AN EXPANSION FOR THE PERMANENT OF A DOUBLY STOCHASTIC MATRIX

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The permanent of an n-square matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of order n. K_n will denote the convex set of all n-square doubly stochastic matrices and K_n^0 its interior. $J_n \in K_n$ will be the matrix with all elements equal to $1/n$. If $M \in K_n^0$, then M lies on a line segment passing through J_n and another $B \in K_n - K_n^0$. This note gives an expansion for the permanent of such a line segment as a weighted average of permanents of matrices in K_n. For a survey article on permanents the reader is referred to Marcus and Minc [3].

A property of permanents needed is that if $C_{ij}(A)$ is the permanent of the matrix obtained by deleting row i and column j of A,

$$\begin{aligned}\operatorname{per} A &= \sum_{j=1}^{n} a_{ij} C_{ij}(A), \qquad i = 1, 2, \cdots, n \\ &= \sum_{i=1}^{n} a_{ij} C_{ij}(A), \qquad j = 1, 2, \cdots, n.\end{aligned}$$

For $1 \leqq r < s \leqq n$, denote by P_{rs} the n-square permutation matrix (p_{ij}) with

$$\begin{aligned} p_{ij} &= 1 \quad \text{if} \quad i = r,\ j = s \\ &\qquad \text{or} \quad i = s,\ j = r \\ &\qquad \text{or} \quad i = j \neq r \neq s, \\ &= 0 \ \text{elsewhere,} \end{aligned}$$

and by P_j, $j = 1, 2, \cdots, \frac{1}{2}n(n-1)$ the idempotent matrices $\frac{1}{2}(I + P_{rs})$, arranged in dictionary order. Multiplication of an n-square matrix by $\frac{1}{2}(I + P_{rs})$ on the right has the effect of replacing columns r and s by their average, multiplication on the left has a similar effect on the rows. $A \in K_n$ implies $AP_j, P_jA \in K_n$, $j = 1, 2, \cdots, \frac{1}{2}n\ (n-1)$.

THEOREM 1. *If* $A \in K_n$, $0 < x \leqq 1$, *then*

$$\text{(1)} \qquad \begin{aligned} \operatorname{per}(xA + (1-x)J_n) &= e^{-y} \sum_{k=0}^{\infty} (y^k k!) Q_k(A), \\ Q_0(A) &= \operatorname{per} A \\ Q_{k+1}(A) &= \{2/n(n-1)\} \sum_j Q_k(AP_j) \\ &= \{2/n(n-1)\} \sum_j Q_k(P_jA), \end{aligned}$$

$k = 0, 1, \cdots$ *and* $y = -2(n-1)\log x$.

PROOF. The proof consists of identifying the coefficient of $y^k/k!$ in the Taylor series expansion of

$$e^y \operatorname{per}(e^{-y/2(n-1)}A + (1 - e^{-y/2(n-1)})J_n)$$

as $Q_k(A)$.

Denote

$$xA + (1-x)J_n = A[x] = A(y)$$

and notice that

$$\begin{aligned} A[x_1][x_2] &= A[x_1 x_2], \\ A(y_1)(y_2) &= A(y_1 + y_2). \end{aligned}$$

$$\begin{aligned} &\frac{d}{dx} Q_0(A[x] \Big|_{x=1} \\ &\quad = \sum_{ij} (a_{ij} - 1/n) C_{ij}(A) \\ &\quad = nQ_0(A) - (1/n) \sum_{ij} C_{ij}(A) \\ &\quad = nQ_0(A) - (1/n) \sum_{jki} a_{ik} C_{ij}(A) \\ &\quad = -(4/n) \sum_{j<k} (\tfrac{1}{2}Q_0(A) + \tfrac{1}{4} \sum_i a_{ik} C_{ij}(A) + \tfrac{1}{4} \sum_i a_{ij} C_{ik}(A)) \\ &\qquad - (1/n) \sum_{ij} a_{ij} C_{ij}(A) + (2n-1) Q_0(A) \\ &\quad = -(4/n) \sum_j Q_0(AP_j) + 2(n-1) Q_0(A) \\ &\quad = -(4/n) \sum_j Q_0(P_jA) + 2(n-1) Q_0(A) \\ &\quad = 2(n-1)(Q_0(A) - Q_1(A)). \end{aligned}$$

More generally

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$$(2)\qquad \frac{d}{dx}Q_0(A[x]) = x^{-1}\frac{d}{dw}Q_0(A[wx])\Big|_{w=1} = x^{-1}\frac{d}{dw}Q_0(A[x][w])\Big|_{w=1} = 2(n-1)x^{-1}(Q_0(A[x]) - Q_1(A[x])).$$

The differential equation (2) is equivalent to

$$(3)\qquad \frac{d}{dy}e^{y}Q_0(A(y)) = e^{y}Q_1(A(y)).$$

By recurrence, using (3)

$$(4)\qquad \frac{d^k}{dy^k}e^{k}Q_0(A(y)) = e^{y}Q_k(A(y)), \qquad k = 1, 2, \cdots.$$

Thus (1) is the Taylor series expansion of $e^{y}Q_0(A(y))$.

If $A \in K_n$, $Q_k(A)$ is the average of the $\{\frac{1}{2}n(n-1)\}^k$ permanents of doubly stochastic matrices in the expansion of

$$A\left(\sum_j P_j\right)^k = A(\tfrac{1}{2}n(n-2)I + \tfrac{1}{2}nJ_n)^k = \{\tfrac{1}{2}n(n-1)\}^k A[(n-2)^k/(n-1)^k].$$

There does not seem to be an analogue of Theorem 1 for determinants. Notice that $\det AP_j = 0$ for any n-square matrix A.

Let $C_k(A)$ be the set of matrices in the expansion of $A(\sum_j P_j)^k$, $k = 0, 1, \cdots$. Because P_j, $j = 1, 2, \cdots, \frac{1}{2}n(n-1)$ are idempotent matrices, $C_k(A)$ and $\bigcup_{l=1}^{k-1} C_l(A)$ are not disjoint. It is appropriate to consider (1) as an expa[illegible] average of permanents of matrices in the expansion of

$$AR_s = A\sum_{l_s \neq l_{s-1}} P_{1_s} \sum_{l_{s-1}\neq l_{s-2}} P_{l_{s-1}} \cdots \sum_{l_1} P_{l_1};$$

this average will be denoted by $U_s(A)$, $s = 0, 1, \cdots$ with $U_0(A) = \operatorname{per} A$.

COROLLARY 1. *If* $A \in K_n$, $0 < x \leqq 1$, *then*

$$(5)\qquad \operatorname{per}(xA + (1-x)J_n) = \sum_{k=0}^{\infty} c_k(x)U_k(A),$$

$c_0(x) = e^{-y}$

$$c_{k+1}(x) = (k!)^{-1}\{(n-2)(n+1)/n(n-1)\}^k e^{-y}\int_0^y v^k e^{2v/n(n-1)}dv,$$

$k = 0, 1, \cdots,$

$$\sum_{k=0}^{\infty} c_k(x) = 1$$

and $y = -2(n-1)\log x$.

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PROOF. $$\sum_{l_{s+1}} P_{l_{s+1}} R_s = \sum_{l_{s+1}=l_s} P_{l_{s+1}} \sum_{l_s \neq l_{s-1}} P_{l_s} \cdots \sum_{l_1} P_{l_1} + \sum_{l_{s+1} \neq l_s} P_{l_{s+1}} \sum_{l_s \neq l_{s-1}} P_{l_s} \cdots \sum_{l_1} P_{l_1} = R_s + R_{s+1}$$

so by recurrence

$$\left(\sum_j P_j\right)^{s+1} = \sum_{l=0}^{s} \binom{s}{l} R_{l+1}. \tag{6}$$

A straightforward use of (6), noting that R_{s+1} has $\frac{1}{2}n(n-1)\{\frac{1}{2}(n+1)(n-2)\}^s$ terms, gives (5).

COROLLARY 2. *If $A \in K_n$, $A \neq J_n$, then for k large enough $Q_k(A) > n!/n^n$ and $Q_k(A) \to n!/n^n$ as $k \to \infty$.*

PROOF. If $A \in K_n$, per $A[x]$ is a polynomial of degree n in x. Marcus and and Newman [1] have shown that the coefficient of x is zero and the coefficient of x^2 is strictly positive if $A \neq J_n$.

$$\text{per}\, A[x] = n!/n^n + \sum_{r=2}^{n} a_r x^r$$

implies from (4)

$$Q_k(A) = n!/n^n + \sum_{r=2}^{n} a_r (1 - r/2(n-1))^k.$$

The corollary follows because $a_2 > 0$.

Corollary 2 is not unexpected because if $B \in K_n$ and

$$\| B - J_n \|^2 = \sum_{ij} (b_{ij} - 1/n)^2, \text{ then}$$

$$\| BP_j - J_n \| \leqq \| B - J_n \|, \; j = 1, 2, \cdots, \tfrac{1}{2}n(n-1)$$

with equality holding if and only if $BP_j = B$.

Marcus and Newman [2] have essentially shown that if $A \in K_n$ is positive semi-definite and symmetric then the coefficients of x^r, $r = 2, \cdots, n$ in per $A[x]$ are non-negative. In this case $Q_k(A)$ is monotonically decreasing to $n!/n^n$.

An unsolved conjecture of van der Waerden's [4] states that

$$\underset{A \in K_n}{\text{minimum}} \; \text{per}\, A = n!/n^n$$

and is attained uniquely when $A = J_n$. Marcus and Newman [1] have shown that if per B is minimal over K_n and $B \in K_n^0$, then $B = J_n$. This result can also be established from (1) because if $B \in K_n^0$, then $B = xA + (1-x)J_n$ for some $A \in K_n$, $0 \leqq x < 1$, and since per B is minimal,

$$\operatorname{per} B = Q_k(A),\ k = 0, 1, \cdots,$$

which, together with Corollary 2, implies $B = J_n$.

It is possible to obtain some identities among the elements of $\{Q_k(A),\ k = 0, 1, \cdots, A \in K_n\}$ by considering $\operatorname{per} A[x]$ as a polynomial of degree n in x.

THEOREM 2. *If $A \in K_n$, then*

$$\operatorname{per} A[x] = \sum_{t=0}^{n} \sum_{l=t}^{n} \sum_{k=t}^{l} (x-1)^l \binom{2(n-1)}{l-k} (k!)^{-1} \{-2(n-1)\}^t s(k,t) Q_t(A), \tag{7}$$

where $s(k,t)$ are Stirling numbers of the first kind such that

$$u_{(k)} = u(u-1)\cdots(u-k+1) = \sum_{t=0}^{k} s(k,t)u^t.$$

PROOF. Consider the expansion of $x^{2(n-1)}x^{-2(n-1)}\operatorname{per} A[x]$ as a polynomial in $(x-1)$.

Suppose

$$\operatorname{per} A[x] = \sum_{j=0}^{n} a_j x^j,$$

then

$$\begin{aligned} \frac{d^k}{dx^k} x^{-2(n-1)} \operatorname{per} A[x] \Big|_{x=1} &= \sum_{j=0}^{n} a_j (j-2(n-1))_{(k)} \\ &= \sum_{t=0}^{k} s(k,t) \sum_{j=0}^{n} a_j (j-2(n-1))^t \\ &= \sum_{t=0}^{k} s(k,t) \{-2(n-1)\}^t Q_t(A), \end{aligned} \tag{8}$$

and (7) follows in a straightforward way from (8).

Other finite expansions are possible, a simpler one that (7) is

$$x^{2(n-1)} \operatorname{per} A[x^{-1}] = \sum_{t=0}^{2(n-1)} Q_t(A) \{2(n-1)\}^t \sum_{l=0}^{2(n-1)} (x-1)^l s(l,t). \tag{9}$$

Placing $x = e^{-y/2(n-1)}$ in (7) (or (9)) and comparing the expansion of $e^y \operatorname{per} A(y)$ with (1) gives identities for $Q_{j+n}(A)$, $j = 1, 2, \cdots$ in terms of $Q_0(A), \cdots, Q_n(A)$.

THEOREM 3. *If M is an n-square matrix whose row and column sums are zero, then*

$$\{(n-2)/2(n-1)\}^k \operatorname{per} M = Q_k(M),\ k = 0, 1, \cdots.$$

PROOF. The proof of (4) extends without change to when A is an n-square matrix whose row and column sums are unity. Placing $A = M + J_n$, the coefficient of x^n in $\operatorname{per} A[x]$ is $\operatorname{per} M$. If

$$\operatorname{per} A[x] = \sum_{i=0}^{n-1} a_i x^i + x^n \operatorname{per} M,$$

then (4) implies

$$(10) \qquad Q_k(A[x]) = \sum_{i=0}^{n-1} a_i \{1 - i/2(n-1)\}^k x^i + \{(n-2)/2(n-1)\}^k x^n \operatorname{per} M.$$

Equating coefficients of x^n in (10) proves the theorem.

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