A REMARK ON CONVEX POLYTOPES

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In this note we wish to present an alternative proof for the following well-known theorem [1, Theorem 16]: every convex polytope X in Euclidean n-dimensional space R^n is the intersection of a finite family of closed half-spaces. It will be supposed that the converse of this theorem has been verified by conventional arguments, namely: every bounded intersection of a finite family of closed half-spaces in R^n is a convex polytope [cf. 1, Theorem 15].

We shall assume that the interior of X (denoted by X^o) is non-empty, since the theorem can be extended to include other cases by standard arguments. We may furthermore suppose, without loss of generality, the origin 0 of R^n to be in X^o.

Now the dual $Y*$ of any set Y in R^n is defined as:

$$Y* = \{y* | y* \cdot y \leq 1 \text{ for all } y \in Y\} .$$

Let $H(Y)$ denote the convex cover of Y ; thus, $X = H(x_1, \ldots, x_k)$ for some finite point set $\{x_1, x_2, \ldots, x_k\} \subset R^n$ by the definition of a convex polytope. It is well known that

$$\{x_1, x_2, \ldots, x_k\} * = \overline{(H(0, x_1, x_2, \ldots, x_k))}*$$

$$= (H(x_1, \ldots, x_k))* = X* \quad (1, \text{p. } 26)$$

where $\overline{H(0, x_1, \ldots, x_k)}$ denotes the closure of $H(0, x_1, \ldots, x_k)$. On the other hand

$$\{x_1, x_2, \ldots, x_k\} * = (\bigcup_{i=1}^{k} \{x_i\})* = \bigcap_{i=1}^{k} \{x_i\} * .$$

https://doi.org/10.4153/CMB-1965-064-1 Published online by Cambridge University Press

But $\{x_i\}^*$ is the closed half-space $\{y \mid y \cdot x_i \leq 1\}$ [1, p. 26]. Thus, X^* is the intersection of a finite family of closed half-spaces and, since $0 \in X^o$, it is also bounded [cf. 1, p. 26]. It follows, therefore, that X^* is a convex polytope.

Now $0 \in (X^*)^o$ since X is bounded [1, p. 26]; applying the first part of our argument to X^* we may conclude that X^{**} is the intersection of a finite family of closed half-spaces. Since $X^{**} = X$ (1, p. 26), this yields the result sought.

It should be pointed out that our argument actually shows that X is the intersection of closed half-spaces bounded by the extreme supporting hyperplanes and that the extreme points [hyperplanes] of X correspond to the extreme hyperplanes [points] of X^* if 0 is chosen to be in X^o.

REFERENCE

1. H. G. Eggleston, "Convexity", (Cambridge, 1958).

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https://doi.org/10.4153/CMB-1965-064-1 Published online by Cambridge University Press