^\infty(S_\mu)$ which converges to a function f in $H^\infty(S_\mu)$ uniformly on compact subsets of S_μ. Suppose further that the operators $f_\alpha(T)$ are uniformly bounded on X. Then $f_\alpha(T)u$ converges to $f(T)u$ for all u in X, and consequently $f(T)$ is a bounded linear operator on X, and $\|f(T)\| \leq \sup_\alpha \|f_\alpha(T)\|$.*

We use here the 'variable constant convention', according to which c, c_1, $\ldots$, denote constants (in $\mathbb{R}^+$) which may vary from one occurrence to the next. In a given formula, the constant does not depend on variables expressly quantified after the formula, but it may depend on variables quantified (implicitly or explicitly) before. Thus, in Definition 1.1, c may depend on X, T, ω, and μ, but not on z.

2. H^∞ functional calculus in Banach spaces

Let T be a one-one type-ω operator acting in a complex Banach space X having dense domain and dense range. Let $\langle \cdot, \cdot \rangle$ denote the bilinear pairing between X and X^*. We wish to describe some necessary and sufficient conditions for T to

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have a bounded H^∞ functional calculus. Our method uses a modification of the Cauchy kernel and an associated modification in the definition of the Riesz-Dunford functional calculus. Initially to simplify matters we restrict our attention to functions in $H^\infty(S_\mu) \cap L^1(\partial S_\mu, |dz|/|z|)$. For $\mu > \omega$, $h \in H^\infty(S_\mu) \cap L^1(\partial S_\mu, |dz|/|z|)$ and ζ in the interior of S_μ,

$$h(\zeta) = \frac{1}{2\pi i} \int_{\partial S_\mu} \frac{h(z)}{(z-\zeta)} dz,$$

and

$$h(T) = \frac{1}{2\pi i} \int_{\partial S_\mu} \frac{h(z)}{(z-T)} dz. \tag{1}$$

(Since T is type-ω, the second integral converges absolutely in operator norm.)

However, many other kernels besides $(z-\zeta)^{-1}$ will reproduce the values of holomorphic functions and provide formulas for holomorphic functions of type-ω operators. For example, Cauchy's theorem shows that,

$$h(\zeta) = \frac{1}{2\pi i} \int_{\partial S_\mu} \frac{h(z) z^{1/2} \zeta^{1/2}}{(z-\zeta)} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\partial S_\mu} \frac{2h(z) z \zeta}{(z^2-\zeta^2)} \frac{dz}{z}, \tag{2}$$

and

$$h(T) = \frac{1}{2\pi i} \int_{\partial S_\mu} \frac{h(z) z^{1/2} T^{1/2}}{(z-T)} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\partial S_\mu} \frac{2h(z) z T}{(z^2-T^2)} \frac{dz}{z}. \tag{3}$$

Note that for the second equalities to hold in the above, $\mu < \pi/2$. Also, using (1), one sees that $\|z^{1/2}T^{1/2}(z-T)^{-1}\|$ and $\|zT(z^2-T^2)^{-1}\|$ are uniformly bounded in z, so that the integrals in (3) converge absolutely in the operator norm.

In these formulas the measure dz has been replaced by dz/z and the kernels have changed. The advantage is, that while $(z-\zeta)^{-1}$ is not integrable with respect to $|dz|$ on ∂S_μ, both $z^{1/2}\zeta^{1/2}(z-\zeta)^{-1}$, and $2z\zeta(z^2-\zeta^2)^{-1}$ are integrable with respect to $|dz|/|z|$. This allows one to unambiguously extend the formulas in (2) to all of $H^\infty(S_\mu)$. Furthermore, the L^1 norms of these kernels depend only on the argument of ζ, a fact critical when ζ is replaced by an operator T in order to make estimates about the norm of $h(T)$.

Fix $\nu > \omega$, $h \in H^\infty(S_\nu) \cap L^1(\partial S_\nu, |dz|/|z|)$, $u \in X$, and $v \in X^*$ with $\|u\| = \|v\| = 1$. Using (3) one has

$$|\langle h(T)u, v\rangle| = \left| \frac{1}{2\pi i} \int_{\partial S_\nu} \left\langle \frac{h(z) z^{1/2} T^{1/2}}{(z-T)} u, v \right\rangle \frac{dz}{z} \right|$$
$$\leq \frac{1}{2\pi} \int_{\partial S_\nu} \left| \left\langle \frac{h(z) z^{1/2} T^{1/2}}{(z-T)} u, v \right\rangle \right| \frac{|dz|}{|z|}$$
$$\leq \frac{1}{2\pi} \|h\|_\infty \int_{\partial S_\nu} \left| \left\langle \frac{z^{1/2} T^{1/2}}{(z-T)} u, v \right\rangle \right| \frac{|dz|}{|z|}. \tag{4}$$

Hence if for all $u \in X$, and $v \in X^*$,

$$\int_{\partial S_\nu} \left| \left\langle \frac{z^{1/2} T^{1/2}}{(z-T)} u, v \right\rangle \right| \frac{|dz|}{|z|} \leq c \|u\| \|v\|, \tag{5}$$

then $\|h(T)\| \leq c \|h\|_\infty$ for all $h \in H^\infty(S_\nu) \cap L^1(\partial S_\nu, |dz|/|z|)$. This implies via the Convergence Lemma that T has a bounded $H^\infty(S_\nu)$ functional calculus. In fact we have the following theorem which is a version of [1, Theorems 4.2 and 4.4].

THEOREM 2.1. *Let T be a one-one type-ω operator acting in a complex Banach space X having dense domain and dense range. If* (5) *holds for all $u \in X$ and $v \in X^*$ then T has a bounded $H^\infty(S_\nu)$ functional calculus. Conversely if T has a bounded $H^\infty(S_\mu)$ functional calculus, then* (5) *holds for all $u \in X$, and $v \in X^*$.*

PROOF. We have proven the first part. To see the second part, fix $\nu > \mu > \omega$, $u \in X$, and $v \in X^*$. For $z \in \partial S_\nu$, let $a(z, u, v)$ be the unimodular function determined by the relation

$$\frac{1}{2\pi i} \int_{\partial S_\nu} a(z, u, v) \left\langle \frac{z^{1/2} T^{1/2}}{(z-T)} u, v \right\rangle \frac{dz}{z} = \frac{1}{2\pi} \int_{\partial S_\nu} \left| \left\langle \frac{z^{1/2} T^{1/2}}{(z-T)} u, v \right\rangle \right| \frac{|dz|}{|z|}.$$

For $\zeta \in S_\mu$, define a holomorphic function $F_{u,v}(\zeta)$ by the formula,

$$F_{u,v}(\zeta) = \int_{\partial S_\nu} a(z, u, v) \frac{z^{1/2} \zeta^{1/2}}{(z-\zeta)} \frac{dz}{z}. \tag{6}$$

One easily sees that

$$\sup_{\zeta \in S_\mu} |F_{u,v}(\zeta)| \leq c \frac{1}{(\nu - \mu)}. \tag{7}$$

Now using (4) and (6) one has that

$$\int_{\partial S_\nu} \left| \left\langle \frac{z^{1/2} T^{1/2}}{(z-T)} u, v \right\rangle \right| \frac{|dz|}{|z|} = \langle F_{u,v,h}(T) u, v \rangle,$$

and thus by (7), (5) holds if T has a bounded $H^\infty(S_\mu)$ functional calculus.

To see that Theorem 2.1 is similar to [1, Theorems 4.2 and 4.4] note that with a suitable change of variables the integral in (5) can be written as two integrals over $\mathbb{R}^+$ with respect to dt/t, corresponding to the upper and lower rays of ∂S_μ. Then, upon replacing t by $1/t$, the kernel $z^{1/2}T^{1/2}(z-T)^{-1}$ becomes,

$$\psi_+(tT) = e^{i\nu/2}t^{1/2}T^{1/2}/(e^{i\nu} - tT) \tag{8}$$

for the upper ray, and

$$\psi_-(tT) = e^{-i\nu/2}t^{1/2}T^{1/2}/(e^{-i\nu} - tT) \tag{9}$$

for the lower ray. Note that for all $z \in S_\mu$ one has

$$|\psi_{+,-}(z)| \le c|z|^{1/2}/(1+|z|);$$

thus $\psi_{+,-}(z) \in \Psi(S_\mu)$ as defined in [1]. Rewriting (5) we obtain,

$$\int_0^\infty \big(|\langle\psi_+(tT)u, v\rangle| + |\langle\psi_-(tT)u, v\rangle|\big)\frac{dt}{t} \le c\|u\|\|v\|. \tag{10}$$

3. H^∞ functional calculus for operators on L^p.

We now wish to apply the modified Cauchy kernels to L^p spaces. Accordingly, let Ω be a domain in $\mathbb{R}^n$ and let T be a one-one type-ω operator acting on $L^p(\Omega)$ having dense domain and dense range. Let q be the conjugate exponent to p and let T^* be the adjoint of T with respect to the bilinear pairing $\langle\cdot,\cdot\rangle$ between L^p and L^q. Let $\psi_{+,-}(tT)$ be defined by formulas (8) and (9) above. Fix $\nu > \mu > \omega$, $f \in H^\infty(S_\nu) \cap L^1(\partial S_\mu, |dz|/|z|)$, $u \in L^p$, $v \in L^q$, and $\zeta \in S_\mu$. Using Section 2 and the above definitions one has,

$$\begin{aligned}
|\langle f(T)u, v\rangle| &\le \frac{1}{2\pi}\int_{\partial S_\mu}\left|\left\langle \frac{f(z)z^{1/2}T^{1/2}}{(z-T)}u, v\right\rangle\right|\frac{|dz|}{|z|} \\
&\le c\|f\|_\infty \int_0^\infty (|\langle\psi_+(tT)u, v\rangle| + |\langle\psi_-(tT)u, v\rangle|)\,dt/t \\
&= c\|f\|_\infty \int_0^\infty \big(|\langle\psi_+^{1/2}(tT)u, \psi_+^{1/2}(tT^*)v\rangle| + |\langle\psi_-^{1/2}(tT)u, \psi_-^{1/2}(tT^*)v\rangle|\big)dt/t
\end{aligned}$$

Considering just the first term of this last integral and using the fact that we are working in function space gives,

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$$\begin{aligned}\int_0^\infty &|\langle\psi_+^{1/2}(tT)u,\psi_+^{1/2}(tT^*)v\rangle|dt/t\\ &\le\int_\Omega\int_0^\infty|\psi_+^{1/2}(tT)u(x)\psi_+^{1/2}(tT^*)v(x)|\frac{dt}{t}dx\\ &\le\int_\Omega\left\{\int_0^\infty|\psi_+^{1/2}(tT)u(x)|^2\frac{dt}{t}\right\}^{1/2}\left\{\int_0^\infty|\psi_+^{1/2}(tT^*)v(x)|\frac{dt}{t}\right\}^{1/2}dx\\ &\le\left\|\left\{\int_0^\infty|\psi_+^{1/2}(tT)u(\cdot)|^2\frac{dt}{t}\right\}^{1/2}\right\|_p\left\|\left\{\int_0^\infty|\psi_+^{1/2}(tT^*)v(\cdot)|\frac{dt}{t}\right\}^{1/2}\right\|_q.\end{aligned}$$

This last quadratic expression, which we have arrived at without intricate transforms and estimates, is a special case of the quadratic expressions [1]. Using the above and the Convergence Lemma one can prove the following version of [1, Corallary 6.8]:

THEOREM 3.1. *Let T be a one-one type-ω operator acting on $L^p(\Omega)$ having dense domain and dense range. If for all $u\in L^p$ and $v\in L^q$,*

(11)
$$\left\|\left\{\int_0^\infty|\psi_+^{1/2}(tT)u(\cdot)|^2\frac{dt}{t}\right\}^{1/2}\right\|_p+\left\|\left\{\int_0^\infty|\psi_-^{1/2}(tT)u(\cdot)|^2\frac{dt}{t}\right\}^{1/2}\right\|_p\le c\|u\|_p,$$

and

(12)
$$\left\|\left\{\int_0^\infty|\psi_+^{1/2}(tT^*)v(\cdot)|\frac{dt}{t}\right\}^{1/2}\right\|_q+\left\|\left\{\int_0^\infty|\psi_-^{1/2}(tT^*)v(\cdot)|\frac{dt}{t}\right\}^{1/2}\right\|_q\le c\|v\|_q$$

then T has a bounded $H^\infty(S_\nu)$ functional calculus.

The converse of this theorem also holds but the methods employed here will not yield a proof. For completeness, and use in Section 4, we state the converse proved in [1].

THEOREM 3.2. *Let T be a one-one type-ω operator acting on $L^p(\Omega)$ having dense domain and dense range. Suppose T has a bounded $H^\infty(S_\mu)$ functional calculus. Then for all $u\in L^p$ and $v\in L^q$,* (11) *and* (12) *hold.*

(Note that the angles have changed so that this is not a precise converse.)

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4. H^∞ matricial functional calculus

Let $\mathscr{H}$ be a complex Hilbert space. In this section we show how by doing little more than adding indices to the formulas in the previously sections one can extend the bounded $H^\infty(S_\nu)$ functional calculus for an operator T acting $\mathscr{H}$ to a bounded $H^\infty(S_\nu; \mathbb{C}^{n\times n})$ functional calculus. Let $(\mathscr{H})^{(n)}$ denote the space of n-tuples of vectors in $\mathscr{H}$. For $u \in (\mathscr{H})^{(n)}$ set $\|u\|_{2,2} = c\left(\sum_{m=1}^n \|u_m\|_2^2\right)^{1/2}$.

With this definition we have the following theorem.

THEOREM 4.1. *Let T be a one-one type-ω operator acting on a complex Hilbert space $\mathscr{H}$ having dense domain and dense range. Suppose T has a bounded $H^\infty(S_\mu)$ functional calculus. Then for $n = 1, 2, \ldots$, $[f_{i,j}(z)] \in H^\infty(S_\nu; \mathbb{C}^{n\times n})$, and $u \in (L^2)^{(n)}$,*

$$|\langle [f_{i,j}(T)]u, u\rangle| \le c\|[f_{i,j}]\|_\infty \|u\|_{2,2}^2,$$

where c does not depend on n.

PROOF. To show T has a bounded $H^\infty(S_\nu; \mathbb{C}^{n\times n})$ functional calculus it suffices to consider T restricted to a closed separable invariant subspace generated by a set of n vector $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and the functional calculus of T: namely, the closure of a subspace of the form $\mathscr{H}_\Gamma = \left\{\sum_{i=1}^n h_i(T)\gamma_i : h_i \in H^\infty(S_\mu),\ 1 \le i \le n\right\}$. Thus, since any separable Hilbert space is unitarily equivalent to $L^2(\Omega)$ for some $\Omega \subset \mathbb{R}^n$, we may assume without loss of generality that $\mathscr{H} = L^2(\Omega)$.

Fix $[f_{i,j}(z)] \in H^\infty(S_\nu; \mathbb{C}^{n\times n})$, and $u \in (L^2)^{(n)}$. We assume also that

$$\|[f_{i,j}]\|_\infty \le 1. \tag{13}$$

Estimating in an analogous fashion to the previous section one has

$$\begin{aligned}
|\langle [f_{i,j}(T)]u, u\rangle| &\le \frac{1}{2\pi}\int_{\partial S_\mu} \left|\left\langle \frac{z^{1/2}T^{1/2}}{(z-T)}[f_{i,j}(z)]u, u\right\rangle\right| \frac{|dz|}{|z|} \\
&\le c\int_0^\infty \left(|\langle \psi_+(tT)[f_{i,j}(e^{i\nu}t^{-1})]u, u\rangle| + |\langle \psi_-(tT)[f_{i,j}(e^{-i\nu}t^{-1})]u, u\rangle|\right)\frac{dt}{t} \\
&= c\int_0^\infty (|\langle \psi_+^{1/2}(tT)[f_{i,j}(e^{i\nu}t^{-1})]u, \psi_+^{1/2}(tT^*)u\rangle| \\
&\quad + |\langle \psi_-^{1/2}(tT)[f_{i,j}(e^{-i\nu}t^{-1})]u, \psi_-^{1/2}(tT^*)u\rangle|)\frac{dt}{t}.
\end{aligned} \tag{14}$$

Now, using (13) and Hölder's inequality, one has that the first term of (14) is bounded by

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$$\int_\Omega \int_0^\infty \left(\sum_{m=1}^n |\psi_+^{1/2}(tT)u_m(x)|^2\right)^{1/2} \left(\sum_{m=1}^n |\psi_+^{1/2}(tT^*)u_m(x)|^2\right)^{1/2} \frac{dt}{t} dx$$
$$\le c \int_\Omega \left\{\sum_{m=1}^n \int_0^\infty |\psi_+^{1/2}(tT)u_m(x)|^2 \frac{dt}{t}\right\}^{1/2} \left\{\sum_{m=1}^n \int_0^\infty |\psi_+^{1/2}(tT^*)u_m(x)|^2 \frac{dt}{t}\right\}^{1/2} dx$$
$$\le c \left\| \left\{\sum_{m=1}^n \int_0^\infty |\psi_+^{1/2}(tT)u_m(x)|^2 \frac{dt}{t}\right\}^{1/2} \right\|_2 \left\| \left\{\sum_{m=1}^n \int_0^\infty |\psi_+^{1/2}(tT^*)u_m(x)|^2 \frac{dt}{t}\right\}^{1/2} \right\|_2 .$$

Using Theorem 3.2 and the Convergence Lemma one has that the above and the analogous expression for $\psi_-^{1/2}$ are bounded by $c\|u\|_{2,2}^2$, proving the theorem.

In [9] Paulsen showed that if T has a uniformly bounded $H^\infty(S_\nu; \mathbb{C}^{n\times n})$ functional calculus then T is similar to an operator B with functional calculus constant 1. That is, $T = LBL^{-1}$, where L and L^{-1} are bounded and B satisfies $\|f(B)\| \le \|f\|_\infty$. Thus Theorem 4.1 shows that if T has a bounded $H^\infty(S_\mu)$ functional calculus, then T is similar to an operator with functional calculus constant 1. Note that, as mentioned in the introduction, this has been obtained independently by Le Merdy [8].

5. Comments

Theorems 2.1, 3.1 and 3.2 can be generalized in the manner alluded to in Section 2. The kernel $z^{1/2}\zeta^{1/2}(z-\zeta)^{-1}$ is not the only modification of the Cauchy kernel which reproduces the values of holomorphic functions and is integrable with respect to $|dz|/|z|$ on ∂S_ν. Any other kernel with those properties would yield similar theorems ($2z\zeta(z^2-\zeta^2)^{-1}$ for example). The modification of the Cauchy kernel to get better integrability properties extends to the Clifford setting as well [4]. The author, in joint work with McIntosh, has developed an alternate and more direct approach to Theorem 3.2, by discretizing the square function estimates on the sector [3].

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