Canad. J. Math. Vol. **53** (5), 2001 pp. 979–1030

Ranks of Algebras of Continuous C^*-Algebra Valued Functions

Masaru Nagisa, Hiroyuki Osaka and N. Christopher Phillips

Abstract. We prove a number of results about the stable and particularly the real ranks of tensor products of C^*-algebras under the assumption that one of the factors is commutative. In particular, we prove the following:

(1) If X is any locally compact σ-compact Hausdorff space and A is any C^*-algebra, then $\mathrm{RR}\big(C_0(X)\otimes A\big) \leq \dim(X) + \mathrm{RR}(A)$.
(2) If X is any locally compact Hausdorff space and A is any purely infinite simple C^*-algebra, then $\mathrm{RR}\big(C_0(X)\otimes A\big) \leq 1$.
(3) $\mathrm{RR}\big(C([0,1])\otimes A\big) \geq 1$ for any nonzero C^*-algebra A, and $\mathrm{sr}\big(C([0,1]^2)\otimes A\big) \geq 2$ for any unital C^*-algebra A.
(4) If A is a unital C^*-algebra such that $\mathrm{RR}(A) = 0$, $\mathrm{sr}(A) = 1$, and $K_1(A) = 0$, then $\mathrm{sr}\big(C([0,1])\otimes A\big) = 1$.
(5) There is a simple separable unital nuclear C^*-algebra A such that $\mathrm{RR}(A) = 1$ and $\mathrm{sr}\big(C([0,1])\otimes A\big) = 1$.

0 Introduction

The (topological) stable rank of Rieffel [31] and the real rank of Brown and Pedersen [5] are noncommutative generalizations of the dimension of a compact Hausdorff space. While it has been known for some time that the covering dimension satisfies $\dim(X\times Y) \leq \dim(X)+\dim(Y)$ for compact Hausdorff spaces X and Y (see Proposition 9.3.2 of [25]), little is known about the analogous situation for C^*-algebras, namely the stable and real ranks of tensor products of C^*-algebras. In this paper, we investigate the real rank, and to some extent the stable rank, of tensor products of C^*-algebras under the assumption that one of the factors is commutative.

If A is a C^*-algebra, we denote its real rank by $\mathrm{RR}(A)$ and its stable rank by $\mathrm{sr}(A)$. (We use the term stable rank because it has been shown [14] that for a C^*-algebra, Rieffel's topological stable rank is equal to the (Bass) stable rank of the algebra.) Our main results are then as follows:

(1) If X is any locally compact σ-compact Hausdorff space and A is any C^*-algebra, then $\mathrm{RR}\big(C_0(X)\otimes A\big) \leq \dim(X) + \mathrm{RR}(A)$. (Corollary 1.10.)
(2) If X is any locally compact Hausdorff space and A is any purely infinite simple C^*-algebra, then $\mathrm{RR}\big(C_0(X)\otimes A\big) \leq 1$. (Theorem 3.11.)

Received by the editors November 6, 1999; revised April 25, 2001.
Research of the second author partially supported by the Japanese Society for the Promotion of Science in cooperation with the Danish Research Academy/DANVIS. Research of the third author partially supported by NSF grants DMS 9400904 and DMS 9706850.
AMS subject classification: Primary: 46L05, 46L52; secondary: 46L80, 19A13, 19B10.

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

(3) $\mathrm{RR}\big(C([0,1])\otimes A\big) \geq 1$ for any nonzero C^*-algebra A, and $\mathrm{sr}\big(C([0,1]^2)\otimes A\big) \geq 2$ for any unital C^*-algebra A. (Propositions 5.1 and 5.3.)
(4) If A is a unital C^*-algebra such that $\mathrm{RR}(A) = 0$, such that $\mathrm{sr}(A) = 1$, and such that $K_1(A) = 0$, then $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$. (Theorem 4.3.)
(5) There is a simple separable unital nuclear C^*-algebra A such that $\mathrm{RR}(A) = 1$ and $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$. (Example 5.8.)

The result (1) is an analog and generalization of the inequality $\dim(X \times Y) \leq \dim(X) + \dim(Y)$. We do not expect equality because this can fail even in the case of compact metric spaces (see [30]), and also for $A = M_n$ [4] or for purely infinite simple A (result (2) above). In fact, even the inequality can fail for a tensor product of two noncommutative C^*-algebras. In [16] and in [24] there are examples of two separable nuclear C^*-algebras A and B such that

$$\mathrm{RR}(A) = \mathrm{RR}(B) = 0 \quad \text{and} \quad \mathrm{RR}(A \otimes B) = 1.$$

As corollaries to (1), we give several related results. The one most closely resembling the inequality for dimensions of products is Corollary 1.12: $\mathrm{RR}\big(C_0(X)\otimes A\big) \leq \mathrm{RR}\big(C_0(X)\big) + \mathrm{RR}(A)$ for any unital A and any X.

The same methods as for real rank prove that $\mathrm{sr}\big(C_0(X) \otimes A\big) \leq \dim(X) + \mathrm{sr}(A)$ for any locally compact σ-compact Hausdorff space X. While for real rank, this inequality is best possible, for stable rank the conjecture is that $\mathrm{sr}\big(C_0(X) \otimes A\big) \leq \langle \frac{\dim(X)}{2} \rangle + \mathrm{sr}(A)$, where $\langle \alpha \rangle$ is the least integer n satisfying $\alpha \leq n$. This conjecture remains open.

The result (2) on purely infinite simple C^*-algebras should be compared with the theorem that $\mathrm{RR}(B\otimes K) \leq 1$ for any B (Proposition 3.3 of [4]), and with the theorem that $\mathrm{RR}(B \otimes A) \leq 1$ for any B and any separable nuclear purely infinite simple C^*-algebra A (Corollary 1.4 of [28]). The basic idea is that tensoring with a C^*-algebra containing matrix algebras of arbitrarily large size in an essential way should reduce the real rank to at most 1. We do not, however, know whether, say, $\mathrm{RR}(B \otimes_{\min} A) \leq 1$ for any B and any (separable) purely infinite simple C^*-algebra A.

Along the way to the proof of (2), we prove the following result on hereditary subalgebras which may be of independent interest (Corollary 2.13). Let A be a separable purely infinite simple C^*-algebra, let $X \subset [0,1]^n$ be closed, and let $D \subset C(X) \otimes A$ be a hereditary subalgebra whose image under every point evaluation map is nonzero and nonunital. Then D is stable. (The restriction to subsets of $[0,1]^n$ is probably unnecessary.)

The results (3), (4), and (5) are the main part of a closer investigation of tensor products with $C([0,1])$. One expects, for example, that $C([0,1]) \otimes A$ should never be "zero dimensional", and the inequality $\mathrm{RR}\big(C([0,1]) \otimes A\big) \geq 1$ (for any C^*-algebra $A \neq 0$) in (3) can be interpreted as saying exactly that. Moreover, if A is "zero dimensional", then one expects $C([0,1]) \otimes A$ to be "one dimensional". One version of this statement follows from (1) and (3): if $\mathrm{RR}(A) = 0$ and $A \neq 0$, then $\mathrm{RR}\big(C([0,1]) \otimes A\big) = 1$. The result (4) is another, more interesting, version. Moreover, we show that $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$ implies both $\mathrm{sr}(A) = 1$ and $K_1(A) = 0$.

One might therefore hope that $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$ would also imply $\mathrm{RR}(A) = 0$. Unfortunately, as our result (5) shows, this is not true.

This paper is organized as follows. In the first section we prove that $\mathrm{RR}\big(C_0(X) \otimes A\big) \leq \dim(X) + \mathrm{RR}(A)$ for X locally compact and σ-compact, and other results related to (1). In the second section we prove the result on hereditary subalgebras of $C(X) \otimes A$ for A purely infinite and simple. We use this (actually, a more precise form) in the third section to derive the bound (2) on $\mathrm{RR}\big(C_0(X) \otimes A\big)$. Section 4 contains the proof that $\mathrm{RR}(A) = 0$, $\mathrm{sr}(A) = 1$, and $K_1(A) = 0$ imply $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$. Section 5 contains the lower bound results (3) and the counterexample (5).

Some of this work was carried out during visits by the second two authors to Københavns Universitet during the Fall Semester 1995, and they are grateful to that institution for its hospitality. The third author would also like to thank Purdue University; some of the work reported here was done during a sabbatical year there in 1997–1998.

Throughout this paper, $\dim(X)$ means the covering dimension of X (Section 3.1 of [25]). We recall, however, that all three common dimensions agree for compact metric spaces. (This follows from Corollary 4.5.10 of [25].) We use [25] as our standard reference for dimension theory. Note, however, that the terminology there is unusual. ("Bicompact" in [25] is what is usually called compact Hausdorff, and "Tihonov" in [25] is what is usually called completely regular, that is, points are closed and can be separated from closed sets by continuous functions.) We use without comment the standard identifications of tensor products with algebras of continuous functions: $C(X) \otimes A \cong C(X, A)$ for compact X and $C_0(X) \otimes A \cong C_0(X, A)$ for locally compact X. We denote by A_{sa} the set of selfadjoint elements of a C^*-algebra A, by $U(A)$ and $\mathrm{inv}(A)$ the groups of unitaries and invertible elements (when A is unital), and by $U_0(A)$ and $\mathrm{inv}_0(A)$ the identity components of these groups. The unitization of a C^*-algebra A is A^+. This means an identity is added even if one is already present. We write $\widetilde{A}$ for the algebra in which the identity is added only if A is nonunital.

1 Real Rank of $C_0(X) \otimes A$

The main result in this section is that if A is unital and X is compact, then $\mathrm{RR}\big(C(X) \otimes A\big) \leq \mathrm{RR}(A) + \dim(X)$. The various formulations involving spaces that are only locally compact and C^*-algebras without identities are then derived from this result by compactifying and passing to ideals.

The basic case is $X = [0,1]$, which is done by a direct argument. The case $X = [0,1]^n$ follows by induction, and the case of a finite complex follows by attaching cells. We pass to a general compact space X by realizing it as an approximate inverse limit of finite complexes with dimension at most $\dim(X)$, following Mardešić and Rubin [21]. (It is known that an exact realization of this type is not in general possible.)

Lemma 1.1 *Let A be a unital C^*-algebra with $\mathrm{RR}(A) = n$. For any $\varepsilon > 0$, $N \geq n$, and $a_0, a_1, \dots, a_N \in A_{\mathrm{sa}}$, there exist $b_0, b_1, \dots, b_N \in A_{\mathrm{sa}}$ such that $\|a_i - b_i\| < \varepsilon$ for $0 \leq i \leq N$ and $\sum_{j=k}^{k+n} b_j^2$ is invertible for $0 \leq k \leq N - n$.*

Proof We prove this statement by induction. In the case $N = n + 1$, it follows immediately from the assumption $\mathrm{RR}(A) = n$.

We assume that it is valid for N, and we prove it for $N+1$. Since $\mathrm{RR}(A) = n$, we can choose $c_0, c_1, \ldots, c_n \in A_{\mathrm{sa}}$ such that $\|a_i - c_i\| < \frac{\varepsilon}{2}$ for $0 \leq i \leq n$ and $c_0^2 + c_1^2 + \cdots + c_n^2$ is invertible. Therefore there exists a positive number δ such that

$$c_0^2 + c_1^2 + \cdots + c_n^2 \geq \delta \cdot 1_A.$$

Choose $\alpha > 0$ satisfying

$$(2 \max_{0 \leq i \leq n} \|c_i\| + \alpha)\alpha \leq \frac{\delta}{2n}.$$

By the induction assumption, there exist $b_1, b_2, \ldots, b_{N+1} \in A_{\mathrm{sa}}$ such that $\|c_i - b_i\| < \min(\alpha, \frac{\varepsilon}{2})$ for $1 \leq i \leq n$, $\|a_i - b_i\| < \min(\alpha, \frac{\varepsilon}{2})$ for $n+1 \leq i \leq N+1$, and $\sum_{j=k}^{k+n} b_j^2$ is invertible for $1 \leq k \leq N+1-n$. We set $b_0 = c_0$; then $\|a_i - b_i\| < \varepsilon$ for $0 \leq i \leq N+1$. Moreover, for $0 \leq i \leq n$,

$$\|b_i^2 - c_i^2\| \leq (\|b_i\| + \|c_i\|)\|b_i - c_i\| \leq (2 \max_{0 \leq i \leq n} \|c_i\| + \alpha)\alpha \leq \frac{\delta}{2n},$$

so we have

$$b_0^2 + b_1^2 + \cdots + b_n^2 \geq \frac{\delta}{2} \cdot 1_A.$$

This completes the proof. ■

Theorem 1.2 *Let A be a unital C^*-algebra. Then* $\mathrm{RR}\big(C([0,1]) \otimes A\big) \leq \mathrm{RR}(A) + 1$.

Proof Without loss of generality, we assume $\mathrm{RR}(A) = n < \infty$. Let $f_0, f_1, \ldots, f_{n+1}$ be selfadjoint elements in $C([0,1], A)$. Let $\varepsilon > 0$. By uniform continuity, there is $\delta > 0$ such that $\|f_i(s) - f_i(t)\| < \frac{\varepsilon}{3}$ whenever $|s - t| < \delta$ and $0 \leq i \leq n+1$. Choose a positive integer N such that $\frac{1}{N} < \delta$. Define $t_k = \frac{k}{N(n+2)}$. Note that $t_0 = 0$ and $t_{N(n+2)} = 1$. Define selfadjoint elements in A by

$$a_{k(n+2)+j} = f_j(t_{k(n+2)+j})$$

for $0 \leq k < N$ and $0 \leq j \leq n+1$. Further define

$$a_{N(n+2)+j} = f_j(1)$$

for $0 \leq j \leq n+1$. By Lemma 1.1, we can choose elements $b_0, b_1, \ldots, b_{N(n+2)+n+1}$ in A_{sa} such that $\|a_i - b_i\| < \frac{\varepsilon}{3}$ for all i, and such that $\sum_{j=k}^{k+n} b_j^2$ is invertible for $0 \leq k \leq N(n+2)+1$.

For each integer l, we define the function $h_l^{(0)} \colon \mathbb{R} \to [0,1]$ by

$$h_l^{(0)}(t) = \begin{cases} 0 & t \leq t_{l-n-2} \\ N(n+2)(t - t_{l-n-2}) & t_{l-n-2} \leq t \leq t_{l-n-1} \\ 1 & t_{l-n-1} \leq t \leq t_l \\ -N(n+2)(t - t_{l+1}) & t_l \leq t \leq t_{l+1} \\ 0 & t_{l+1} \leq t. \end{cases}$$

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

Thus $h_l^{(0)}$ is continuous, piecewise linear, equal to 1 on $[t_{l-n-1}, t_l]$, and equal to 0 on $(-\infty, t_{l-n-2}] \cup [t_{l+1}, \infty)$. Set $h_l = h_l^{(0)}|_{[0,1]}$. Define

$$S_j = \{l \in \mathbb{Z} : l = j \bmod (n+2)\}$$

and

$$S = \{l \in \mathbb{Z} : h_l = h_l^{(0)}|_{[0,1]} \neq 0\} = \{0, 1, \ldots, N(n+2) + n + 1\}.$$

Note that $\sum_{l \in S_j} h_l^{(0)}(t) = 1$ for $0 \leq j \leq n+1$ and $t \in \mathbb{R}$, so $\sum_{l \in S \cap S_j} h_l(t) = 1$ for $0 \leq j \leq n+1$ and $t \in [0,1]$.

We now define functions $g_0, g_1, \ldots, g_{n+1} \in \big(C([0,1]) \otimes A\big)_{\mathrm{sa}}$ by

$$g_j(t) = \sum_{l \in S \cap S_j} h_l(t) b_l$$

for $t \in [0,1]$. Note that if $l \in S \cap S_j$ then $a_l = f_j(s)$ for some $s \in \operatorname{supp}(h_l)$. Since the support of h_l has length at most $(n+3)/\big(N(n+2)\big) < 2/N$, the choice of N implies that $\|f_j(t) - a_l\| < 2\varepsilon/3$ whenever $h_l(t) \neq 0$. Therefore

$$\|g_j - f_j\| \leq \max_{l \in S} \|b_l - a_l\| + \sup\{\|a_l - f_j(t)\| : l \in S \cap S_j, t \in \operatorname{supp}(h_l)\} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Moreover, for any $t \in [0,1]$, there is k such that $h_k(t), h_{k+1}(t), \ldots, h_{k+n}(t)$ are all equal to 1. It follows that $n+1$ of the $n+2$ elements $g_j(t)$ are exactly the elements $b_k, b_{k+1}, \ldots, b_{k+n}$, whence

$$\sum_{j=0}^{n+1} g_j(t)^2 \geq \sum_{i=k}^{n+k} b_i^2.$$

The right hand side is invertible, so the left hand side is too. Thus $\sum_{j=0}^{n+1} g_j(t)^2$ is invertible. ∎

We now recall the definition of the pullback (or fibered product).

Definition 1.3 Let A, B, and C be C^*-algebras, and let $\varphi\colon A \to C$ and $\psi\colon B \to C$ be homomorphisms. Define

$$A \oplus_{(C,\varphi,\psi)} B = \{(a,b) \in A \oplus B : \varphi(a) = \psi(b)\}.$$

When φ and ψ are understood, we simply write $A \oplus_C B$.

Lemma 1.4 *Let X_0 be a compact Hausdorff space, and let $X = X_0 \cup_h D^n$ be the compact Hausdorff space obtained by attaching an n-cell D^n to X_0 via the attaching map $h\colon S^{n-1} \to X_0$. (Here S^{n-1} is the boundary of D^n.) Let A_0 be any C^*-algebra, set $A = C(X_0) \otimes A_0$, $B = C(D^n) \otimes A_0$, and $C = C(S^{n-1}) \otimes A_0$, and define $\varphi\colon A \to C$ and $\psi\colon B \to C$ by $\varphi(f) = f \circ h$ for $f\colon X_0 \to \mathbb{C}$ continuous and $\psi(f) = f|_{S^{n-1}}$ for $f\colon D^n \to \mathbb{C}$ continuous. Then*

$$A \oplus_{(C,\varphi,\psi)} B \cong C(X_0 \cup_h D^n) \otimes A_0.$$

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

Proof Let $\tilde{h}\colon D^n \to X_0 \cup_h D^n$ be the obvious extension of $h\colon S^{n-1} \to X_0$. Then the isomorphism $\alpha\colon C(X_0 \cup_h D^n, A_0) \to A \oplus_{(C,\varphi,\psi)} B$ is given by $\alpha(f) = (f|_{X_0}, f \circ \tilde{h})$. ■

We need a result on the real rank of pullbacks. The result is stated in Proposition 1.3 of [23], but the proof given there contains an error: too much surjectivity is assumed. We are grateful to Takashi Sakamoto for calling our attention to this.

The following lemma will be used in the proof of the estimate for the real rank of a pullback. It is well known, and we omit its proof. (Compare with the introduction to [10].)

Lemma 1.5 *Let A be a unital C^*-algebra, and let $a_0, a_1, \ldots, a_n \in A$. Then the following are equivalent:*

(1) There are $c_0, c_1, \ldots, c_n \in A$ such that $c_0a_0 + c_1a_1 + \cdots + c_na_n$ is invertible.
(2) There are $c_0, c_1, \ldots, c_n \in A$ such that $c_0a_0 + c_1a_1 + \cdots + c_na_n = 1$.
(3) $a_0^*a_0 + a_1^*a_1 + \cdots + a_n^*a_n$ is invertible.

Proposition 1.6 *Let A, B, and C be unital C^*-algebras, let $\varphi\colon A \to C$ be a unital homomorphism, and let $\psi\colon B \to C$ be a surjective unital homomorphism. Then*

$$\mathrm{RR}(A \oplus_C B) \le \max\bigl(\mathrm{RR}(A), \mathrm{RR}(B)\bigr).$$

Proof Without loss of generality $\mathrm{RR}(A)$ and $\mathrm{RR}(B)$ are finite. Let

$$n = \max\bigl(\mathrm{RR}(A), \mathrm{RR}(B)\bigr).$$

Let

$$(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n) \in (A \oplus_C B)_{\mathrm{sa}},$$

and let $\varepsilon > 0$. We approximate the (a_j, b_j) within ε by selfadjoint elements $(r_j, y_j) \in A \oplus_C B$ such that $\sum_{j=0}^{n}(r_j, y_j)^2$ is invertible.

Without loss of generality assume $\varepsilon < \frac{3}{4}$. Since $\mathrm{RR}(A) \le n$, there exist elements $r_0, r_1, \ldots, r_n \in A_{\mathrm{sa}}$ such that $\|r_j - a_j\| < \frac{\varepsilon}{3}$ and $\sum_{j=0}^{n} r_j^2$ is invertible. Since ψ is surjective, and $\|\varphi(r_j) - \psi(b_j)\| < \frac{\varepsilon}{3}$, there exist $t_0, t_1, \ldots, t_n \in B_{\mathrm{sa}}$ such that $\psi(t_j) = \varphi(r_j) - \psi(b_j)$ and $\|t_j\| < \frac{\varepsilon}{3}$. Set $w_j = b_j + t_j$, which is in B_{sa} and satisfies $\psi(w_j) = \varphi(r_j)$ and $\|w_j - b_j\| < \frac{\varepsilon}{3}$ for all j. Since

$$\sum_{j=0}^{n} \psi(w_j)^2 = \sum_{j=0}^{n} \varphi(r_j)^2$$

is invertible, we can apply Lemma 1.5 in C and use the surjectivity of ψ to find $f_0, f_1, \ldots, f_n \in B$ such that the element

$$d = 1 - \sum_{j=0}^{n} f_j w_j \in B$$

is in $\ker(\psi)$. Set

$$M_1 = \max(\|f_0\|, \|f_1\|, \dots, \|f_n\|).$$

Since $\mathrm{RR}(B) \leq n$, there are also $x_0, x_1, \dots, x_n \in B_{\mathrm{sa}}$ such that $\sum_{j=0}^{n} x_j^2$ is invertible and

$$\|x_j - w_j\| < \frac{\varepsilon}{3(n+1)(M_1+1)}.$$

Apply Lemma 1.5 again to find $g_0, g_1, \dots, g_n \in B$ such that

$$\sum_{j=0}^{n} g_j x_j = 1.$$

Set

$$M_2 = \max(\|g_0\|, \|g_1\|, \dots, \|g_n\|),$$

and set

$$N = \max(\|w_0\|, \|w_1\|, \dots, \|w_n\|, \|x_0\|, \|x_1\|, \dots, \|x_n\|).$$

Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $\ker(\psi)$ which is quasicentral for B. (See Theorem 1 of [3].) Choose λ large enough that $\|(1-e_\lambda)d\| < \frac{\varepsilon}{3}$. Define

$$r = d + e_\lambda\Big(1 - d - \sum_{j=0}^{n} g_j w_j\Big) \in \ker(\psi).$$

Choose $\delta > 0$ small enough that if D is a C^*-algebra, and if $s \in D_{\mathrm{sa}}$ and $z \in D$ satisfy

$$s \geq 0, \quad \|s\| \leq 1, \quad \|z\| \leq N+1, \quad \text{and} \quad \|sz - zs\| < \delta,$$

then

$$\|s^{1/2}z - zs^{1/2}\| < \frac{\varepsilon}{6(n+1)(M_1+M_2+1)}.$$

Choose μ large enough that

$$\|e_\mu w_j - w_j e_\mu\| < \delta, \quad \|e_\mu x_j - x_j e_\mu\| < \delta, \quad \text{and} \quad \|r(1-e_\mu)\| < \frac{\varepsilon}{3}.$$

Now define

$$y_j = (1-e_\mu)^{1/2} w_j (1-e_\mu)^{1/2} + e_\mu^{1/2} x_j e_\mu^{1/2} \in B_{\mathrm{sa}} \quad \text{and} \quad h_j = (1-e_\lambda) f_j + e_\lambda g_j \in B.$$

Clearly $\psi(y_j) = \psi(w_j) = \varphi(r_j)$. We will show below that $\|y_j - b_j\| < \varepsilon$, and that $\sum_{j=0}^{n} h_j y_j$ is invertible. Lemma 1.5 then implies that $\sum_{j=0}^{n} y_j^2$ is invertible, so that $\sum_{j=0}^{n} (r_j, y_j)^2$ is invertible, and we have $\|(r_j, y_j) - (a_j, b_j)\| < \varepsilon$. Thus, we will have shown that $\mathrm{RR}(A \oplus_C B) \leq \max\big(\mathrm{RR}(A), \mathrm{RR}(B)\big)$.

We start our proof of the required conditions with a preliminary estimate. Since $\|x_j\| \leq N+1$, the choices of δ and μ imply that

$$\|e_\mu^{1/2} x_j - x_j e_\mu^{1/2}\| < \frac{\varepsilon}{6(n+1)(M_1+M_2+1)}.$$

Therefore
$$\|e_\mu^{1/2} x_j e_\mu^{1/2} - x_j e_\mu\| < \frac{\varepsilon}{6(n+1)(M_1+M_2+1)}.$$
Also,
$$\|(1-e_\mu)w_j - w_j(1-e_\mu)\| = \|e_\mu w_j - w_j e_\mu\| < \delta,$$
so similarly
$$\|(1-e_\mu)^{1/2} w_j (1-e_\mu)^{1/2} - w_j(1-e_\mu)\| < \frac{\varepsilon}{6(n+1)(M_1+M_2+1)}.$$

Now we estimate $\|y_j - b_j\|$. Using the definition of y_j and the preliminary estimate, we have:
$$\begin{aligned}
\|y_j - b_j\| &\leq \|y_j - w_j\| + \|w_j - b_j\| \\
&< 2 \cdot \frac{\varepsilon}{6(n+1)(M_1+M_2+1)} + \|w_j(1-e_\mu) + x_j e_\mu - w_j\| + \|w_j - b_j\| \\
&< \frac{\varepsilon}{3} + \|(x_j - w_j)e_\mu\| + \|w_j - b_j\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3(n+1)(M_1+1)} + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

It remains to prove that $\sum_{j=0}^n h_j y_j$ is invertible. We calculate:
$$\begin{aligned}
\sum_{j=0}^n h_j w_j &= (1-e_\lambda)\sum_{j=0}^n f_j w_j + e_\lambda \sum_{j=0}^n g_j w_j = (1-e_\lambda)(1-d) + e_\lambda \sum_{j=0}^n g_j w_j \\
&= 1 - \Big[d + e_\lambda \Big(1 - d - \sum_{j=0}^n g_j w_j \Big) \Big] = 1 - r.
\end{aligned}$$

Also, since
$$\|x_j - w_j\| < \frac{\varepsilon}{3(n+1)(M_1+1)}, \quad \sum_{j=0}^n f_j w_j = 1 - d, \quad \text{and} \quad \sum_{j=0}^n g_j x_j = 1,$$
we have
$$\begin{aligned}
&\Big\| \sum_{j=0}^n h_j x_j - [1 - (1-e_\lambda)d] \Big\| \\
&\qquad = \Big\| (1-e_\lambda)\sum_{j=0}^n f_j x_j + e_\lambda \sum_{j=0}^n g_j x_j - [1-(1-e_\lambda)d] \Big\| \\
&\qquad = \Big\| (1-e_\lambda)\sum_{j=0}^n f_j(x_j - w_j) \Big\| \\
&\qquad < (n+1)\max_j \|f_j\| \cdot \frac{\varepsilon}{3(n+1)(M_1+1)} \leq \frac{\varepsilon}{3}.
\end{aligned}$$

Using our preliminary estimate above, we get

$$\begin{aligned}\|h_j y_j - [h_j w_j(1-e_\mu) + h_j x_j e_\mu]\| &\le 2 \cdot \frac{\|h_j\|\varepsilon}{6(n+1)(M_1+M_2+1)} \\ &\le \frac{(\|f_j\| + \|g_j\|)\varepsilon}{3(n+1)(M_1+M_2+1)} \le \frac{\varepsilon}{3(n+1)}.\end{aligned}$$

Therefore

$$\begin{aligned}\Big\| 1 - \sum_{j=0}^{n} h_j y_j \Big\| &< \frac{\varepsilon}{3} + \Big\| 1 - \Big(\sum_{j=0}^{n} h_j w_j\Big)(1-e_\mu) - \Big(\sum_{j=0}^{n} h_j x_j\Big) e_\mu \Big\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|1 - (1-r)(1-e_\mu) - [1-(1-e_\lambda)d]e_\mu\| \\ &= \frac{2\varepsilon}{3} + \|r(1-e_\mu) + (1-e_\lambda)de_\mu\| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < 1,\end{aligned}$$

where the last inequality follows from the assumption $\varepsilon < \frac{3}{4}$. So $\sum_{j=0}^{n} h_j y_j$ is invertible.

We have completed the proof. ■

Proposition 1.7 *Let A be a unital C^*-algebra, and let X be a finite* CW*-complex of dimension n. Then*

$$\mathrm{RR}\big(C(X) \otimes A\big) = \mathrm{RR}\big(C([0,1]^n) \otimes A\big).$$

Proof The algebra $C([0,1]^n) \otimes A$ is a quotient of $C(X) \otimes A$, so certainly

$$\mathrm{RR}\big(C(X) \otimes A\big) \ge \mathrm{RR}\big(C([0,1]^n) \otimes A\big).$$

The proof of the reverse inequality is by induction on the number of cells of X. If there is only one cell, it is a 0-cell, $n = 0$, and

$$\mathrm{RR}\big(C(X) \otimes A\big) = \mathrm{RR}(A) = \mathrm{RR}\big(C([0,1]^n) \otimes A\big).$$

For the induction step, we can assume X is obtained by adjoining an n-cell D^n to a finite CW-complex X_0 with $\dim(X_0) \le n$ and for which the result is already known to hold. By Lemma 1.4, we have

$$C(X) \otimes A \cong \big(C(X_0) \otimes A\big) \oplus_{C(S^{n-1}) \otimes A} \big(C(D^n) \otimes A\big).$$

Proposition 1.6 therefore implies that

$$\begin{aligned}\mathrm{RR}\big(C(X) \otimes A\big) &\le \max\Big(\mathrm{RR}\big(C(X_0) \otimes A\big), \mathrm{RR}\big(C(D^n) \otimes A\big)\Big) \\ &= \max\Big(\mathrm{RR}\big(C(X_0) \otimes A\big), \mathrm{RR}\big(C([0,1]^n) \otimes A\big)\Big) \\ &\le \mathrm{RR}\big(C([0,1]^n) \otimes A\big).\end{aligned}$$

■

We now pass from finite CW-complexes to compact Hausdorff spaces. For this, we use the notion of an approximate inverse system of compact metric spaces, due to Mardešić and Rubin [21, Definition 1]. An approximate inverse system of compact metric spaces consists of a directed set Λ with no maximal element, for each $\lambda \in \Lambda$ a compact metric space X_λ with metric d_λ and a real number $\varepsilon_\lambda > 0$, and for each $\lambda, \lambda' \in \Lambda$ with $\lambda \leq \lambda'$ a not necessarily continuous function $p_{\lambda\lambda'} \colon X_{\lambda'} \to X_\lambda$. Moreover, the following conditions must be satisfied:

(1) $d_{\lambda_1}\big(p_{\lambda_1\lambda_2} \circ p_{\lambda_2\lambda_3}(x), p_{\lambda_1\lambda_3}(x)\big) \leq \varepsilon_{\lambda_1}$ for $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and $x \in X_{\lambda_3}$.
(2) $p_{\lambda\lambda} = \mathrm{id}$ for all λ.
(3) For all $\lambda \in \Lambda$ and all $\eta > 0$ there is $\lambda' \geq \lambda$ such that for all $\lambda_2 \geq \lambda_1 > \lambda'$ and all $x \in X_{\lambda_2}$, we have $d_\lambda\big(p_{\lambda\lambda_1} \circ p_{\lambda_1\lambda_2}(x), p_{\lambda\lambda_2}(x)\big) \leq \eta$.
(4) For all $\lambda \in \Lambda$ and all $\eta > 0$, there is $\lambda' \geq \lambda$ such that for all $\lambda'' \geq \lambda'$ and all $x, x' \in X_{\lambda''}$, if $d_{\lambda''}(x, x') \leq \varepsilon_{\lambda''}$ then $d_\lambda\big(p_{\lambda\lambda''}(x), p_{\lambda\lambda''}(x')\big) \leq \eta$.

The (inverse) limit [21, Definition 2] $X = \lim(X_\lambda, \varepsilon_\lambda, p_{\lambda\lambda'}, \Lambda)$ is the subspace of $\prod_{\lambda\in\Lambda} X_\lambda$ defined by

$$X = \Big\{ x = (x_\lambda) \in \prod_{\lambda\in\Lambda} X_\lambda : x_\lambda = \lim_{\lambda' \geq \lambda} p_{\lambda,\lambda'}(x_{\lambda'}) \text{ for all } \lambda \in \Lambda \Big\},$$

with the relative product topology. (See also Theorem 2 of [21].)

Lemma 1.8 *Let $(X_\lambda, \varepsilon_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an approximate inverse system of compact metric spaces, with limit X. Let $p_\lambda \colon X \to X_\lambda$ be the restriction to X of the projection $\prod_{\lambda\in\Lambda} X_\lambda \to X_\lambda$. Let A be a C^*-algebra, and let $\alpha_\lambda \colon C(X_\lambda) \otimes A \to C(X) \otimes A$ be given by $\alpha_\lambda(f) = f \circ p_\lambda$. Then for any $f_1, f_2, \ldots, f_n \in C(X) \otimes A$ and any $\varepsilon > 0$, there exist $\lambda \in \Lambda$ and $g_1, g_2, \ldots, g_n \in C(X_\lambda) \otimes A$ such that $\|\alpha_\lambda(g_m) - f_m\| < \varepsilon$ for $1 \leq m \leq n$.*

Proof Choose an open cover $\{U_1, U_2, \ldots, U_k\}$ of X and, for each m, elements $a_1^{(m)}, a_2^{(m)}, \ldots, a_k^{(m)} \in A$ such that $\|f_m(x) - a_i^{(m)}\| < \varepsilon$ whenever $x \in U_i$. By Theorem 3 of [21], there exist $\lambda \in \Lambda$ and an open cover $\{V_1, V_2, \ldots, V_l\}$ of X_λ such that $\{p_\lambda^{-1}(V_1), p_\lambda^{-1}(V_2), \ldots, p_\lambda^{-1}(V_l)\}$ refines $\{U_1, U_2, \ldots, U_k\}$. That is, for any $i \in \{1, 2, \ldots, l\}$ there exists $j(i) \in \{1, 2, \ldots, k\}$ such that $p_\lambda^{-1}(V_i) \subset U_{j(i)}$. Let $\{h_i : 1 \leq i \leq l\}$ be a partition of unity subordinate to $\{V_i\}$. (We can take $\mathrm{supp}(h_i) \subset V_i$ by choosing an arbitrary partition of unity subordinate to $\{V_i\}$, assigning to each function in it some i such that V_i contains its support, and taking h_i to be the sum of all the functions assigned to i.) Note that $\{h_i \circ p_\lambda\}$ is a partition of unity subordinate to the open cover $\{p_\lambda^{-1}(V_1), p_\lambda^{-1}(V_2), \ldots, p_\lambda^{-1}(V_l)\}$ of X. Now define $g_m \in C(X_\lambda) \otimes A$ by

$$g_m(x) = \sum_{i=1}^{l} h_i(x) a_{j(i)}^{(m)}$$

for $x \in X_\lambda$. Then for $x \in X$ we have

$$\big\| g_m\big(p_\lambda(x)\big) - f_m(x)\big\| \leq \sum_{i=1}^{l} h_i\big(p_\lambda(x)\big) \|a_{j(i)}^{(m)} - f_m(x)\| < \varepsilon,$$

since $\|a_j^{(m)} - f_m(x)\| < \varepsilon$ whenever $x \in U_j$. This shows that $\|\alpha_\lambda(g_m) - f_m\| < \varepsilon$. ■

In the following results, $\widetilde{A}$ denotes A if A is unital and the unitization A^+ of A if A is not unital. By definition, we have $\mathrm{RR}(A) = \mathrm{RR}(\widetilde{A})$.

Theorem 1.9 *Let X be a locally compact Hausdorff space and let βX be its Stone-Čech compactification. Let $n = \dim(\beta X)$. Then for any C^*-algebra A we have*

$$\mathrm{RR}\big(C_0(X) \otimes A\big) \le \mathrm{RR}\big(C([0,1]^n) \otimes \widetilde{A}\big) \le \mathrm{RR}(A) + \dim(\beta X).$$

Proof The inequality $\mathrm{RR}\big(C([0,1]^n) \otimes \widetilde{A}\big) \le \mathrm{RR}(\widetilde{A}) + \dim(\beta X)$ follows from Theorem 1.2. Since $\mathrm{RR}(A) = \mathrm{RR}(\widetilde{A})$, this gives the second half of the inequality. For the first half of the inequality, we first assume X is compact and A is unital. In this case, $\dim(\beta X) = \dim(X)$.

By Theorem 5 of [21], there exists an approximate inverse system of compact metric spaces $(X_\lambda, \varepsilon_\lambda, p_{\lambda\lambda'}, \Lambda)$, with limit X, such that each X_λ is a polyhedron (and thus in particular a finite CW-complex) of dimension at most n. It follows from Proposition 1.7 that $\mathrm{RR}\big(C(X_\lambda) \otimes A\big) \le \mathrm{RR}\big(C([0,1]^n) \otimes A\big)$.

Let $N = \mathrm{RR}\big(C([0,1]^n) \otimes A\big)$, let $a_0, a_1, \dots, a_N \in \big(C(X) \otimes A\big)_{\mathrm{sa}}$, and let $\varepsilon > 0$. By Lemma 1.8, there is $\lambda \in \Lambda$, a unital homomorphism $\alpha_\lambda \colon C(X_\lambda) \otimes A \to C(X) \otimes A$, and $b_0, b_1, \dots, b_N \in C(X_\lambda) \otimes A$, such that $\|\alpha_\lambda(b_j) - a_j\| < \frac{\varepsilon}{2}$ for $0 \le j \le N$. Replacing b_j by $\frac{1}{2}(b_j + b_j^*)$, we may assume each b_j is selfadjoint without increasing $\|\alpha_\lambda(b_j) - a_j\|$. By Proposition 1.7, there are $c_0, c_1, \dots, c_N \in (C(X_\lambda) \otimes A)_{\mathrm{sa}}$ such that $\|c_j - b_j\| < \frac{\varepsilon}{2}$ for $0 \le j \le N$ and such that $\sum_{j=0}^N c_j^2$ is invertible. Then the elements $\alpha_\lambda(c_0), \alpha_\lambda(c_1), \dots, \alpha_\lambda(c_N)$ are in $\big(C(X) \otimes A\big)_{\mathrm{sa}}$, and satisfy $\|\alpha_\lambda(c_j) - a_j\| < \varepsilon$ and $\sum_{j=0}^N \alpha_\lambda(c_j)^2$ is invertible. This proves that $\mathrm{RR}\big(C(X) \otimes A\big) \le N$.

Now we consider the first half of the inequality for general A. The case we have already done gives

$$\mathrm{RR}\big(C(\beta X) \otimes \widetilde{A}\big) \le \mathrm{RR}\big(C([0,1]^n) \otimes \widetilde{A}\big).$$

Moreover, the real rank of an ideal is at most the real rank of the algebra containing it, by Theorem 1.4 of [11]. So

$$\mathrm{RR}\big(C_0(X) \otimes A\big) \le \mathrm{RR}\big(C(\beta X) \otimes \widetilde{A}\big) \le \mathrm{RR}\big(C([0,1]^n) \otimes \widetilde{A}\big). \qquad ■$$

We could use any other compactification in place of βX and obtain the same estimate.

The theorem is the best possible for general C^*-algebras, as is seen by taking $A = C([0,1]^m)$.

Corollary 1.10 *Let X be a normal locally compact Hausdorff space (in particular, a σ-compact locally compact Hausdorff space), and let $n = \dim(X)$. Then for any C^*-algebra A we have*

$$\mathrm{RR}\big(C_0(X) \otimes A\big) \le \mathrm{RR}\big(C([0,1]^n) \otimes \widetilde{A}\big) \le \mathrm{RR}(A) + \dim(X).$$

Proof For a normal space X, we have $\dim(\beta X) = \dim(X)$ by Proposition 6.4.3 or Corollary 10.1.7 of [25]. ■

In Remark 6.4.6 of [25] there is a locally compact Hausdorff space X satisfying $\dim(X) = 1$ and $\dim(\beta X) = 0$. Theorem 1.4 of [11] implies that $\mathrm{RR}\big(C_0(X)\big) \leq \mathrm{RR}\big(C(\beta X)\big)$. Therefore, for this X, we get $\mathrm{RR}\big(C_0(X)\big) = 0 < \dim(X)$.

We don't know if the opposite inequality $\dim(\beta X) > \dim(X)$ can occur.

It turns out that the correct notion of dimension to use in this context is the *modified covering dimension*, which we denote by $\dim_0(X)$. It is defined by restricting the open sets in the usual definition of covering dimension (Definition 3.1.1 of [25]) to be complements of zero sets of continuous real valued functions on the space. See Definition 10.1.3 of [25], where it is called $\partial\mathrm{im}(X)$. We have $\dim_0(X) = \dim(X)$ for normal X (Proposition 10.1.6 of [25]) and $\dim(\beta X) = \dim_0(X)$ for any completely regular X (Theorem 10.1.4 of [25]). In particular, we have the following corollary.

Corollary 1.11 *Let X be a locally compact Hausdorff space and let $n = \dim_0(X)$. Then for any C^*-algebra A we have*

$$\mathrm{RR}\big(C_0(X) \otimes A\big) \leq \mathrm{RR}\big(C([0,1]^n) \otimes \widetilde{A}\big) \leq \mathrm{RR}(A) + \dim_0(X).$$

Letting X^+ denote the one point compactification of X, using Theorem 1.4 of [11] on $C_0(X) \otimes A$ as an ideal in $C(X^+) \otimes \widetilde{A}$, and noting that $\mathrm{RR}\big(C_0(X)\big) = \mathrm{RR}\big(C_0(X)^+\big) = \mathrm{RR}\big(C(X^+)\big)$, we also obtain the following corollary:

Corollary 1.12 *Let X be a locally compact Hausdorff space and let $n = \mathrm{RR}\big(C_0(X)\big)$. Then for any C^*-algebra A we have*

$$\mathrm{RR}\big(C_0(X) \otimes A\big) \leq \mathrm{RR}\big(C([0,1]^n) \otimes \widetilde{A}\big) \leq \mathrm{RR}(A) + \mathrm{RR}\big(C_0(X)\big).$$

All the steps in the proof of Theorem 1.9, in the lemmas leading up to it, and in the corollaries, go through just as easily for the stable rank in place of the real rank. (The proofs require the result that if J is an ideal in a C^*-algebra B, then $\mathrm{sr}(J) \leq \mathrm{sr}(B)$. This is Theorem 4.4 of [31].) We state the final result, and, for later use, a special case of the analog of Proposition 1.7. We omit the analogs of the corollaries. Note, however, that for stable rank this is presumably not the best possible: one hopes that $\mathrm{sr}\big(C([0,1]^2) \otimes A\big) \leq \mathrm{sr}(A) + 1$ for any A. (Compare with Question 1.8 of [31].) We also point out that the analog for stable rank of Theorem 1.2 has already been proved by Rieffel (Corollary 7.2 of [31]), using different methods.

Theorem 1.13 *Let X be a locally compact Hausdorff space and let $n = \dim(\beta X)$. Then for any C^*-algebra A we have*

$$\mathrm{sr}\big(C_0(X) \otimes A\big) \leq \mathrm{sr}\big(C([0,1]^n) \otimes \widetilde{A}\big) \leq \mathrm{sr}(A) + \dim(\beta X).$$

Lemma 1.14 *Let A be a unital C^*-algebra. Then*

$$\mathrm{sr}\big(C([0,1]) \otimes A\big) = \mathrm{sr}\big(C(S^1) \otimes A\big).$$

Proof The proof is the same as for Proposition 1.7. ■

2 Hereditary Subalgebras of Algebras of Continuous Functions

The results of this section will play an important technical role in the estimation of $\mathrm{RR}\big(C(X)\otimes A\big)$ when A is purely infinite simple.

In this section, we will essentially always regard elements of $C(X)\otimes A$ as A-valued functions on X. We therefore write $C(X, A)$ rather than $C(X)\otimes A$.

Let A be a separable purely infinite simple C^*-algebra. If D is a hereditary subalgebra of A, then D is either unital or stable (by Theorem 1.2 of [34]). Now let X be a compact Hausdorff space, and let D be a hereditary subalgebra of $C(X, A)$. It is easy to construct examples in which D is neither unital nor stable, by arranging to have the images of D under the point evaluation maps $\mathrm{ev}_x\colon C(X, A)\to A$ be sometimes unital and sometimes stable. In this section we show that when D is full, and at least for $X\subset[0,1]^n$ closed, this is the worst that can happen. Specifically, if every image $\mathrm{ev}_x(D)$ is nonzero and nonunital, then D is stable. This generalizes a result implicit in Section 2 of [18] (see the remark before Proposition 2.6 of [18]), which is the case $X=[0,1]$. The proof we give here is essentially an adaptation of the methods there to an induction argument on n for the case $X=[0,1]^n$.

Presumably the restriction $X\subset[0,1]^n$ can be dropped by a direct limit argument. However, for the purposes of this paper it is more convenient to restrict to subsets of $[0,1]^n$ until the end of the proof of the estimate for $\mathrm{RR}\big(C(X)\otimes A\big)$.

Notation 2.1 **([18])** If B is a C^*-algebra, X is a compact Hausdorff space, and D is a hereditary subalgebra of $C(X, B)$, we let $\mathrm{ev}_x\colon C(X, B)\to B$ be the evaluation map at $x\in X$, and define

$$D_x=\mathrm{ev}_x(D)=\{b(x): b\in D\}.$$

Recall (see Lemma 2.3 of [18]) that each D_x is a hereditary subalgebra of B, and that if $b\in C(X,B)$ satisfies $b(x)\in D_x$ for all $x\in X$, then $b\in D$.

We now give three important lemmas on projections.

Lemma 2.2 *Let X, B, and D be as in Notation 2.1, and let $x_0\in X$. Let $p_0\in D_{x_0}$ be a nonzero projection. Then there exists an open set U containing x_0 and a continuous projection valued function $p\colon U\to B$ such that $p(x_0)=p_0$.*

Proof Choose $b\in D_{\mathrm{sa}}$ such that $b(x_0)=p_0$. Set $U=\left\{x\in X:\frac{1}{2}\notin\mathrm{sp}\big(b(x)\big)\right\}$. For $x\in U$, set $p(x)=\chi_{(\frac{1}{2},\infty)}\big(b(x)\big)$. ■

Lemma 2.3 *For every $\varepsilon>0$ and positive integer d, there is $\delta>0$ such that the following holds. Let X be compact Hausdorff, let A be a C^*-algebra, let $U_1,\dots,U_n$ be open sets which cover X and such that any distinct $d+2$ of the U_j have empty intersection, and let $p_j\colon U_j\to A$ be projection valued continuous functions such that $\|p_j(x)p_k(x)\|<\delta$ for $j\neq k$ and $x\in U_j\cap U_k$. Then there exist open sets $V_j\subset U_j$ which cover X, and projection valued continuous functions $q_j\colon V_j\to A$ such that $q_j(x)q_k(x)=0$ for $j\neq k$ and $x\in V_j\cap V_k$, such that $\|q_j(x)-p_j(x)\|<\varepsilon$ for $x\in V_j$, and such that for every x and j, the projection $q_j(x)$ is in the C^*-subalgebra of A generated by those $p_k(x)$ for which $x\in U_k$.*

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

Proof Set $\varepsilon_{d+1} = \varepsilon$. Given $\varepsilon_{r+1} > 0$, choose $\varepsilon_r > 0$ so that $\varepsilon_r \leq \varepsilon_{r+1}$ and such that if e is a projection in a C^*-algebra A and $a \in A_{\mathrm{sa}}$ satisfies $\|a - e\| < 5 \cdot 4^r \cdot \varepsilon_r$, then $\frac{1}{2} \notin \mathrm{sp}(a)$ and $\|\chi_{(\frac{1}{2},\infty)}(a) - e\| < \varepsilon_{r+1}$. Choose $\delta = \varepsilon_1$.

Choose open sets V_j with $V_j \subset \overline{V_j} \subset U_j$ and such that the V_j still cover X. Further choose open sets $U_j^{(k)}$ with

$$V_j = U_j^{(n)} \subset \overline{U_j^{(n)}} \subset U_j^{(n-1)} \subset \overline{U_j^{(n-1)}} \subset \cdots \subset U_j^{(1)} \subset \overline{U_j^{(1)}} \subset U_j^{(0)} = U_j.$$

We now construct families of projection valued functions $q_1^{(k)}, \ldots, q_k^{(k)}$, by induction on k, which satisfy:

(1) $q_j^{(k)} \colon \overline{U_j^{(k)}} \to A$ is continuous.
(2) $q_i^{(k)}(x) q_j^{(k)}(x) = 0$ for $i \neq j$ and $x \in \overline{U_i^{(k)}} \cap \overline{U_j^{(k)}}$.
(3) $\|q_j^{(k)}(x) - p_j(x)\| < \varepsilon_r$ whenever x is in no more than r of the sets $U_1, \ldots, U_k$.
(4) $q_j^{(k)}(x)$ is in the C*-subalgebra of A generated by those $p_l(x)$ for which $x \in U_l$.

We start the induction by taking $q_1^{(1)}$ to be the restriction of p_1 to $\overline{U_1^{(1)}}$.

Given $q_1^{(k)}, \ldots, q_k^{(k)}$ satisfying the properties above, we let $q_j^{(k+1)}$ be the restriction of $q_j^{(k)}$ to $\overline{U_j^{(k+1)}}$ for $1 \leq j \leq k$. The construction of $q_{k+1}^{(k+1)}$ requires a further induction.

Choose continuous functions $f_1, \ldots, f_k \colon X \to [0,1]$ such that $f_j = 1$ on $\overline{U_j^{(k+1)}}$ and $f_j = 0$ outside the set $U_j^{(k)}$. For $x \in \overline{U_{k+1}^{(k+1)}}$ successively define $a_0(x) = p_{k+1}(x)$, and

$$a_j(x) = f_j(x)\big(1 - q_j^{(k)}(x)\big)\, a_{j-1}(x)\big(1 - q_j^{(k)}(x)\big) + \big(1 - f_j(x)\big)\, a_{j-1}(x).$$

Then

$$\|a_{j+1}(x)\| \leq \|a_j(x)\| \leq \cdots \leq \|a_0(x)\| \leq 1.$$

Note that if $x \notin U_{j+1}^{(k)}$ then $a_{j+1}(x) = a_j(x)$. If $x \in U_{j+1}^{(k)}$, then

$$\begin{aligned}
\|a_{j+1}(x) - a_j(x)\| &\leq \Big\| \big(1 - q_{j+1}^{(k)}(x)\big)\, a_j(x) \big(1 - q_{j+1}^{(k)}(x)\big) - a_j(x) \Big\| \\
&\leq 2\|q_{j+1}^{(k)}(x) - p_{j+1}(x)\| + 2\|a_j(x) - p_{k+1}(x)\| \\
&\qquad + \Big\| \big(1 - p_{j+1}(x)\big)\, p_{k+1}(x) \big(1 - p_{j+1}(x)\big) - p_{k+1}(x) \Big\| \\
&\leq 2\|q_{j+1}^{(k)}(x) - p_{j+1}(x)\| + 2\|a_j(x) - p_{k+1}(x)\| \\
&\qquad + 3\|p_{j+1}(x) p_{k+1}(x)\|.
\end{aligned}$$

Therefore

$$\|a_{j+1}(x) - p_{k+1}(x)\| < 2\|q_{j+1}^{(k)}(x) - p_{j+1}(x)\| + 3\|a_j(x) - p_{k+1}(x)\| + 3\delta.$$

Suppose now that $x \in U_{k+1}$ is in exactly r of the sets $U_1, \ldots, U_k$. Then $r \leq d$ by hypothesis. We have $a_{j+1}(x) \neq a_j(x)$ for at most r values of j, say $j(1) < j(2) < \cdots < j(s)$ with $s \leq r$. We have $\|q_{j+1}^{(k)}(x) - p_{j+1}(x)\| < \varepsilon_r$ for these values of j. Therefore

$$\|a_{j(1)+1}(x) - p_{k+1}(x)\| < 2\varepsilon_r + 3\delta,$$

$$\|a_{j(2)+1}(x) - p_{k+1}(x)\| < 2\varepsilon_r + 3(2\varepsilon_r + 3\delta) + 3\delta = 4(2\varepsilon_r + 3\delta),$$

$$\|a_{j(3)+1}(x) - p_{k+1}(x)\| < 2\varepsilon_r + 3\big(4(2\varepsilon_r + 3\delta)\big) + 3\delta \leq (4^2)(2\varepsilon_r + 3\delta),$$

etc. So

$$\|a_k(x) - p_{k+1}(x)\| < (4^s)(2\varepsilon_r + 3\delta) \leq (4^r)(5\varepsilon_r).$$

It follows from the choice of ε_r that

$$q_{k+1}^{(k+1)}(x) = \chi_{(\frac{1}{2},\infty)}\big(a_k(x)\big)$$

is defined and satisfies

$$\|q_{k+1}^{(k+1)}(x) - p_{k+1}(x)\| < \varepsilon_{r+1}.$$

This defines $q_{k+1}^{(k+1)}$, and verifies (3) of the induction hypothesis for $k+1$ for those x which are in U_{k+1}. If $x \notin U_{k+1}$, then (3) for $k+1$ follows immediately from (3) for k. For (2), we note that, because any two of the projections $q_j^{(k)}(x)$ are orthogonal where both are defined, it follows that

$$q_j^{(k)}(x)a_k(x) = a_k(x)q_j^{(k)}(x) = 0$$

whenever $x \in \overline{U_j^{(k+1)}}$. Therefore

$$q_j^{(k)}(x)q_{k+1}^{(k+1)}(x) = q_{k+1}^{(k+1)}(x)q_j^{(k)}(x) = 0$$

whenever $x \in \overline{U_j^{(k+1)}}$. Conditions (1) and (4) are immediate.

We have completed the inductive construction of the projections $q_1^{(k)}, \ldots, q_k^{(k)}$ for $1 \leq k \leq n$. The proof of the lemma is completed by taking $q_j = q_j^{(n)}|_{V_j}$ for $1 \leq j \leq n$. ■

Lemma 2.4 *Let A be a C^*-algebra, let $a, h \in A$ be selfadjoint elements with $0 \leq a \leq 1$ and $0 \leq h \leq 1$, and let $q \in A$ be a projection. Then $\|qa - q\| \leq 12\|qhah - q\|^{1/3}$.*

Proof Represent A faithfully on a Hilbert space H. Note that $\|qa - q\| = \|aq - q\|$ and $\|qhah - q\| = \|hahq - q\|$. It suffices to show that if $\xi \in qH$, then $\|a\xi - \xi\| \leq 12\|ha h\xi - \xi\|^{1/3}$.

We first claim that if $b \in L(H)$ satisfies $0 \leq b \leq 1$ and $\eta \in H$ satisfies $\|\eta\| = 1$, then $\|b\eta - \eta\| \leq 4(1 - \|b\eta\|)^{1/3}$. To see this, set $\delta = 1 - \|b\eta\|$, let $\rho = \delta^{1/3}$, and let $p \in L(H)$ be the spectral projection for b corresponding to $[1-\rho, 1]$. Then

$$\begin{aligned}(1-\delta)^2 = \|b\eta\|^2 &= \|bp\eta\|^2 + \|b(1-p)\eta\|^2 \leq \|p\eta\|^2 + (1-\rho)^2\|(1-p)\eta\|^2 \\ &= 1 - \|(1-p)\eta\|^2 + (1-\rho)^2\|(1-p)\eta\|^2 = 1 - \rho(2-\rho)\|(1-p)\eta\|^2.\end{aligned}$$

It follows that

$$\|(1-p)\eta\| \leq \sqrt{\left(\frac{2-\delta}{2-\rho}\right)\left(\frac{\delta}{\rho}\right)} \leq \sqrt{2}\delta^{1/3}.$$

So

$$\|b\eta - \eta\| \leq \|b\|\,\|(1-p)\eta\| + \|bp\eta - p\eta\| + \|(1-p)\eta\| \leq 2\sqrt{2}\delta^{1/3} + \rho < 4\delta^{1/3}.$$

This proves the claim.

Now let $\xi \in qH$ satisfy $\|\xi\| = 1$. Since $\|a\|, \|h\| \leq 1$, we have

$$\|h\xi\| \geq \|hah\xi\| \geq 1 - \|hah\xi - \xi\|.$$

Applying the claim to h and ξ, we get $\|h\xi - \xi\| \leq 4\|hah\xi - \xi\|^{1/3}$. If $h\xi = 0$, then $\|a\xi - \xi\| \leq 12\|hah\xi - \xi\|^{1/3}$ for trivial reasons. Otherwise, with $\eta = \frac{1}{\|h\xi\|}h\xi$, we have

$$\|a\eta\| \geq \frac{1}{\|h\xi\|}\|hah\xi\| \geq \frac{1}{\|h\xi\|}(1 - \|hah\xi - \xi\|) \geq 1 - \|hah\xi - \xi\|,$$

so

$$\|ah\xi - h\xi\| = \|h\xi\|\,\|a\eta - \eta\| \leq \|h\xi\| \cdot 4\|hah\xi - \xi\|^{1/3} \leq 4\|hah\xi - \xi\|^{1/3}.$$

Therefore

$$\|a\xi - \xi\| \leq \|a\|\,\|\xi - h\xi\| + \|ah\xi - h\xi\| + \|h\xi - \xi\| \leq 12\|hah\xi - \xi\|^{1/3}. \qquad \blacksquare$$

Lemma 2.5 *Let A be a purely infinite simple C^*-algebra, let $p_1, \ldots, p_n \in A$ be nonzero projections, and let $\varepsilon > 0$. Then:*

(1) There exist mutually orthogonal nonzero projections $q_1, \ldots, q_n \in A$ such that $\|p_k q_k - q_k\| < \varepsilon$ for $1 \leq k \leq n$.
(2) There exist nonzero projections $e_1 \leq p_1, \ldots, e_n \leq p_n$ such that $\|e_j e_k\| < \varepsilon$ for $j \neq k$.

Proof We prove part (1). Choose an irreducible representation π of A on a Hilbert space H. By induction, we construct a sequence $\xi_1, \ldots, \xi_n$ of orthogonal unit vectors in H with $\pi(p_j)\xi_j = \xi_j$ for $1 \leq j \leq n$. Choose ξ_1 to be any unit vector in $\pi(p_1)H$. Suppose we are given $\xi_1, \ldots, \xi_j$. The subspace $\pi(p_{j+1})H$ is infinite dimensional since $\pi(A)$ contains no compact operators. Therefore it must nontrivially intersect the finite codimension subspace $\operatorname{span}(\xi_1, \ldots, \xi_j)^{\perp}$, and we take ξ_{j+1} to be any unit vector in the intersection.

Let $e_j \in L(H)$ be the projection onto $\mathbb{C}\xi_j$, and let $e = e_1 + \cdots + e_n$ be the projection onto $\operatorname{span}(\xi_1, \ldots, \xi_n)$. Let

$$L = \{a \in A : \pi(a)e = 0\} \quad \text{and} \quad N = \{a \in A : \pi(a)e = e\pi(a)\}.$$

Then L is a left ideal of A, N is a C*-subalgebra of A, and $L \cap L^*$ is an ideal in N. Define a unital completely positive map $T\colon A \to L(eH) \cong M_n$ by $T(a) = e\pi(a)e$. Then $T|_N$ is a homomorphism with kernel $L \cap L^*$. The Kadison Transitivity Theorem implies that it is surjective. Indeed, let $u \in L(eH)$ be unitary. Since π is injective, Theorem 5.4.5 of [15] provides a unitary $v \in \widetilde{A}$ such that $e\pi(v)e = u$. Since $\pi(v)$ and $e\pi(v)e$ are both unitary, e must commute with $\pi(v)$. So $v \in \widetilde{N}$. If we let $\widetilde{T}\colon \widetilde{N} \to L(eH)$ be the unitization of $T|_N$, then $\widetilde{T}(v) = u$. This shows that the image of $\widetilde{N}$ contains all unitaries in $L(eH)$, and so is all of $L(eH)$. The image of N is an ideal of codimension at most 1. We may clearly assume $n \geq 2$; then $L(eH)$ has no proper ideals of codimension at most 1, so $T|_N$ must be surjective.

By Theorem 4.6 of [19] (essentially Proposition 2.6 of [1]), there are $h_1, \ldots, h_n \in N$ satisfying $T(h_j) = e_j$, $0 \leq h_j \leq 1$, and $h_j h_k = 0$ for $j \neq k$. Then

$$\|h_j p_j h_j\| \geq \|e\pi(h_j)\pi(p_j)\pi(h_j)e\| = \|e_j \pi(p_j) e_j\| = \langle \pi(p_j)\xi_j, \xi_j \rangle \geq 1,$$

so $\|h_j p_j h_j\| = 1$. The proof of Lemma 1.7 of [6] provides a projection $q_j \in \overline{h_j A h_j}$ such that $\|q_j h_j p_j h_j - q_j\| < (\varepsilon/12)^3$. Since $h_j h_k = 0$ for $j \neq k$, we also have $q_j q_k = 0$ for $j \neq k$. Moreover, $\|q_j p_j - q_j\| < \varepsilon$ by the previous lemma.

Part (2) is an easy consequence of part (1) (using a different value of ε). ■

We will combine the following statement (quoted here for easy reference) with an orthogonality trick (made possible by the previous lemma) to produce elements in a full hereditary subalgebra of $C(X, A)$ whose spectrum at every point in X contains $[0, 1]$.

Lemma 2.6 **([2])** *Let A be a simple C^*-algebra which is not isomorphic to $K(H)$ for any Hilbert space H. Then there is $a \in A_{\mathrm{sa}}$ such that* $\mathrm{sp}(a) = [0, 1]$.

Proof See p. 61 of [2]. ■

Lemma 2.7 *Let A be a purely infinite simple C^*-algebra, let X be a finite dimensional compact metric space, and let D be a hereditary subalgebra of $C(X, A)$ such that $D_x \neq 0$ for all $x \in X$. Then there exists $c \in D_{\mathrm{sa}}$ such that $\mathrm{sp}\big(c(x)\big) = [0, 1]$ for all $x \in X$.*

Proof Let $d = \dim(X)$. Choose δ for this value of d and for $\varepsilon = \frac{1}{4}$ in Lemma 2.3. Also require $\delta < 1$. Choose $\rho > 0$ so that if $e_1, e_2, f \in A$ are projections satisfying $f \leq e_2$ and $\|e_1 - e_2\| < 2\rho$, then $\|f - \chi_{(\frac{1}{2}, \infty)}(e_1 f e_1)\| < \frac{1}{3}\delta$. For each $x \in X$ choose a nonzero projection $p_x \in D_x$, and use Lemma 2.2 to find an open set U_x containing x and a continuous projection valued function $q_x\colon U_x \to A$ with $q_x(x) = p_x$ and $q_x(y) \in D_y$ for all $y \in U_x$. Since q_x is continuous, we may assume, by reducing the size of U_x, that $\|q_x(y) - q_x(x)\| < \rho$ for all $y \in U_x$. Since $\dim(X) = d$ and X is compact, there is a finite open cover $V_1, \ldots, V_n$ of X which refines the cover $\{U_x : x \in X\}$ and such that any $d+2$ of the sets V_j have empty intersection. Define e_j to be the restriction to V_j of some q_x for which $V_j \subset U_x$. Note that $\|e_j(x) - e_j(y)\| < 2\rho$

for all $x, y \in V_j$. Choose $x_j \in V_j$, use Lemma 2.5 (2) to choose nonzero projections $f_j^{(0)} \leq e_j(x_j)$ such that $\|f_j^{(0)} f_k^{(0)}\| < \frac{1}{3}\delta$ for $j \neq k$, and define $f_j\colon V_j \to A$ by

$$f_j(x) = \chi_{(\frac{1}{2},\infty)}\big(e_j(x) f_j^{(0)} e_j(x)\big).$$

Then $\|f_j(x) - f_j^{(0)}\| < \frac{1}{3}\delta$, so for $j \neq k$ and $x \in V_j \cap V_k$ we have $\|f_j(x) f_k(x)\| < \delta$. Also $f_j(x) \in D_x$ for $x \in V_j$, and $f_j(x) \neq 0$. Use the previous lemma to find open subsets $W_1, \ldots, W_n$ which cover X, and continuous projection valued functions $r_j\colon W_j \to A$ such that $r_j(x) r_k(x) = 0$ for $j \neq k$ and $x \in W_j \cap W_k$, such that $r_j(x) \in D_x$, and such that $\|r_j(x) - f_j(x)\| < \frac{1}{4}$.

Note that, using $f_j(x_j) = f_j^{(0)}$, we have

$$\begin{aligned}\|r_j(x) - r_j(x_j)\| &\leq \|r_j(x) - f_j(x)\| + \|f_j(x) - f_j(x_j)\| + \|f_j(x_j) - r_j(x_j)\| \\ &< \frac{1}{4} + \frac{1}{3}\delta + \frac{1}{4} < 1.\end{aligned}$$

Standard methods therefore give a continuous unitary function $u_j\colon W_j \to A$ such that $u_j(x) r_j(x) u_j(x)^* = r_j(x_j)$ for all $x \in W_j$. Use Lemma 2.6 to choose $a_j \in [r_j(x_j) A r_j(x_j)]_{\mathrm{sa}}$ with $\mathrm{sp}(a_j) = [0,1]$. Choose continuous functions $h_j\colon X \to [0,1]$ which vanish outside W_j and such that for every $x \in X$ there is at least one j with $h_j(x) = 1$. Now set $c(x) = \sum_{j=1}^n h_j(x) u_j(x)^* a_j u_j(x)$. Note that for any x any two nonzero summands are orthogonal. It is now easy to check that $\mathrm{sp}\big(c(x)\big) = [0,1]$ for all x. Also, $c(x) \in D_x$ because D_x is hereditary, and it follows from Lemma 2.3 of [18] that $c \in D$. ■

The following two lemmas deal with straightening out projection valued functions.

Lemma 2.8 **(Zhang)** *Let A be a separable purely infinite simple C^*-algebra, let $D \subset A$ be a nonzero hereditary subalgebra, and let X be a compact Hausdorff space. Let $p, q \in C(X, D)$ be projections such that, for all $x \in X$, $p(x)$ and $q(x)$ are neither zero nor the identity of D (if it has one). If p is homotopic to q in $C(X, A)$, then p is homotopic to q in $C(X, D)$.*

Proof Let

$$P = \{p \in A : p \text{ is a projection and } p \neq 0, 1_A\}$$

and

$$Q = \{p \in D : p \text{ is a projection and } p \neq 0, 1_D\}.$$

We denote by $\pi_n(Y, y_0)$ the set (group if $n \geq 1$) of homotopy classes of maps from the sphere S^n to Y which are pointed in the sense that they send the north pole x_n of S^n to y_0. Regardless of what A is, for any projection $p_0 \in P$ there is a canonical map $\gamma_{n,A}\colon \pi_n(P, p_0) \to K_n(A)$ (where n in $K_n(A)$ is taken mod 2). It is defined by sending a map $p\colon S^n \to P$ to the class $[p] - [p_0] \in K_0\big(C(S^n, A)\big)$ (regarding p_0 as a constant projection), observing that this class is in the image of $K_0\big(C_0(S^n \setminus \{x_n\}, A)\big) \to K_0\big(C(S^n, A)\big)$, and applying Bott periodicity. According to Theorem B of [35], when A is purely infinite and simple, $\gamma_{n,A}$ is an isomorphism for all n and all $p_0 \in P$. Similarly, $\gamma_{n,D}\colon \pi_n(Q, q_0) \to K_n(D)$ is an isomorphism for all n and all $q_0 \in Q$. Since D is a full hereditary subalgebra of the separable C^*-algebra A, the inclusion induces isomorphisms $K_n(D) \to K_n(A)$ for all n. Therefore the inclusion induces isomorphisms $\pi_n(Q, q_0) \to \pi_n(P, q_0)$ for all n and q_0. That is, the inclusion of Q in P is a weak homotopy equivalence.

We now observe that P is homotopy equivalent to an open subset of the Banach space A_{sa}. Let

$$U = \left\{ a \in A_{\mathrm{sa}} : \operatorname{dist}(a, P) < \frac{1}{4} \right\},$$

and define $f_t\colon U \to U$ by $f_t(a) = ta + (1-t)\chi_{(\frac{1}{2},\infty)}(a)$ for $t \in [0,1]$. Then $f_0(U) \subset P$. Letting i be the inclusion of P in U, we see that $f_0 \circ i = \mathrm{id}_P$ and $t \mapsto f_t$ is a homotopy from $i \circ f_0$ to id_U. Thus P is homotopy equivalent to U. Similarly Q is homotopy equivalent to an open subset of the Banach space D_{sa}.

Lemma IV.5.2 and Theorem IV.5.3 of [20] now imply that both P and Q are homotopy equivalent to CW-complexes. Theorem IV.3.3 of [20] therefore implies that the inclusion of Q in P is a homotopy equivalence. The desired conclusion is immediate. ∎

Lemma 2.9 *Let A be a separable purely infinite simple C^*-algebra, and let X be a compact Hausdorff space. Let $D \subset C(X, A)$ be a hereditary subalgebra. Assume that D contains an approximate identity of projections $p_1, p_2, \dots$ such that $p_{k+1}(x) > p_k(x)$ for all x, and such that each p_k is homotopic in $C(X, A)$ to a constant projection. Then $D \cong C(X, K \otimes A)$, and the isomorphism can be chosen to have the form $\varphi(a)(x) = \varphi_x\big(a(x)\big)$ for isomorphisms $\varphi_x\colon D_x \to K \otimes A$, and to send each p_k to a constant projection in $C(X, A)$.*

Proof Fix $x_0 \in X$. Define

$$B = \overline{\bigcup_{n=1}^{\infty} p_n(x_0) A p_n(x_0)}.$$

Standard arguments show that $B \cong K \otimes A$, so we prove the lemma for B in place of $K \otimes A$.

We inductively construct unitaries $u_n \in C(X, A)$, each homotopic to 1, such that

$$p_n(x_0) = u_n(x)\big[\,[u_{n-1}(x)u_{n-2}(x)\cdots u_1(x)]\,p_n(x)[u_{n-1}(x)u_{n-2}(x)\cdots u_1(x)]^*\big]\,u_n(x)^*$$

and $u_n(x)p_{n-1}(x_0) = p_{n-1}(x_0)u_n(x) = p_{n-1}(x_0)$. This will prove the result with

$$\varphi_x(c) = [u_n(x)u_{n-1}(x)\cdots u_1(x)]c[u_n(x)u_{n-1}(x)\cdots u_1(x)]^*$$

for $c \in p_n(x)D_x p_n(x)$, extended to all of D_x by continuity.

The construction of u_1 is immediate from the fact that p_1 is homotopic in $C(X, A)$ to the constant function with value $p_1(x_0)$. Given $u_1, \ldots, u_n$, set

$$q = [u_n(x)u_{n-1}(x) \cdots u_1(x)]p_{n+1}(x)[u_n(x)u_{n-1}(x) \cdots u_1(x)]^*.$$

Then q is homotopic to the constant function $p_{n+1}(x_0)$, and both are nowhere zero projections in $C\big(X, [1 - p_n(x_0)]A[1 - p_n(x_0)]\big)$. It suffices to show that there is a homotopy in $C\big(X, [1 - p_n(x_0)]A[1 - p_n(x_0)]\big)$ from q to the constant function $p_{n+1}(x_0)$. This follows from Lemma 2.8. ■

The next two lemmas are the heart of our argument, which is an induction using the methods of Lemma 2.5 and Proposition 2.6 of [18] in the induction step.

Lemma 2.10 **(Compare with Lemma 2.5 of [18])** *Let B be a separable C^*-algebra, let $\alpha, \beta \in \mathbb{R}$, and let $D \subset C([\alpha, \beta], B)$ be a hereditary subalgebra. For each t, assume that D_t is full in B, and that $D_t \cong C(X, K \otimes A)$ for some contractible compact Hausdorff space X and some purely infinite simple C^*-algebra A. Let $p \in D$, $e_\alpha \in D_\alpha$, and $e_\beta \in D_\beta$ be projections such that, for $i = \alpha, \beta$ we have $e_i > p(i)$ and $[e_i - p(i)] = 0$ in $K_0(B)$. Then there exists a projection $q \in D$ such that $q \geq p$, $q(\alpha) = e_\alpha$, and $q(\beta) = e_\beta$.*

Proof Without loss of generality, assume $\alpha = 0$ and $\beta = 1$. Further, replace D by $(1 - p)D(1 - p)$. This allows us to assume that $p = 0$. Since X is contractible, $p(t)$ is unitarily equivalent in $D_t \cong C(X, K \otimes A)$ to a constant projection. The new D_t is therefore still isomorphic to $C(X, K \otimes A)$.

For each $t \in (0, 1)$, choose a nonzero projection $e_t \in D_t$ with $[e_t] = 0$ in $K_0(D_t)$. Take e_0 and e_1 to be as already given. Use Lemma 2.2 to choose a continuous projection valued function $f_t \colon U_t \to B$, for some open interval U_t containing t, such that $f_t(s) \in D_s$ and $f_t(t) = e_t$. Now observe that $K_0(D_s) \to K_0(B)$ is an isomorphism for all s. Since $f_t(s)$ is homotopic to e_t in B, and $[e_t] = 0$ in $K_0(D_t)$, it follows that $[f_t(s)] = 0$ in $K_0(D_s)$ for all s.

By passing to a finite subcover of the collection $\{U_t : t \in [0, 1]\}$, we can find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and continuous projection valued functions $q_i \colon [t_{i-1}, t_i] \to B$ such that $q_i(t) \in D_t$ and $[q_i(t)] = 0$ in $K_0(D_t)$ for all t. Note that $q_i(t_{i-1})$ and $q_{i-1}(t_{i-1})$ are homotopic in $D_{t_{i-1}} \cong C(X, K \otimes A)$. Indeed, X is contractible, and $q_i(t_{i-1})(x_0)$ and $q_{i-1}(t_{i-1})(x_0)$ are nonzero projections in a purely infinite simple C^*-algebra with the same K_0-class. Similarly, $q_1(0)$ is homotopic to e_0 and $q_n(1)$ is homotopic to e_1.

Apply Lemma 2.4 of [18], with $X = [0, t_1]$, with the hereditary subalgebra there being the image $D|_{[0,t_1]}$ of D under the restriction map $C([0, 1], B) \to C([0, t_1], B)$, with $Z = \{t_1\}$, $x_0 = 0$, $p = q_1$, and $e = e_0$. This gives a projection $f_1 \in D|_{[0,t_1]}$ such that $f_1(0) = e_0$ and $f_1(t_1) = q_1(t_1)$. In the same manner, for $2 \leq i \leq n - 1$ find projections $f_i \in D|_{[t_{i-1},t_i]}$ (using notation analogous to the above) such that $f_i(t_{i-1}) = q_{i-1}(t_{i-1})$ and $f_i(t_i) = q_i(t_i)$. Set $s = \frac{1}{2}(t_{n-1}+1)$, and use the same method to find projections $f_n \in D|_{[t_{n-1},s]}$ such that $f_n(t_{n-1}) = q_{n-1}(t_{n-1})$ and $f_n(s) = q_n(s)$, and $f_{n+1} \in D|_{[s,1]}$ such that $f_{n+1}(s) = q_n(s)$ and $f_{n+1}(1) = e_1$. (To get f_{n+1}, we take

$Z = \{s\}$, not $\{1\}$.) Then the definition

$$q(t) = \begin{cases} f_i(t) & t \in [t_{i-1}, t_i],\ 1 \leq i \leq n-1 \\ f_n(t) & t \in [t_{n-1}, s] \\ f_{n+1}(t) & t \in [s, 1] \end{cases}$$

gives a continuous projection valued function on $[0, 1]$ satisfying $q_0 = e_0$, $q_1 = e_1$, and $q_t \in D_t$ for all t. We have $q \in D$ by Lemma 2.3 of [18]. ■

Lemma 2.11 *Let $X = [0, 1]^n$, and let A be a separable purely infinite simple C^*-algebra. Then for every $b \in C(X, A)_{\mathrm{sa}}$ such that $\mathrm{sp}\big(b(x)\big) = [0, 1]$ for all $x \in X$, the hereditary subalgebra $D = \overline{bC(X, A)b}$ has an approximate identity of projections, each of whose images in $C(X, A)$ is homotopic to a constant projection with trivial K_0-class. Moreover, the approximate identity can be chosen to be of the form $(p_k)_{k=1}^{\infty}$ with $p_k(x) < p_{k+1}(x)$ for all k and all $x \in X$.*

Proof The proof is a modification of part of the proof of Proposition 2.6 of [18]. There are enough differences and improvements that we give the full proof here.

The proof is by induction on n. So assume the result is known for a particular value of n, and all $b \in C([0, 1]^n, A)$ satisfying the conditions of the lemma. We prove it for $n+1$. We write $[0, 1]^{n+1} = [0, 1]^n \times [0, 1]$, and we accordingly write $b(x, t)$ with $x \in [0, 1]^n$ and $t \in [0, 1]$, *etc.* The notation $b(-, t)$ refers to the function $x \mapsto b(x, t)$, which is an element of $C([0, 1]^n, A)$. If $E \subset C([0, 1]^{n+1}, A)$ is a hereditary subalgebra, and $t \in [0, 1]$, we take $E_t \subset C([0, 1]^n, A)$ to be $\{a(-, t) : a \in E\}$. That is, we evaluate in the last coordinate only.

The main part of the proof is to show that for any $\varepsilon > 0$, there is a projection $q \in D$, homotopic in $C([0, 1]^{n+1}, A)$ to a constant projection with trivial K_0-class, such that $\|qb - b\| < \varepsilon$.

Standard functional calculus arguments show that there are functions β_1, β_2 from $[0, \infty)$ to $[0, \infty]$ which are nondecreasing and satisfy $\lim_{t \to 0} \beta_i(t) = 0$, such that if p and q are projections in a C^*-algebra C such that $\|pq - q\| < \eta$, then:

(1) There exists a projection $p' \in C$ such that $p' \geq q$ and $\|p' - p\| < \beta_1(\eta)$.
(2) There exists a projection $q' \in C$ and a unitary path $t \mapsto v_t \in C^+$ such that $p \geq q'$, $\|q' - q\| < \beta_2(\eta)$, $v_0 = 1$, $v_1 q' v_1^* = q$, and $\|v_t - 1\| < \beta_2(\eta)$ for all t.

Moreover, the claimed elements in both parts depend continuously on p and q. (The notation *etc.* follows Lemma 2.1 of [18].)

Choose $\delta > 0$ such that

$$3\beta_2\big(15\delta + \beta_1(\delta)\big) + 19\delta < \min\left(\varepsilon, \frac{1}{2}\right).$$

For $0 \leq k \leq 5$, let $g_k \colon [0, \infty) \to [0, 1]$ be the continuous function which is equal to 0 on $[0, k\delta]$, equal to 1 on $\big[(k+1)\delta, \infty\big)$, and linear on $[k\delta, (k+1)\delta]$. Note that $g_{k+1} g_k = g_{k+1}$. Define $b_k = g_k(b) \in D$, and note that $\mathrm{sp}\big(b_k(x)\big) = [0, 1]$ for all $x \in X$

(since $\delta < \frac{1}{38}$). Define $D^{(k)} = \overline{b_k C([0,1]^{n+1}, A) b_k}$. Then, in particular, the induction hypothesis applies to the hereditary subalgebras

$$D_t^{(k)} = \overline{b_k(-,t) C([0,1]^n, A) b_k(-,t)}.$$

It follows from Lemma 2.9 that $D_t^{(k)} \cong C([0,1]^n, K \otimes A)$. Since $[0,1]^n$ is contractible and nonzero projections in $K \otimes A$ with the same K_0-class are homotopic, we see that nonzero projections in $D_t^{(k)}$ with the same K_0-class are homotopic.

Choose a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

of $[0,1]$ such that

$$\|b(-,t) - b(-,t_i)\| < \delta \quad \text{and} \quad \|b(-,t) - b(-,t_{i-1})\| < \delta$$

for all $t \in [t_{i-1}, t_i]$ and

$$\|b_k(-,t) - b_k(-,t_i)\| < \delta \quad \text{and} \quad \|b_k(-,t) - b_k(-,t_{i-1})\| < \delta$$

for all $t \in [t_{i-1}, t_i]$ and for $0 \leq k \leq 5$. The induction assumption provides projections

$$e_i^{(0)} \in D_{t_i}^{(4)} \quad \text{and} \quad \tilde{e}_i^{(1)} \in D_{t_i}^{(1)}$$

such that

$$\|e_i^{(0)} b_5(-,t_i) - b_5(-,t_i)\| < \delta \quad \text{and} \quad \|\tilde{e}_i^{(1)} b_2(-,t_i) - b_2(-,t_i)\| < \delta,$$

and such that $e_i^{(0)}$ and $\tilde{e}_i^{(1)}$ are both homotopic in $C([0,1]^n, A)$ to constant projections with trivial K_0-class. Since $b_2 b_3 = b_3$ and $b_3 e_i^{(0)} = e_i^{(0)}$, it follows that $\|\tilde{e}_i^{(1)} e_i^{(0)} - e_i^{(0)}\| < \delta$. Therefore there exists a projection $e_i^{(1)} \in D_{t_i}^{(1)}$ such that

$$\|e_i^{(1)} - \tilde{e}_i^{(1)}\| < \beta_1(\delta) \quad \text{and} \quad e_i^{(1)} \geq e_i^{(0)}.$$

Also, since $\beta_1(\delta) < \frac{1}{2}$, it follows that $e_i^{(1)}$ is homotopic to $\tilde{e}_i^{(1)}$, and thus to a constant projection in $C([0,1]^n, A)$ with trivial K_0-class.

Temporarily fix i, and work over the interval $[t_{i-1}, t_{i+1}]$. Note that

$$b_3(-,t) e_i^{(0)} b_3(-,t) \in D_t^{(3)}$$

and satisfies

$$\|b_3(-,t) e_i^{(0)} b_3(-,t) - e_i^{(0)}\| < 2\delta < \frac{1}{4}$$

(since $\delta < \frac{1}{38}$). Therefore functional calculus yields a projection $f_i^{(0)}(t) \in D_t^{(3)}$, depending continuously on $t \in [t_{i-1}, t_{i+1}]$, such that

$$\|f_i^{(0)}(t) - e_i^{(0)}\| < 4\delta.$$

Similarly, there is a projection $f_i^{(1)}(t) \in D_t^{(0)}$, depending continuously on $t \in [t_{i-1}, t_{i+1}]$, such that

$$\|f_i^{(1)}(t) - e_i^{(1)}\| < 4\delta.$$

It follows that

$$\begin{aligned}\|f_i^{(1)}(t)f_i^{(0)}(t) - f_i^{(0)}(t)\| &\leq 2\|f_i^{(0)}(t) - e_i^{(0)}\| + \|f_i^{(1)}(t) - e_i^{(1)}\| \\ &< 3(4\delta) < 15\delta + \beta_1(\delta)\end{aligned}$$

for $t \in [t_{i-1}, t_{i+1}]$. Next,

$$\begin{aligned}\|e_{i-1}^{(1)}e_i^{(0)} - e_i^{(0)}\| &= \|e_{i-1}^{(1)}b_2(-,t_i)e_i^{(0)} - b_2(-,t_i)e_i^{(0)}\| \\ &\leq \|e_{i-1}^{(1)}b_2(-,t_i) - b_2(-,t_i)\| \\ &\leq 2\|b_2(-,t_i) - b_2(-,t_{i-1})\| + \|e_{i-1}^{(1)}b_2(-,t_{i-1}) - b_2(-,t_{i-1})\| \\ &< 2\delta + \big(\delta + \beta_1(\delta)\big).\end{aligned}$$

An estimate similar to the one at the beginning of this paragraph therefore shows that

$$\|f_{i-1}^{(1)}(t)f_i^{(0)}(t) - f_i^{(0)}(t)\| < 12\delta + \big(3\delta + \beta_1(\delta)\big) = 15\delta + \beta_1(\delta)$$

for $t \in [t_{i-1}, t_i]$.

Using the function β_2 from the beginning of the proof, we obtain a projection $r_i(t) \in D_t$ and a unitary path $s \mapsto v_s(t) \in D_t^+$ for $t \in [t_{i-1}, t_i]$ and $s \in [0,1]$, both varying continuously with t, such that $r_i(t) \leq f_i^{(1)}(t)$, $v_0(t) = 1$, $v_1(t)r_i(t)v_1(t)^* = f_i^{(0)}(t)$, and

$$\|r_i(t) - f_i^{(0)}(t)\| < \beta_2\big(15\delta + \beta_1(\delta)\big) \quad \text{and} \quad \|v_s(t) - 1\| < \beta_2\big(15\delta + \beta_1(\delta)\big).$$

We similarly obtain $r_i'(t)$ and $v_s'(t)$ satisfying all the same conditions, except with $f_i^{(1)}(t)$ replaced by $f_{i-1}^{(1)}(t)$. We now define a continuous projection q_i on $[t_{i-1}, t_i]$ by letting $t_i' = \frac{1}{2}(t_{i-1} + t_i)$ and setting

$$q_i(t) = v_{1-\alpha}'(t)^* f_i^{(0)}(t) v_{1-\alpha}'(t)$$

for $t = (1-\alpha)t_{i-1} + \alpha t_i'$ and $\alpha \in [0,1]$, and

$$q_i(t) = v_\alpha(t)^* f_i^{(0)}(t) v_\alpha(t)$$

for $t = (1-\alpha)t_i' + \alpha t_i$ and $\alpha \in [0,1]$. This gives $q_i(t_i') = f_i^{(0)}(t_i')$ (with either definition), and

$$q_i(t_{i-1}) = r_i'(t_{i-1}) \leq f_{i-1}^{(1)}(t_{i-1}) \quad \text{and} \quad q_i(t_i) = r_i(t_i) \leq f_i^{(1)}(t_i).$$

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

Let $t \in [t_i', t_i]$. Then for suitable $\alpha \in [0,1]$, we have

$$\begin{aligned}\|q_i(t) - e_i^{(0)}\| &\leq \|q_i(t) - r_i(t)\| + \|r_i(t) - f_i^{(0)}(t)\| + \|f_i^{(0)}(t) - e_i^{(0)}\| \\ &< 2\|v_\alpha(t) - 1\| + \beta_2\big(15\delta + \beta_1(\delta)\big) + 4\delta < 3\beta_2\big(15\delta + \beta_1(\delta)\big) + 4\delta.\end{aligned}$$

A similar estimate holds for $t \in [t_{i-1}, t_i']$. Since $3\beta_2\big(15\delta + \beta_1(\delta)\big) + 4\delta < \frac{1}{2}$, it follows that $[q_i(t)] = 0$ in $K_0\big(C([0,1]^n, A)\big)$ for all $t \in [t_{i-1}, t_i]$. Since D_t is full in $C([0,1]^n, A)$, we conclude that $[q_i(t)] = 0$ in $K_0(D_t)$ as well.

Since $D_t \cong C([0,1]^n, K \otimes A)$, and since $[0,1]^n$ is contractible and A is purely infinite, there are projections $g_i \in D_{t_i}$ such that $g_i > f_i^{(1)}(t_i)$ and $[g_i] = 0$ in $K_0(D_{t_i})$. Lemma 2.10 provides a continuous projection $q_i' \colon [t_{i-1}, t_i] \to C([0,1]^n, A)$ such that $q_i'(t) \in D_t$, $q_i'(t_{i-1}) = g_{i-1}$, $q_i'(t_i) = g_i$, and $q_i'(t) \geq q_i(t)$ for all $t \in [t_{i-1}, t_i]$. Now define $q \colon [0,1] \to C([0,1]^n, A)$ by $q(t) = q_i'(t)$ for $t \in [t_{i-1}, t_i]$. Then q is well defined and continuous, since at the overlap points t_i both definitions yield g_i. Furthermore, $q \in D$ by Lemma 2.3 of [18], and $q(t) \geq q_i(t)$ whenever $t \in [t_{i-1}, t_i]$.

We now estimate, for $t \in [t_{i-1}, t_i]$:

$$\begin{aligned}&\|q(t)b(-,t) - b(-,t)\| \\ &\quad\leq \|q_i(t)b(-,t) - b(-,t)\| \\ &\quad\leq \Big[3\beta_2\big(15\delta + \beta_1(\delta)\big) + 4\delta\Big] + 2\delta + \|e_i^{(0)}b(-,t_i) - b(-,t_i)\|.\end{aligned}$$

Also, since $\|b_5(-,t_i)b(-,t_i) - b(-,t_i)\| < 6\delta$, we get

$$\|e_i^{(0)}b(-,t_i) - b(-,t_i)\| \leq 12\delta + \Big\|\big(e_i^{(0)}b_5(-,t_i) - b_5(-,t_i)\big)b(-,t_i)\Big\| \leq 12\delta + \delta.$$

So

$$\|q(t)b(-,t) - b(-,t)\| < 3\beta_2\big(15\delta + \beta_1(\delta)\big) + 19\delta < \varepsilon.$$

Moreover, the choice of q ensures that $[q(0)] = 0$ in $K_0(D_0)$, and so also in $K_0\big(C([0,1]^n, A)\big)$. Since $[0,1]^{n+1}$ is contractible, it is immediate that q is homotopic in $C([0,1]^{n+1}, A)$ to a constant projection with trivial K_0-class.

We now know that, for any $\varepsilon > 0$, there is a suitable projection $q \in D$ such that $\|qb - b\| < \varepsilon$. It is easy to get from this that for any polynomial $c = \sum_{k=0}^n \alpha_k b^k$ in b and any $\varepsilon > 0$, there is a suitable projection $q \in D$ such that $\|qc - c\| < \varepsilon$. Standard arguments show that this is also true for any c in the closure of the set of such polynomials. In particular, it is true for the elements $b^{1/n}$ for $n \in \mathbb{N}$. Since these form an approximate identity for D, one easily checks that for $c_1, \ldots, c_m \in D$, there is a projection $q \in D$ such that $\|qc_k - c_k\| < \varepsilon$ for all k. (Without loss of generality assume $\|c_k\| \leq 1$. Choose n such that $\|b^{1/n}c_k - c_k\| < \frac{\varepsilon}{2}$, and choose q such that $\|qb^{1/n} - b^{1/n}\| < \frac{\varepsilon}{2}$.) That is, D has an approximate identity consisting of projections which are homotopic in $C([0,1]^{n+1}, A)$ to constant projections with trivial K_0-class.

Standard arguments now show that D has a nondecreasing approximate identity $(p_k)_{k=1}^\infty$ of projections which are homotopic in $C([0,1]^{n+1}, A)$ to constant projections with trivial K_0-class. Since $\mathrm{sp}(b) = [0,1]$, the hereditary subalgebra D is not unital.

Passing to a subsequence, we may therefore assume that $p_k \neq p_{k+1}$ for all k. For each k there is therefore some $x_0 \in [0,1]^{n+1}$ such that $p_k(x_0) < p_{k+1}(x_0)$. Since $p_k(x) \leq p_{k+1}(x)$ for all x and $[0,1]^{n+1}$ is connected, we get $p_k(x) < p_{k+1}(x)$ for all $x \in [0,1]^{n+1}$. This completes the induction, and proves the lemma. ∎

Theorem 2.12 *Let A be a separable purely infinite simple C^*-algebra, let X be a closed subset of $[0,1]^n$ for some n, and let $D \subset C(X,A)$ be a hereditary subalgebra such that D_x is nonzero and nonunital for every $x \in X$. Then D has an approximate identity consisting of projections whose images in $C(X,A)$ are homotopic to constant projections with trivial K_0-classes.*

Proof We first consider the special case $X = [0,1]^n$.

Clearly D is separable, so it has a strictly positive element b. Without loss of generality we may take $\|b\| \leq 1$. Then $\{b^{1/n}\}$ is an approximate identity for D. As at the end of the previous proof, it suffices to show that for all $\varepsilon > 0$ there is a projection q of the desired form satisfying $\|qb - b\| < \varepsilon$.

Let $f\colon [0,\infty) \to [0,1]$ be the continuous function such that f is 0 on $[0, \frac{\varepsilon}{9}]$, f is 1 on $[\frac{\varepsilon}{6}, \infty)$, and f is linear on $[\frac{\varepsilon}{9}, \frac{\varepsilon}{6}]$. Let $g\colon [0,\infty) \to [0,1]$ be the continuous function such that g is 0 on $[\frac{\varepsilon}{9}, \infty)$ and at 0, such that $g(\frac{\varepsilon}{18}) = 1$, and g is linear on $[0, \frac{\varepsilon}{18}]$ and on $[\frac{\varepsilon}{18}, \frac{\varepsilon}{9}]$. Then $\|f(b)b - b\| < \frac{\varepsilon}{6}$. Also, since D_x is nonunital for all x, the point 0 is not isolated in $\operatorname{sp}\bigl(b(x)\bigr)$. Therefore $g(b)(x) \neq 0$ for all x. So Lemma 2.7 provides $c \in \bigl(\overline{g(b)C(X,A)g(b)}\bigr)_{\mathrm{sa}}$ such that $\operatorname{sp}\bigl(c(x)\bigr) = [0,1]$ for all x.

We estimate $\bigl\| \bigl(c + f(b)\bigr) b - b \bigr\|$. First, $\{g(b)^{1/n}\}$ is an approximate identity for $\overline{g(b)C(X,A)g(b)}$, so

$$\|cb\| = \lim_{n\to\infty} \|cg(b)^{1/n}b\| \leq \limsup_{n\to\infty} \|g(b)^{1/n}b\| \leq \frac{\varepsilon}{9} < \frac{\varepsilon}{6}.$$

Therefore

$$\bigl\| \bigl(c + f(b)\bigr) b - b \bigr\| < 2\left(\frac{\varepsilon}{6}\right) = \frac{\varepsilon}{3}.$$

Since c is orthogonal to $f(b)$ and $\operatorname{sp}\bigl(f(b)(x)\bigr) \subset [0,1]$, we have $\operatorname{sp}\Bigl(\bigl(c+f(b)\bigr)(x)\Bigr) = [0,1]$ for all x. By the previous lemma, there is thus a projection q of the desired form such that $\bigl\| q\bigl(c + f(b)\bigr) - \bigl(c + f(b)\bigr) \bigr\| < \frac{\varepsilon}{3}$. Now

$$\|qb-b\| \leq \bigl\| q\bigl(c+f(b)\bigr) - \bigl(c+f(b)\bigr) \bigr\| \, \|b\| + 2\bigl\| \bigl(c+f(b)\bigr) b - b \bigr\| < \frac{\varepsilon}{3} + 2\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

This completes the proof for $X = [0,1]^n$.

We now reduce the general case to the special case. Let $X \subset [0,1]^n$, and let $D \subset C(X,A)$ be a hereditary subalgebra such that D_x is nonzero and nonunital for all x. Embed A in $M_2(A)$ as the upper left corner, and in this way consider D to be a hereditary subalgebra of $C\bigl(X, M_2(A)\bigr)$, with D_x as before, but now thought of as a subalgebra of $M_2(A)$. Let B be a nonunital hereditary subalgebra of $M_2(A)$ which contains A. Define a hereditary subalgebra $E \subset C\bigl([0,1]^n, M_2(A)\bigr)$ to be the set of continuous functions $a\colon [0,1]^n \to M_2(A)$ such that $a(x) \in B$ for all x and $a(x) \in D_x$

for $x \in X$. Note that the restriction map $a \mapsto a|_X$ is a surjective map from E to D. In particular, $E_x = D_x$ for $x \in X$, and $E_x = B$ otherwise. The special case proved above applies to E, and provides an approximate identity for E consisting of projections whose images in $C\big([0,1]^n, M_2(A)\big)$ are homotopic to constant projections. Restricting to X, we obtain an approximate identity for D consisting of projections whose images in $C(X, A)$ are homotopic to constant projections. The homotopy can be chosen in $C(X, A)$ by Lemma 2.8. ■

Corollary 2.13 *Let A be a separable purely infinite simple C^*-algebra, let X be a closed subset of $[0,1]^n$ for some n, and let $D \subset C(X, A)$ be a hereditary subalgebra such that D_x is nonzero and nonunital for every $x \in X$. Then $D \cong C(X, K \otimes A)$, and the isomorphism can be chosen to have the form $\varphi(a)(x) = \varphi_x\big(a(x)\big)$ for isomorphisms $\varphi_x \colon D_x \to K \otimes A$.*

Proof This is immediate from Theorem 2.12 and Lemma 2.9. ■

As a first illustration of the value of this result, we prove the following generalization of a part of Lemma 2.5 to projection valued functions.

Lemma 2.14 *Let A be a purely infinite simple C^*-algebra, let X be a closed subset of $[0,1]^n$ for some n, let $p_1, p_2 \in C(X, A)$ be nowhere vanishing projections, and let $\varepsilon > 0$. Then there exist orthogonal nowhere vanishing projections $q_1, q_2 \in C(X, A)$, each homotopic in $C(X, A)$ to a constant projection with trivial K_0-class, such that $\|p_k q_k - q_k\| < \varepsilon$ for $k = 1, 2$.*

Proof Cover X by finitely many open sets $U_1, \dots, U_N$ such that

$$\|p_1(x) - p_1(y)\| < \frac{\varepsilon}{2(n+1)} \quad \text{and} \quad \|p_2(x) - p_2(y)\| < \frac{\varepsilon}{2(n+1)}$$

for all k and $x, y \in U_k$. Since $X \subset [0,1]^n$, we have $\dim(X) \leq n$. (See Proposition 3.1.5 of [25].) Refining our cover (and relabelling), we may therefore assume that any distinct $n+2$ of the U_k have empty intersection.

Choose $x_k \in U_k$. Use Lemma 2.5 (1) to find $2N$ mutually orthogonal nonzero projections $e_1, \dots, e_N, f_1, \dots, f_N \in A$ such that

$$\|p_1(x_k)e_k - e_k\| < \frac{\varepsilon}{2(n+1)} \quad \text{and} \quad \|p_2(x_k)f_k - f_k\| < \frac{\varepsilon}{2(n+1)}$$

for $1 \leq k \leq N$. It follows that

$$\|p_1(x)e_k - e_k\| < \frac{\varepsilon}{n+1} \quad \text{and} \quad \|p_2(x)f_k - f_k\| < \frac{\varepsilon}{n+1}$$

for all $x \in U_k$.

Choose $a_k \in (e_k A e_k)_{\mathrm{sa}}$ and $b_k \in (f_k A f_k)_{\mathrm{sa}}$ such that $\mathrm{sp}(a_k) = \mathrm{sp}(b_k) = [0,1]$ for all k. (See Lemma 2.6.) Choose continuous functions $h_k \colon X \to [0,1]$ such that h_k vanishes outside U_k and for every $x \in X$ there is some k with $h_k(x) = 1$. Define

$$a(x) = h_1(x)a_1 + \cdots + h_n(x)a_n \quad \text{and} \quad b(x) = h_1(x)b_1 + \cdots + h_n(x)b_n$$

for $x \in X$. Since the e_k and f_k are mutually orthogonal, one readily checks that $\mathrm{sp}\big(a(x)\big) = \mathrm{sp}\big(b(x)\big) = [0,1]$ for all $x \in X$. It follows from Theorem 2.12 that the hereditary subalgebras $\overline{aC(X,A)a}$ and $\overline{bC(X,A)b}$ have approximate identities of projections which are homotopic in $C(X,A)$ to constant projections with trivial K_0-classes. In particular, there are projections $q_1 \in \overline{aC(X,A)a}$ and $q_2 \in \overline{bC(X,A)b}$, each homotopic in $C(X,A)$ to a constant projection, such that $q_1(x) \neq 0$ and $q_2(x) \neq 0$ for all $x \in X$. Since $ab = 0$, we have $q_1 q_2 = 0$.

We now estimate $\|p_i(x)q_i(x) - q_i(x)\|$. Fix $x \in X$. Let $S = \{k : x \in U_k\}$. Set

$$e_S = \sum_{k \in S} e_k \quad \text{and} \quad f_S = \sum_{k \in S} f_k.$$

Then $e_S a(x) = a(x)$, so $e_S q_1(x) = q_1(x)$. Furthermore,

$$\|p_1(x)e_S - e_S\| \leq \sum_{k \in S} \|p_1(x)e_k - e_k\| < \varepsilon,$$

because the choice of the sets U_k ensures that S has at most $n+1$ elements. So $\|p_1(x)q_1(x) - q_1(x)\| < \varepsilon$. Similarly $\|p_2(x)q_2(x) - q_2(x)\| < \varepsilon$. ∎

3 Real Rank of $C(X) \otimes A$ when A is Purely Infinite

In this section, we show that $\mathrm{RR}\big(C(X) \otimes A\big) \leq 1$ for any compact Hausdorff space X and any purely infinite simple C^*-algebra A.

As in the previous section, we will essentially always regard elements of $C(X) \otimes A$ as A-valued functions on X, and write $C(X,A)$ rather than $C(X) \otimes A$.

Besides the result of the previous section, the main technical device is the perturbation of a selfadjoint element $a \in C(X,A)$ in such a way that, for every $x \in X$, the set $\mathrm{sp}\big(a(x)\big)$ contains a neighborhood of a certain size of each of its points. At a single point, this can be accomplished by writing $a(x) \approx a_0(x) + \sum_{j=1}^{N} \lambda_j p_j$, an orthogonal sum in which $\{\lambda_1, \ldots, \lambda_n\}$ is ε-dense in $\mathrm{sp}\big(a(x)\big)$, and using a scaled version of Lemma 2.6 to replace $\lambda_j p_j$ by an element whose spectrum is a small interval containing λ_j. The problem is to do this simultaneously for all $x \in X$.

We start by giving some useful notation.

Notation 3.1 Let B be a C^*-algebra, and let $P = \{p_1, \ldots, p_n\} \subset A$ be a set of mutually orthogonal projections. For $a \in B$ define the "cutdown by P" to be

$$C_P(a) = \Big(1 - \sum_{k=1}^{n} p_k\Big) a \Big(1 - \sum_{k=1}^{n} p_k\Big) + \sum_{k=1}^{n} p_k a p_k.$$

(Note that we implicitly include the orthogonal complement of the sum of the elements of P.) By an obvious abuse of notation, if $p_1, \ldots, p_n \in A$ are understood, and $S \subset \{1, \ldots, n\}$, we will refer to $C_S(a)$, and similar devices.

Given $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, regarded as a function α on P, we define

$$C_{P,\alpha}(a) = \Big(1 - \sum_{k=1}^{n} p_k\Big) a \Big(1 - \sum_{k=1}^{n} p_k\Big) + \sum_{k=1}^{n} \alpha_k p_k.$$

We will switch back and forth between $C_P(a)$ and $C_{P,\alpha}(a)$, because $C_{P,\alpha}(a)$ is more useful but $C_P(a)$ is easier to work with.

The following lemma gives some useful properties of these expressions. One might hope to have better estimates in some of the conclusions, but that seems to be not always possible.

Lemma 3.2

(1) Let P be a finite set of orthogonal projections, and let $p \mapsto \alpha_p$ be a function from P to $\mathbb{C}$. If $\|a - C_{P,\alpha}(a)\| < \varepsilon$, then $\|a - C_P(a)\| < 2\varepsilon$.

(2) Let P and Q be finite sets of orthogonal projections, and assume that

$$\sum_{p \in P} p = \sum_{q \in Q} q = 1.$$

If each projection in Q is a sum of projections in P, and $\|a - C_P(a)\| < \varepsilon$, then $\|a - C_Q(a)\| < 2\varepsilon$.

(3) Let P and Q be finite sets of orthogonal projections. Let $e = \sum_{p \in P} p$ and $f = \sum_{q \in Q} q$. If e is orthogonal to f, and if

$$\|a - C_P(a)\| < \varepsilon \quad \text{and} \quad \|a - C_Q(a)\| < \eta,$$

then

$$\|a - C_{P \cup Q}(a)\| < \varepsilon + \eta + \min(\varepsilon, \eta).$$

(4) Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be finite sets of orthogonal projections, and let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Assume $q_k \le p_k$ for all k. If $\|a - C_{P,\alpha}(a)\| < \varepsilon$ then (regarding α as a function on Q in the obvious way) $\|a - C_{Q,\alpha}(a)\| < 2\varepsilon$.

Proof (1) For each $p \in P$, we have

$$\|pap - \alpha_p p\| = \big\| p\big(a - C_{P,\alpha}(a)\big) p \big\| \le \|a - C_{P,\alpha}(a)\| < \varepsilon.$$

Using orthogonality of the projections in the second step, we then get

$$\big\| \big(a - C_{P,\alpha}(a)\big) - \big(a - C_P(a)\big) \big\| = \Big\| \sum_{p \in P} (pap - \alpha_p p) \Big\| = \max_{p \in P} \|pap - \alpha_p p\| < \varepsilon.$$

Therefore $\|a - C_P(a)\| < 2\varepsilon$.

(2) Let $c = a - C_P(a)$ and $d = a - C_Q(a)$. A computation, involving partitioning P into the disjoint subsets whose members add up to the various elements of Q, shows that $c - d = \sum_{q \in Q} qcq$. Using orthogonality, we get

$$\|d\| \le \|c\| + \Big\| \sum_{q \in Q} qcq \Big\| = \|c\| + \max_{q \in Q} \|qcq\| \le 2\|c\|.$$

(3) A computation shows that

$$a - C_{P\cup Q}(a) = \big(a - C_P(a)\big) + \big(a - C_Q(a)\big) - (eaf + fae).$$

Now it is easy to check, using orthogonality of e and f, that $\|eaf + fae\| = \max(\|eaf\|, \|fae\|)$. Moreover, e commutes with $C_P(a)$, so $eC_P(a)f = 0$. It follows that $\|eaf\| = \big\| e\big(a - C_P(a)\big) f\big\| < \varepsilon$. Similarly $\|fae\| = \big\| f\big(a - C_P(a)\big) e\big\| < \varepsilon$. So $\|a - C_{P\cup Q}(a)\| < 2\varepsilon + \eta$. The same reasoning shows that $\|a - C_{P\cup Q}(a)\| < \varepsilon + 2\eta$.

(4) A computation shows that

$$\big(a - C_{P,\alpha}(a)\big) - \big(a - C_{Q,\alpha}(a)\big) = (1-q)\big(a - C_{P,\alpha}(a)\big)(1-q).$$

Therefore

$$\begin{aligned}\|a - C_{Q,\alpha}(a)\| &\le \|a - C_{P,\alpha}(a)\| + \big\|(1-q)\big(a - C_{P,\alpha}(a)\big)(1-q)\big\| \\ &\le 2\|a - C_{P,\alpha}(a)\|.\end{aligned}$$

■

We denote by $\operatorname{dist}(x, S)$ the distance between a point x and a subset S in a metric space.

Lemma 3.3 *Let X be a compact Hausdorff space with finite covering dimension. Let A be a unital purely infinite simple C^*-algebra, and let $a \in C(X, A)_{\mathrm{sa}}$. Then there exist N, nonzero mutually orthogonal projections $p_1, \dots, p_N \in A$, numbers $\alpha_1, \dots, \alpha_N \in \mathbb{R}$, and open sets $U_1, \dots, U_N, V_1, \dots, V_N \subset X$ (not necessarily distinct) satisfying the following properties:*

(1) $\overline{V}_l \subset U_l$ for all l.

(2) For every $x \in X$ and $\lambda \in \operatorname{sp}\big(a(x)\big)$, we have

$$\operatorname{dist}(\lambda, \{\alpha_l : l \text{ satisfies } x \in V_l\}) < \varepsilon,$$

and, for every $x \in U_l$, we have $\operatorname{dist}\Big(\alpha_l, \operatorname{sp}\big(a(x)\big)\Big) < \varepsilon$.

(3) For every $x \in X$, with $P(x) = \{p_l : l \text{ satisfies } x \in V_l\}$, and $\alpha|_{P(x)}$ interpreted as a function on $P(x)$ in the obvious way, we have

$$\big\| a(x) - C_{P(x),\alpha|_{P(x)}}\big(a(x)\big)\big\| < \varepsilon.$$

Proof Let $d = \dim(X)$. Choose $\delta > 0$ such that $(20d + 35)\delta \le \varepsilon$.

For each $x \in X$, choose a finite set $S_x = \{\beta_x^{(1)}, \dots, \beta_x^{(m(x))}\} \subset \operatorname{sp}\big(a(x)\big)$ such that $\operatorname{dist}(\lambda, S_x) < \frac{\varepsilon}{2}$ for $\lambda \in \operatorname{sp}\big(a(x)\big)$. Since $\mathrm{RR}(A) = 0$, we can approximate $a(x)$ arbitrarily closely by a selfadjoint element with finite spectrum, and then find nonzero mutually orthogonal projections $q_x^{(1)}, \dots, q_x^{(m(x))} \in A$ such that, with $e_x = \sum_{j=1}^{m(x)} q_x^{(j)}$, we have

$$\left\| a(x) - \left[(1 - e_x)a(x)(1 - e_x) + \sum_{j=1}^{m(x)} \beta_x^{(j)} q_x^{(j)}\right] \right\| < \delta.$$

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

Then there exist $x_1, \ldots, x_n \in X$ and open sets $W_1, \ldots, W_n \subset X$, with $x_k \in W_k$, such that the W_k cover X and for $x \in W_k$ we have

$$\left\| a(x) - \left[(1 - e_{x_k}) a(x) (1 - e_{x_k}) + \sum_{j=1}^{m(x_k)} \beta_{x_k}^{(j)} q_{x_k}^{(j)} \right] \right\| < 2\delta,$$

and also $\operatorname{dist}(\lambda, S_{x_k}) < \varepsilon$ for $\lambda \in \operatorname{sp}\bigl(a(x)\bigr)$ and $\operatorname{dist}\Bigl(\beta_{x_k}^{(j)}, \operatorname{sp}\bigl(a(x)\bigr)\Bigr) < \varepsilon$ for $1 \leq j \leq m(x)$. Since $\dim(X) = d$, we can refine the open cover $\{W_1, \ldots, W_n\}$ to obtain a new finite open cover such that any distinct $d + 2$ of of its elements have empty intersection. Renaming things, we may assume that we have open sets $Y_1, \ldots, Y_n \subset X$, for each k a set of nonzero mutually orthogonal projections $Q_k = \{q_k^{(1)}, \ldots, q_k^{(m(k))}\} \subset A$, and numbers $\beta_k^{(1)}, \ldots, \beta_k^{(m(k))} \in \mathbb{R}$ such that, regarding $j \mapsto \beta_k^{(j)}$ as a function β_k on Q_k, we have

$$\operatorname{dist}(\lambda, \{\beta_k^{(1)}, \ldots, \beta_k^{(m(k))}\}) < \varepsilon$$

for all $x \in Y_k$ and $\lambda \in \operatorname{sp}\bigl(a(x)\bigr)$,

$$\operatorname{dist}\Bigl(\beta_k^{(j)}, \operatorname{sp}\bigl(a(x)\bigr)\Bigr) < \varepsilon$$

for all $x \in Y_k$ and for $1 \leq j \leq m(k)$, and

$$\bigl\| a(x) - C_{Q_k, \beta_k}\bigl(a(x)\bigr) \bigr\| < 2\delta$$

for all $x \in Y_k$. Then also

$$\|q_k^{(j)} a(x) q_k^{(j)} - \beta_k^{(j)} q_k^{(j)}\| < 2\delta.$$

Let $\rho > 0$ be a suitable small number (chosen below). Use Lemma 2.5 (1) to choose a single large family $\{r_k^{(j)} : 1 \leq k \leq n,\ 1 \leq j \leq m(k)\}$ of nonzero mutually orthogonal projections such that

$$\|q_k^{(j)} r_k^{(j)} - r_k^{(j)}\| < \rho$$

for all k and j. There will then be projections $\widetilde{r}_k^{(j)} \leq q_k^{(j)}$ with $\|\widetilde{r}_k^{(j)} - r_k^{(j)}\| < \widetilde{\rho}$, where $\widetilde{\rho}$ depends only on ρ, and is small if ρ is small. We require that ρ be small enough that

$$\widetilde{\rho} < \min\left(\frac{\delta}{4(\|a\| + 1) \max_k m(k)}, \frac{\delta}{2\|a\| + \max_{j,k} |\beta_k^{(j)}| + 1} \right).$$

Let

$$R_k = \{r_k^{(1)}, \ldots, r_k^{(m(k))}\} \quad \text{and} \quad \widetilde{R}_k = \{\widetilde{r}_k^{(1)}, \ldots, \widetilde{r}_k^{(m(k))}\}.$$

From part (4) of the previous lemma, we get (with an obvious abuse of notation) $\left\| a(x) - C_{\widetilde{R}_k,\beta_k}\big(a(x)\big) \right\| < 4\delta$ whenever $x \in Y_k$. From the estimates above, and using

$$\|r_k^{(j)} a(x) r_k^{(j)} - \widetilde{r}_k^{(j)} a(x) \widetilde{r}_k^{(j)}\| < 2\widetilde{\rho}\|a(x)\|$$

and

$$\left\| \left(1 - \sum_{j=1}^{m(k)} r_k^{(j)}\right) - \left(1 - \sum_{j=1}^{m(k)} \widetilde{r}_k^{(j)}\right) \right\| < m(k)\widetilde{\rho},$$

it now follows that

$$\left\| a(x) - C_{R_k,\beta_k}\big(a(x)\big) \right\| < \left\| a(x) - C_{\widetilde{R}_k,\beta_k}\big(a(x)\big) \right\| + 4m(k)\|a\|\widetilde{\rho} < 5\delta.$$

So part (1) of the previous lemma gives

$$\left\| a(x) - C_{R_k}\big(a(x)\big) \right\| < 10\delta.$$

Also,

$$\|r_k^{(j)} a(x) r_k^{(j)} - \beta_k^{(j)} r_k^{(j)}\| \leq \left\| a(x) - C_{R_k,\beta_k}\big(a(x)\big) \right\| < 5\delta.$$

Now choose open sets $Z_1, \ldots, Z_n$ which still cover X and which satisfy $\overline{Z}_k \subset Y_k$ for all k. Let

$$(U_1, \ldots, U_N) = (Y_1, \ldots, Y_1, Y_2, \ldots, Y_2, \ldots, Y_n, \ldots, Y_n),$$

with the set Y_k being repeated $m(k)$ times. Similarly, let

$$(V_1, \ldots, V_N) = (Z_1, \ldots, Z_1, Z_2, \ldots, Z_2, \ldots, Z_n, \ldots, Z_n),$$
$$(\alpha_1, \ldots, \alpha_N) = (\beta_1^{(1)}, \ldots, \beta_1^{(m(1))}, \beta_2^{(1)}, \ldots, \beta_2^{(m(2))}, \ldots, \beta_n^{(1)}, \ldots, \beta_n^{(m(n))}),$$

and

$$(p_1, \ldots, p_N) = (r_1^{(1)}, \ldots, r_1^{(m(1))}, r_2^{(1)}, \ldots, r_2^{(m(2))}, \ldots, r_n^{(1)}, \ldots, r_n^{(m(n))}).$$

With these choices, conditions (1) and (2) of the lemma are clearly satisfied. We verify (3). Let $x \in X$. Let $P = \{p_l : x \in U_l\}$. Let

$$S = \{k : x \in Y_k\} = \{k(1), \ldots, k(\nu)\}.$$

Thus P is the disjoint union of the sets R_k for $k \in S$. Part (3) of the previous lemma and the relation $\left\| a(x) - C_{R_k}\big(a(x)\big) \right\| < 10\delta$ imply inductively

$$\left\| a(x) - C_{R_{k(1)} \cup \cdots \cup R_{k(\mu)}}\big(a(x)\big) \right\| < (2\mu + 1) \cdot 10\delta$$

for all μ. The choice of the Y_k implies that $\nu \leq d + 1$, and $P = R_{k(1)} \cup \cdots \cup R_{k(\nu)}$, so we get

$$\left\| a(x) - C_P\big(a(x)\big) \right\| < (2d + 3) \cdot 10\delta = (20d + 30)\delta.$$

Also $\|p_l a(x) p_l - \alpha_l p_l\| < 5\delta$ whenever $x \in U_l$ (since $p_l = r_k^{(j)}$ for some suitable k and j, and $\alpha_l = \beta_k^{(j)}$). It follows (using orthogonality) that

$$\left\| a(x) - C_{P,\alpha}\big(a(x)\big) \right\| < (20d + 35)\delta \leq \varepsilon. \qquad \blacksquare$$

Lemma 3.4 *Let X be a compact Hausdorff space with finite covering dimension. Let A be a unital purely infinite simple C^*-algebra, and let $a \in C(X, A)_{\mathrm{sa}}$. Then there exist N, nonzero mutually orthogonal projections $p_1, \dots, p_N \in A$, numbers $\alpha_1, \dots, \alpha_N \in \mathbb{R}$, an element $b \in C(X, A)_{\mathrm{sa}}$, and open sets $U_1, \dots, U_N, V_1, \dots, V_N \subset X$ (not necessarily distinct) satisfying the following properties:*

(1) $\overline{V}_l \subset U_l$ for all l.
(2) For every $x \in X$ and $\lambda \in \mathrm{sp}\big(a(x)\big)$, we have

$$\mathrm{dist}(\lambda, \{\alpha_l : l \textit{ satisfies } x \in V_l\}) < \varepsilon,$$

and, for every $x \in U_l$, we have $\mathrm{dist}\Big(\alpha_l, \mathrm{sp}\big(a(x)\big)\Big) < \varepsilon$.
(3) $p_l b(x) = \alpha_l p_l$ whenever $x \in U_l$.
(4) $\|a - b\| < \varepsilon$.

Proof Apply the previous lemma to a, except using $\frac{\varepsilon}{5}$ in place of ε, and calling the resulting sets $V_l \subset \overline{V}_l \subset W_l$. Choose open sets U_l, Y_l with

$$V_l \subset \overline{V}_l \subset U_l \subset \overline{U}_l \subset Y_l \subset \overline{Y}_l \subset W_l.$$

These choices certainly satisfy conditions (1) and (2). Choose continuous functions $f_l \colon X \to [0, 1]$ which are equal to 1 on $\overline{Y}_l$ and equal to 0 outside W_l.

For $x \in X$ define

$$S(x) = \big\{ l \in \{1, \dots, N\} : x \in W_l \big\}.$$

Inductively define $b_0 = a$ and, given b_l, define

$$\begin{aligned} b_{l+1}(x) &= \big(1 - f_{l+1}(x)\big) b_l(x) + f_{l+1}(x)[(1 - p_{l+1}) b_l(x)(1 - p_{l+1}) + p_{l+1} b_l(x) p_{l+1}] \\ &= \big(1 - f_{l+1}(x)\big) b_l(x) + f_{l+1}(x) C_{\{p_{l+1}\}}\big(b_l(x)\big). \end{aligned}$$

We now claim that $b_l(x)$ is always in the convex hull of the elements $C_T\big(a(x)\big)$ for $T \subset S(x)$. (C_T is to be understood via the obvious abuse of notation for any $T \subset \{1, \dots, N\}$.) This is certainly true for $l = 0$, because $a(x) = C_\varnothing\big(a(x)\big)$. Assume it is known for l. If $x \notin W_{l+1}$, then $b_{l+1}(x) = b_l(x)$, so it is true for $l + 1$. If $x \in W_{l+1}$, then $l \in S(x)$, and it is enough to show that $C_{\{p_{l+1}\}}\big(b_l(x)\big)$ is in this convex hull. Since C_S is always a linear map, and $b_l(x)$ is assumed to be a convex combination of $C_T\big(a(x)\big)$ for various $T \subset S(x)$, this follows from the observation that $C_{\{p_{l+1}\}}\big(C_T(d)\big) = C_{\{l+1\} \cup T}(d)$ for any d. This completes the induction.

We furthermore note that if $x \in Y_l$, then $f_l(x) = 1$, so the orthogonality of the projections p_k implies, for $k \geq l$, that

$$b_k(x) = (1 - p_l)b_k(x)(1 - p_l) + p_l b_k(x) p_l = (1 - p_l)b_k(x)(1 - p_l) + p_l a(x) p_l.$$

In particular, $b_N(x) = (1 - p_l)b_N(x)(1 - p_l) + p_l a(x) p_l$.

Next, recall that by construction we have

$$\left\| a(x) - C_{S(x),\alpha|_{S(x)}}\big(a(x)\big) \right\| < \frac{\varepsilon}{5}$$

for all x. Lemma 3.2 (1) shows that $\left\| a(x) - C_{S(x)}\big(a(x)\big) \right\| < \frac{2\varepsilon}{5}$ for all x. For $T \subset S(x)$, Lemma 3.2 (2), applied to

$$\{p_l : l \in T\} \cup \Big\{ 1 - \sum_{l \in T} p_l \Big\} \quad \text{and} \quad \{p_l : l \in S(x)\} \cup \Big\{ 1 - \sum_{l \in S(x)} p_l \Big\},$$

shows that $\left\| a(x) - C_T\big(a(x)\big) \right\| < \frac{4\varepsilon}{5}$. The convex combination result above therefore implies that $\|a(x) - b_N(x)\| < \frac{4\varepsilon}{5}$ for all x.

Choose continuous functions $g_l \colon X \to [0, 1]$ which are equal to 1 on $\overline{U}_l$ and equal to 0 outside Y_l. Define

$$b(x) = b_N(x) + \sum_{l=1}^{N} g_l(x)\big(\alpha_l p_l - p_l a(x) p_l\big).$$

For $x \in \overline{U}_l$, this gives $b(x) = (1 - p_l)b(x)(1 - p_l) + \alpha_l p_l$. This is condition (3). For condition (4), we note that $\|p_l a(x) p_l - \alpha_l p_l\| < \frac{\varepsilon}{5}$ for $x \in W_l$, so $\|b(x) - b_N(x)\| < \frac{\varepsilon}{5}$, and $\|b(x) - a(x)\| < \varepsilon$. ∎

Now we are ready to thicken the spectrum.

Lemma 3.5 *Let X be a compact Hausdorff space with finite covering dimension. Let A be a unital purely infinite simple C^*-algebra, and let $a \in C(X, A)_{\mathrm{sa}}$. Then there exists $c \in C(X, A)_{\mathrm{sa}}$ such that, for all $x \in X$,*

(1) $\mathrm{sp}\big(c(x)\big)$ *contains the $\frac{\varepsilon}{3}$-neighborhood of* $\mathrm{sp}\big(a(x)\big)$ *and is contained in the ε-neighborhood of* $\mathrm{sp}\big(a(x)\big)$.

(2) For every $\lambda \in \mathrm{sp}\big(c(x)\big)$ there is an interval I of length ε with $\lambda \in I \subset \mathrm{sp}\big(c(x)\big)$.

(3) $\|c(x) - a(x)\| < \varepsilon$.

Proof Apply the previous lemma with $\frac{\varepsilon}{3}$ in place of ε, but otherwise following the notation there. Using Lemma 2.6, choose elements $d_l \in (p_l A p_l)_{\mathrm{sa}}$ with $\mathrm{sp}(d_l) = [0, 1]$. Choose continuous functions $h_l \colon X \to [0, 1]$ which are equal to 1 on $\overline{V}_l$ and equal to 0 outside U_l. Define

$$c_l(x) = \alpha_l p_l + \frac{2\varepsilon}{3}\big(2h_l(x) - 1\big)\, d_l,$$

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

so that

$$\operatorname{sp}\big(c_l(x)\big) = \left[\alpha_l - \frac{2\varepsilon}{3}h_l(x), \alpha_l + \frac{2\varepsilon}{3}h_l(x)\right].$$

Define

$$c(x) = b(x) + \sum_{l=1}^{N}\big(c_l(x) - \alpha_l p_l\big).$$

We always have $\|c_l(x) - \alpha_l p_l\| \leq \frac{2\varepsilon}{3}$, so clearly $\|c(x) - a(x)\| < \varepsilon$. This is condition (3), and implies that $\operatorname{sp}\big(c(x)\big)$ is contained in the ε-neighborhood of $\operatorname{sp}\big(a(x)\big)$, which is half of (1).

Fix $x \in X$. Then $c(x)$ is the orthogonal sum of selfadjoint elements with spectrum $[\alpha_l - \frac{2\varepsilon}{3}, \alpha_l + \frac{2\varepsilon}{3}]$ (for those l with $x \in \overline{V}_l$), selfadjoint elements with spectrum $[\alpha_l - \beta_l(x), \alpha_l + \beta_l(x)]$ with $0 \leq \beta_l(x) \leq \frac{2\varepsilon}{3}$ (for those l with $x \in U_l \setminus \overline{V}_l$), and a selfadjoint element with spectrum contained in $\operatorname{sp}\big(b(x)\big)$. Its spectrum is the union of these sets. By construction, the $\frac{\varepsilon}{3}$-neighborhood of $\{\alpha_l : x \in V_l\}$ contains $\operatorname{sp}\big(a(x)\big)$. Since $\operatorname{sp}\big(c(x)\big)$ contains the $\frac{2\varepsilon}{3}$-neighborhood of $\{\alpha_l : x \in V_l\}$, it contains the $\frac{\varepsilon}{3}$-neighborhood of $\operatorname{sp}\big(a(x)\big)$. This is the other half of (1).

For (2), the existence of an interval I of length ε is clear for $\lambda \in [\alpha_l - \frac{2\varepsilon}{3}, \alpha_l + \frac{2\varepsilon}{3}]$ with $x \in \overline{V}_l$. If $\lambda \in \operatorname{sp}\big(b(x)\big)$, then λ is in the $\frac{\varepsilon}{3}$-neighborhood of $\operatorname{sp}\big(a(x)\big)$. As we have seen, such a λ must also be in one of the intervals $[\alpha_l - \frac{2\varepsilon}{3}, \alpha_l + \frac{2\varepsilon}{3}]$. It remains to consider $\lambda \in [\alpha_l - \beta_l(x), \alpha_l + \beta_l(x)]$ with $x \in U_l \setminus \overline{V}_l$. By construction, $\operatorname{dist}\Big(\alpha_l, \operatorname{sp}\big(a(x)\big)\Big) < \frac{\varepsilon}{3}$. Therefore $|\alpha_l - \alpha_k| < \frac{2\varepsilon}{3}$ for some k with $x \in V_k$. So $\alpha_l \in [\alpha_k - \frac{2\varepsilon}{3}, \alpha_k + \frac{2\varepsilon}{3}]$. Therefore

$$\lambda \in [\alpha_l - \beta_l(x), \alpha_l + \beta_l(x)] \cup \left[\alpha_k - \frac{2\varepsilon}{3}, \alpha_k + \frac{2\varepsilon}{3}\right],$$

which is an interval of length greater than ε and contained in $\operatorname{sp}\big(c(x)\big)$. ∎

Three miscellaneous lemmas are still required before we prove the main result.

Lemma 3.6 *Let A be a unital purely infinite simple C^*-algebra, and let X be a compact Hausdorff space. Let $p, q_1, q_2 \in C(X, A)$ be projections such that $q_1(x), q_2(x) < p(x)$ for all x, and such that each is homotopic to a nonzero constant projection with trivial K_0-class. Let $Y, Z \subset X$ be disjoint closed sets. Then there exists a unitary $u \in pC(X, A)p$ which is homotopic to p and such that $u(x) = p$ for $x \in Y$ and $u(x)q_1(x)u(x)^* = q_2(x)$ for $x \in Z$.*

Proof Conjugating by a suitable unitary, we may assume that p is a constant projection $x \mapsto p_0$ for some $p_0 \in A$. By Lemma 2.8, q_1 and q_2 are still homotopic in $C(X, p_0Ap_0)$ to constant projections with trivial K_0-classes. Therefore we may replace A by p_0Ap_0, and so assume that $p = 1$.

The projections q_1 and q_2 are homotopic, so there is a unitary path $(t, x) \mapsto v_t(x)$, defined for $t \in [0, 1]$ and $x \in X$, such that $v_0 = 1$ and $v_1(x)q_1(x)v_1(x)^* = q_2(x)$ for all $x \in X$. Choose a continuous function $f: X \to [0,1]$ which is equal to 1 on Z and equal to 0 on Y. Then set $u(x) = v_{f(x)}(x)$ for $x \in X$. ■

Lemma 3.7 *Let A be a unital C^*-algebra, and let $a \in A_{\mathrm{sa}}$. Let $\delta > 0$, let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function with $\operatorname{supp}(f) \subset [-\delta, \delta]$, let $p \in \overline{f(a)Af(a)}$ be a projection, and let $u \in A$ be a unitary such that $u(1-p) = (1-p)u = (1-p)$. Then $\|a - uau^*\| \leq 6\delta$.*

Proof We first estimate $\|ap\|$. Let $\rho > 0$. Let $g: \mathbb{R} \to [0,1]$ be a continuous function supported in $[-(\delta+\rho), \delta+\rho]$ and with $g = 1$ on $[-\delta, \delta]$. Then $g(a)f(a) = f(a)$, so $g(a)p = p$. Therefore $\|ap\| = \|ag(a)p\| \leq \|ag(a)\| \leq \delta + \rho$. Since $\rho > 0$ is arbitrary, we get $\|ap\| \leq \delta$. Similarly $\|pa\| \leq \delta$, so also $\|pap\| \leq \delta$.

Now use $u = (1-p) + pup$ and $1 = (1-p) + p$ to write

$$\begin{aligned}
&\|a - uau^*\| \\
&\quad \leq \|(1-p)a(1-p) - (1-p)a(1-p)\| + \|(1-p)ap - (1-p)apu^*p\| \\
&\qquad + \|pa(1-p) - pupa(1-p)\| + \|pap - pupapu^*p\| \\
&\quad \leq 0 + 2\|ap\| + 2\|pa\| + 2\|pap\| \leq 6\delta.
\end{aligned}$$

■

Lemma 3.8 *Let A be a unital C^*-algebra, and let $a \in A_{\mathrm{sa}}$. Let $p \in A$ be a projection, and suppose $(1-p)a(1-p)$ has an inverse r in $(1-p)A(1-p)$. Let*

$$\varepsilon = \frac{1}{2(2\|a\| + 1 + \|r\|^{-1})\|r\|}.$$

If $b \in A_{\mathrm{sa}}$ and a projection $q \in A$ satisfy $\|a - b\| \leq \varepsilon$ and $\|p - q\| \leq \varepsilon$, then $(1-q)b(1-q)$ has an inverse s in $(1-q)A(1-q)$, and $\|s\| \leq 2\|r\|$.

Proof Let $\alpha = \|r\|$. We calculate

$$\begin{aligned}
&\|[(1-q)b(1-q) + \alpha^{-1}q] - [(1-p)a(1-p) + \alpha^{-1}p]\| \\
&\quad \leq \|a - b\| + \|[(1-q)a(1-q) + \alpha^{-1}q] - [(1-p)a(1-p) + \alpha^{-1}p]\| \\
&\quad \leq \|a - b\| + (2\|a\| + \alpha^{-1})\|p - q\| \leq (2\|a\| + 1 + \alpha^{-1})\varepsilon \\
&\quad = \frac{1}{2\|r\|} = \frac{1}{2\|(r + \alpha p)\|}.
\end{aligned}$$

Since

$$[(1-p)a(1-p) + \alpha^{-1}p]^{-1} = r + \alpha p,$$

it follows that $(1-q)b(1-q) + \alpha^{-1}q$ is invertible with

$$\|[(1-q)b(1-q) + \alpha^{-1}q]^{-1}\| \leq 2\|(r + \alpha p)\| = 2\|r\|.$$

Clearly

$$[(1-q)b(1-q) + \alpha^{-1}q]^{-1} = s + \alpha q$$

with s an inverse for $(1-q)b(1-q)$ in $(1-q)A(1-q)$. ■

Lemma 3.9 *Let A be a unital C^*-algebra, and let $a, b \in A_{\mathrm{sa}}$. Let $p, q \in A$ be orthogonal projections, and suppose $(1-p)a(1-p)$ and $(1-q)b(1-q)$ have inverses r and s in $(1-p)A(1-p)$ and $(1-q)A(1-q)$ respectively. Let*

$$\varepsilon = \frac{1}{4\max(\|a\|, \|b\|)\max(\|r\|, \|s\|)^2}.$$

If

$$\|a - [(1-p)a(1-p) + pap]\| < \varepsilon \quad \text{and} \quad \|b - [(1-q)b(1-q) + qbq]\| < \varepsilon,$$

then $a^2 + b^2$ is invertible.

Proof Let

$$a_0 = (1-p)a(1-p) + pap \quad \text{and} \quad b_0 = (1-q)b(1-q) + qbq.$$

With s^{-2} being interpreted in $(1-q)A(1-q)$, we have

$$[(1-q)b(1-q)]^2 = s^{-2} \geq \|s\|^{-2}(1-q) \geq \|s\|^{-2}p.$$

So

$$\begin{aligned} a_0^2 + b_0^2 &= [(1-p)a(1-p)]^2 + [pap]^2 + [(1-q)b(1-q)]^2 + [qbq]^2 \\ &\geq [(1-p)a(1-p)]^2 + [(1-q)b(1-q)]^2 \geq \|r\|^{-2}(1-p) + \|s\|^{-2}p. \end{aligned}$$

Therefore

$$\|(a_0^2 + b_0^2)^{-1}\| \leq \max(\|r\|^2, \|s\|^2).$$

Using $\|a_0\| \leq \|a\|$, we get

$$\|a^2 - a_0^2\| \leq \|a\|\,\|a - a_0\| + \|a_0\|\,\|a - a_0\| \leq 2\|a\|\,\|a - a_0\| < \frac{1}{2\max(\|r\|^2, \|s\|^2)}.$$

Similarly,

$$\|b^2 - b_0^2\| < \frac{1}{2\max(\|r\|^2, \|s\|^2)}.$$

So

$$\|(a^2 + b^2) - (a_0^2 + b_0^2)\| < \|(a_0^2 + b_0^2)^{-1}\|^{-1}.$$

It follows that $a^2 + b^2$ is invertible. ■

Theorem 3.10 *Let A be a purely infinite simple C^*-algebra, and let X be a compact Hausdorff space. Then $\mathrm{RR}\big(C(X) \otimes A\big) \leq 1$.*

Proof We first consider the case $X \subset [0,1]^n$ for some n and A separable. By Theorem 1.2 of [34], A is either unital or stable. If A is stable, then so is $C(X) \otimes A$, whence $\mathrm{RR}\big(C(X) \otimes A\big) \leq 1$ by Proposition 3.3 of [4]. Therefore, without loss of generality, A is unital.

We have to show that if $a_1, a_2 \in C(X, A)_{\mathrm{sa}}$, then the a_i can be arbitrarily closely approximated by $b_i \in C(X, A)_{\mathrm{sa}}$ such that $b_1^2 + b_2^2$ is invertible. By Lemma 3.5 we may assume without loss of generality that, for some $\delta > 0$, for every $\lambda \in \mathrm{sp}\big(a_i(x)\big)$ there is an interval I of length δ with $\lambda \in I \subset \mathrm{sp}\big(a_i(x)\big)$.

Let $\varepsilon > 0$. Let $M = \max(\|a_1\|, \|a_2\|)$. Choose $\alpha > 0$ with $\alpha < \max(\frac{\delta}{8}, \frac{\varepsilon}{48})$. Let $f \colon \mathbb{R} \to [0,1]$ be the continuous function which is equal to 1 on $[-2\alpha, 2\alpha]$, equal to 0 outside $[-4\alpha, 4\alpha]$, and linear on $[-4\alpha, -2\alpha]$ and on $[2\alpha, 4\alpha]$. Choose $\rho_1 > 0$ such that $\rho_1 < \frac{\varepsilon}{4M}$, and such that if projections p and q in some C^*-algebra satisfy $\|qp - p\| < 2\rho_1$, then there is a projection $e \leq q$ such that

$$\|e - p\| < \min\left(\frac{\alpha}{2M + 1 + 2\alpha}, \frac{\alpha^2}{32M^2}\right).$$

Choose $\rho_2 > 0$ such that

$$\rho_2 < \min\left(\frac{\alpha^2}{24M}, \frac{\alpha}{2M + 1 + 2\alpha}\right),$$

and such that if selfadjoint elements c_1 and c_2 in some C^*-algebra satisfy $\|c_i\| \leq M$ and $\|c_1 - c_2\| < \rho_2$, then $\|f(c_1) - f(c_2)\| < \rho_1$. (Approximate f to within $\frac{\rho_1}{3}$ on $[-M, M]$ by a polynomial g, and choose ρ_2 small enough that $\|g(c_1) - g(c_2)\| < \frac{\rho_1}{3}$.) Choose $\rho_3 > 0$ such that if projections p and q in some unital C^*-algebra satisfy $\|qp - p\| < \rho_3$, then there is a projection $e \leq q$ such that $\|e - p\| < \rho_1$, and there is a unitary u such that $ueu^* = p$ and $\|u - 1\| < \rho_1$.

Define

$$Y_i = \big\{x \in X : \mathrm{sp}\big(a_i(x)\big) \text{ contains at least one of } 3\alpha \text{ and } -3\alpha\big\}$$

and

$$Z_i = \big\{x \in X : \mathrm{sp}\big(a_i(x)\big) \text{ contains at least one of } \alpha \text{ and } -\alpha\big\}.$$

Recall that every point of $\mathrm{sp}\big(a_i(x)\big)$ is contained in an interval in $\mathrm{sp}\big(a_i(x)\big)$ of length at least 8α. It follows, first, that if $x \notin Z_i$ then $a_i(x)$ is invertible. So if $Z_1 \cap Z_2 = \varnothing$ then $a_1^2 + a_2^2$ is already invertible. Thus without loss of generality $Z_1 \cap Z_2 \neq \varnothing$. It also follows that Y_i contains a neighborhood of Z_i.

For each $x \in Z_i$, choose $c_x^{(i)} \in A_{\mathrm{sa}}$ such that $\mathrm{sp}(c_x^{(i)})$ is finite, $\|c_x^{(i)} - a_i(x)\| < \rho_2$, and $\|c_x^{(i)}\| < \|a_i(x)\|$. (This is possible because purely infinite simple C^*-algebras have real rank zero [33].) Let $e_x^{(i)} \in A$ be the spectral projection for $c_x^{(i)}$ corresponding to the interval $[-2\alpha, 2\alpha]$. Note that $(1 - e_x^{(i)})c_x^{(i)}(1 - e_x^{(i)})$ is invertible in $(1 - e_x^{(i)})A(1 - e_x^{(i)})$, and its inverse $d_x^{(i)}$ in that algebra satisfies $\|d_x^{(i)}\| \leq \frac{1}{2\alpha}$. It follows from the choice of ρ_2 that $\big\| f\big(a_i(x)\big) - f(c_x^{(i)})\big\| < \rho_1$. Using $f(c_x^{(i)})e_x^{(i)} = e_x^{(i)}$, we get $\big\| f\big(a_i(x)\big)\, e_x^{(i)} - e_x^{(i)}\big\| < \rho_1$.

For $x \in Y_i$, we have $3\alpha \in \mathrm{sp}\big(a_i(x)\big)$ or $-3\alpha \in \mathrm{sp}\big(a_i(x)\big)$. Since there is an interval in $\mathrm{sp}\big(a_i(x)\big)$ of length more than 8α which contains one of these numbers, at least one of the intervals $[-4\alpha, -3\alpha]$ and $[3\alpha, 4\alpha]$ is contained in $\mathrm{sp}\big(a_i(x)\big)$. Since $f(-4\alpha) = f(4\alpha) = 0$ and $f(-3\alpha), f(3\alpha) \neq 0$, the hereditary subalgebra $\overline{f\big(a_i(x)\big)Af\big(a_i(x)\big)}$ is nonzero and nonunital. By Lemma 2.11, the hereditary subalgebra

$$\overline{f(a_i|_{Y_1\cap Y_2})C(Y_1\cap Y_2, A)f(a_i|_{Y_1\cap Y_2})}$$

has an approximate identity of projections each of which is homotopic to a constant projection with trivial K_0-class. In particular, there exist projections

$$p_i, q_i \in \overline{f(a_i|_{Y_1\cap Y_2})C(Y_1\cap Y_2, A)f(a_i|_{Y_1\cap Y_2})},$$

each homotopic to a constant projection with trivial K_0-class, such that

$$p_i(x) > q_i(x) \quad \text{and} \quad \big\| q_i(x)f\big(a_i(x)\big) - f\big(a_i(x)\big)\big\| < \rho_1$$

for all x. It follows that $\|q_i(x)e_x^{(i)} - e_x^{(i)}\| < 2\rho_1$ for $x \in Z_i \cap Y_1 \cap Y_2$. By the choice of ρ_1, there are projections $\widetilde{e}_x^{(i)} \leq q_i(x)$ such that

$$\|\widetilde{e}_x^{(i)} - e_x^{(i)}\| < \min\left(\frac{\alpha}{2M+1+2\alpha}, \frac{\alpha^2}{32M^2}\right).$$

Use Lemma 2.14 to find nowhere zero orthogonal projections $r_1, r_2 \in C(Y_1\cap Y_2, A)$, each homotopic to a constant projection with trivial K_0-class, such that $\|q_i r_i - r_i\| < \rho_3$. By the choice of ρ_3, there exist projections $\widetilde{r}_i \leq q_i$ and unitaries $u_i^{(0)} \in C(Y_1\cap Y_2, A)$ such that

$$\|\widetilde{r}_i - r_i\| < \rho_1, \quad u_i^{(0)}\widetilde{r}_i(u_i^{(0)})^* = r_i, \quad \text{and} \quad \|u_i^{(0)} - 1\| < \rho_1.$$

Applying the Tietze extension theorem to $\log(u_i^{(0)})$ and exponentiating, we obtain unitaries $u_i \in C(X, A)$ such that

$$u_i|_{Y_1\cap Y_2} = u_i^{(0)} \quad \text{and} \quad \|u_i - 1\| < \rho_1.$$

Use Lemma 3.6 to find unitaries $v_i \in C(Y_1\cap Y_2, A)$ such that $v_i(1-p_i) = (1-p_i)v_i = (1-p_i)$, v_i is equal to 1 on $\partial(Y_1\cap Y_2)$, and $v_i(x)q_i(x)v_i(x)^* = \widetilde{r}_i(x)$ for $x \in Z_1 \cap Z_2$. (The boundary is taken in X. Since Z_i is contained in the interior of Y_i, the sets $\partial(Y_1\cap Y_2)$ and $Z_1\cap Z_2$ are disjoint.) Extend v_i to a unitary in $C(X, A)$ by taking $v_i(x) = 1$ for $x \notin Y_1 \cap Y_2$.

Now define $b_i = u_i v_i a_i v_i^* u_i^*$. We show that $\|b_i - a_i\| < \varepsilon$ and $b_1^2 + b_2^2$ is invertible. We have

$$\|b_i - a_i\| \leq 2\|1-u_i\|\,\|a_i\| + \|v_i a_i v_i^* - a_i\|.$$

The first term is at most $2\rho_1 M < \frac{\varepsilon}{2}$. By Lemma 3.7, the second term is at most $6(4\alpha) < \frac{\varepsilon}{2}$. So $\|b_i - a_i\| < \varepsilon$, as desired.

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

As we saw above, if $x \notin Z_i$ then $a_i(x)$ is invertible. So also $b_i(x)$ is invertible. Thus $b_1(x)^2 + b_2(x)^2$ is invertible for $x \notin Z_1 \cap Z_2$. So let $x \in Z_1 \cap Z_2$. We have

$$\|[(1 - e_x^{(i)})c_x^{(i)}(1 - e_x^{(i)})]^{-1}\| = \|d_x^{(i)}\| \leq \frac{1}{2\alpha}$$

(the inverse being taken in $(1 - e_x^{(i)})A(1 - e_x^{(i)})$),

$$\|\widetilde{e}_x^{(i)} - e_x^{(i)}\| < \min\left(\frac{\alpha}{2M + 1 + 2\alpha}, \frac{\alpha^2}{32M^2}\right),$$

and

$$\|c_x^{(i)} - a_i(x)\| < \rho_2 < \min\left(\frac{\alpha}{2M + 1 + 2\alpha}, \frac{\alpha^2}{24M}\right).$$

Applying Lemma 3.8, we find that $(1 - \widetilde{e}_x^{(i)})a_i(x)(1 - \widetilde{e}_x^{(i)})$ is invertible in $(1 - \widetilde{e}_x^{(i)})A(1 - \widetilde{e}_x^{(i)})$, and

$$\|[(1 - \widetilde{e}_x^{(i)})a_i(x)(1 - \widetilde{e}_x^{(i)})]^{-1}\| \leq \frac{1}{\alpha}.$$

Moreover,

$$\begin{aligned}
&\|a_i(x) - [(1 - \widetilde{e}_x^{(i)})a_i(x)(1 - \widetilde{e}_x^{(i)}) + \widetilde{e}_x^{(i)}a_i(x)\widetilde{e}_x^{(i)}]\| \\
&\quad \leq \|[(1 - \widetilde{e}_x^{(i)})a_i(x)(1 - \widetilde{e}_x^{(i)}) + \widetilde{e}_x^{(i)}a_i(x)\widetilde{e}_x^{(i)}] \\
&\qquad - [(1 - e_x^{(i)})a_i(x)(1 - e_x^{(i)}) + e_x^{(i)}a_i(x)e_x^{(i)}]\| \\
&\qquad + 3\|a_i(x) - c_x^{(i)}\| + \|c_x^{(i)} - [(1 - e_x^{(i)})c_x^{(i)}(1 - e_x^{(i)}) + e_x^{(i)}c_x^{(i)}e_x^{(i)}]\| \\
&\quad < 4\|\widetilde{e}_x^{(i)} - e_x^{(i)}\|M + 3\rho_2 + 0 \\
&\quad < 4M\left(\frac{\alpha^2}{32M^2}\right) + 3\left(\frac{\alpha^2}{24M}\right) = \frac{\alpha^2}{4M}.
\end{aligned}$$

We conjugate the elements considered above by $u_i v_i$; this does not change the norm estimates. Now

$$u_1(x)v_1(x)\widetilde{e}_x^{(1)}v_1(x)^*u_1(x)^* \leq r_1(x) \quad \text{and} \quad u_2(x)v_2(x)\widetilde{e}_x^{(2)}v_2(x)^*u_2(x)^* \leq r_2(x)$$

are orthogonal projections, so Lemma 3.9 implies that

$$\left(u_1(x)v_1(x)a_1(x)v_1(x)^*u_1(x)^*\right)^2 + \left(u_2(x)v_2(x)a_2(x)v_2(x)^*u_2(x)^*\right)^2 = b_1(x)^2 + b_2(x)^2$$

is invertible.

This proves the special case $X \subset [0,1]^n$ and A separable. This case covers all finite complexes X. By Theorem 10.1 in Chapter 10 of [9], a general compact Hausdorff space is an inverse limit of finite complexes. Therefore the result holds for arbitrary compact Hausdorff X by taking direct limits. Finally, if A is not separable, we write it as a direct limit of separable purely infinite simple C^*-algebras. (The proof of Theorem 4.3.11 of [27] shows that if A is a purely infinite simple C^*-algebra and $S \subset A$ is a separable subset, then there is a separable purely infinite simple C^*-algebra B with $S \subset B \subset A$. It follows that A is the direct limit of its separable purely infinite simple subalgebras.) The corresponding direct limit expression for algebras of functions proves the theorem in full generality. ∎

Corollary 3.11 *Let A be a purely infinite simple C^*-algebra, and let X be a locally compact Hausdorff space. Then* $\mathrm{RR}\big(C_0(X) \otimes A\big) \leq 1$.

Proof Let X^+ be the one point compactification of X. Apply Theorem 1.4 of [11] (passage to ideals does not increase real rank) to $C_0(X) \otimes A$ as an ideal in $C(X^+) \otimes A$. ∎

4 Stable Rank of $C([0,1]) \otimes A$

This section is devoted to the proof that if $\mathrm{RR}(A) = 0$, $\mathrm{sr}(A) = 1$, and $K_1(A) = 0$ (a kind of "strong zero dimensionality" condition—note that if X is zero dimensional then $K^1(X) = 0$), then $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$. The hypotheses are satisfied by AF algebras, but are also satisfied by some nonnuclear C^*-algebras, and some nuclear C^*-algebras whose K_0-groups contain torsion; these cannot be AF. See Examples 4.4 and 4.5 at the end of this section.

Lemma 4.1 *For every $\varepsilon > 0$ there is $\delta > 0$ such that whenever A is a unital C^*-algebra, $u, v \in A$ are unitaries, and $p \in A$ is a projection, with $\|up - vp\| < \delta$, then there is a path $t \mapsto z_t$ of unitaries in A with $z_0 = 1$, $z_1 up = vp$, and $\|z_t - 1\| < \varepsilon$ for all $t \in [0,1]$.*

Proof Let $\varepsilon > 0$. Choose $\delta_0 > 0$ small enough that whenever z is a unitary in a unital C^*-algebra B with $\|z - 1\| < \delta_0$, then there is a continuous unitary path $t \mapsto z_t \in U(B)$ such that

$$z_0 = 1, \quad z_1 = z, \quad \text{and} \quad \sup_{t \in [0,1]} \|z_t - 1\| < \varepsilon.$$

Choose $\delta > 0$ small enough that whenever e and f are projections in a unital C^*-algebra B with $\|e - f\| < 2\delta$, then there is $y \in U(B)$ such that $yey^* = f$ and $\|y - 1\| < \frac{\delta_0}{2}$. We further require $\delta < \frac{\delta_0}{2}$.

Let u, v, and p be as in the hypotheses, and assume that $\|up - vp\| < \delta$. Then $\|upu^* - vpv^*\| < 2\delta$. Choose a unitary $y \in A$ as in the previous paragraph, for $e = upu^*$ and $f = vpv^*$, and satisfying $\|y - 1\| < \frac{\delta_0}{2}$. Set $w = yu(1-p)u^*$, which is a partial isometry in A such that $w^*w = u(1-p)u^*$ and $ww^* = v(1-p)v^*$. Then $\tilde{w} = vpu^* + w$ is a unitary such that $\tilde{w}up = vp$. Moreover,

$$\|\tilde{w} - 1\| \leq \|vpu^* - upu^*\| + \|w - u(1-p)u^*\| \leq \|up - vp\| + \|y - 1\| < \delta_0.$$

Hence there is a path $t \mapsto z_t$ in $U(A)$ such that $z_0 = 1$, $z_1 = \tilde{w}$, and $\|z_t - 1\| < \varepsilon$. This is the required path. ∎

Lemma 4.2 *Let A be a unital C^*-algebra with $K_1(A) = 0$, $\mathrm{sr}(A) = 1$, and $\mathrm{RR}(A) = 0$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $a, b \in \mathrm{inv}(A)$ satisfy $\|a\|, \|b\| \leq 1$ and $\|a - b\| < \delta$, then there is a continuous path $t \mapsto c_t$ in $\mathrm{inv}(A)$ such that*

$$c_0 = a, \quad c_1 = b, \quad \textit{and} \quad \|c_t - a\| < \varepsilon.$$

Proof Let $\varepsilon > 0$. Set $\rho = \frac{\varepsilon}{4}$. Choose $\delta_0 > 0$ with $\delta_0 < \frac{\varepsilon}{4}$, and also so small that, following Lemma 4.1, if $\|up - vp\| < 4\delta_0/\rho$, then the unitary path $t \mapsto z_t$ there satisfies $\|z_t - 1\| < \frac{\varepsilon}{2}$. Now choose $\delta > 0$ with $\delta \leq \delta_0 \leq \frac{\varepsilon}{4}$, and also so small that if $a, b \in A$ satisfy $\|a\|, \|b\| \leq 1$ and $\|a - b\| < \delta$, then $\||a| - |b|\| < \delta_0$. (Recall that $|a| = (a^*a)^{1/2}$.)

Let $a, b \in \mathrm{inv}(A)$ satisfy $\|a\|, \|b\| \leq 1$ and $\|a - b\| < \delta$. We are going to construct an invertible path from a to b, within the ε-ball $B_\varepsilon(a) = \{c \in A : \|c - a\| < \varepsilon\}$, in five stages: from $a = r_0$ to r_1 to r_2 to r_3 to r_4 and finally to $r_5 = b$.

Define $u = a(a^*a)^{-1/2}$ and $v = b(b^*b)^{-1/2}$, so that $a = u|a|$ and $b = v|b|$ are the polar decompositions of a and b respectively.

Define a continuous invertible path from $r_0 = a = u|a|$ to $r_1 = u|b|$ by $t \mapsto u\big(t|b| + (1-t)|a|\big)$. Clearly this path is invertible and has the correct values at $t = 0$ and $t = 1$. Moreover, from the choices above, we have $\||a| - |b|\| < \delta_0$. Therefore

$$\Big\| u\big(t|b| + (1-t)|a|\big) - a\Big\| = t\||a| - |b|\| < \delta_0 < \varepsilon.$$

Also note that $\|r_1 - a\| < \delta_0$.

Since $\mathrm{RR}(A) = 0$, there is a positive invertible element $b_0 \in A_{\mathrm{sa}}$ such that $\mathrm{sp}(b_0)$ is finite and $\|b_0 - |b|\| < \delta$. Define a continuous invertible path from $r_1 = u|b|$ to $r_2 = ub_0$ by $t \mapsto u\big(tb_0 + (1-t)|b|\big)$. Following the reasoning of the previous paragraph, this path is invertible and satisfies

$$\Big\| u\big(tb_0 + (1-t)|b|\big) - a\Big\| \leq \Big\| u\big(tb_0 + (1-t)|b|\big) - u|b|\Big\| + \|u|b| - a\| < \delta + \delta_0 < \varepsilon.$$

Also note that $\|r_2 - a\| < \delta + \delta_0$. Similarly define a continuous invertible path from $r_4 = vb_0$ to $r_5 = b = v|b|$ by $t \mapsto v\big((1-t)b_0 + t|b|\big)$. This path satisfies

$$\Big\| v\big((1-t)b_0 + t|b|\big) - a\Big\| \leq \Big\| v\big((1-t)b_0 + t|b|\big) - b\Big\| + \|b - a\| < 2\delta < \varepsilon,$$

and in particular $\|r_4 - a\| < 2\delta$.

Let $\chi_{[\rho,\infty)}$ be the characteristic function of the interval $[\rho, \infty)$. Define $p = \chi_{[\rho,\infty)}(b_0)$, which is well defined because b_0 has finite spectrum. Let $f\colon [0, 1] \to [0, \rho^{-1}]$ be a continuous function such that $f(t) = t^{-1}$ for $t \geq \rho$. Then $b_0 f(b_0)p = p$ and $\|f(b_0)\| \leq \rho^{-1}$. Therefore

$$\begin{aligned}\|up - vp\| &= \|ub_0 f(b_0)p - vb_0 f(b_0)p\| \leq \rho^{-1}\|ub_0 - vb_0\| \\ &= \rho^{-1}\|r_2 - r_4\| \leq \rho^{-1}(\|r_2 - a\| + \|r_4 - a\|) < \rho^{-1}(\delta_0 + 3\delta) \leq 4\delta_0\rho^{-1}.\end{aligned}$$

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

By the choice of δ_0 above, and following Lemma 4.1, there is a unitary path $t \mapsto z_t$ such that $z_0 = 1$, $z_1 up = vp$, and $\|z_t - 1\| < \frac{\varepsilon}{2}$. Now define a continuous invertible path from $r_2 = ub_0$ to $r_3 = z_1 ub_0$ by $t \mapsto z_t ub_0$. We have

$$\|z_t ub_0 - a\| \leq \|z_t ub_0 - ub_0\| + \|ub_0 - a\| < \frac{\varepsilon}{2} + \delta + \delta_0 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

We have the polar decompositions $r_3 = (z_1 u)b_0$ and $r_4 = vb_0$, in which the projection p commutes with b_0 and satisfies $v^* z_1 up = pv^* z_1 u = p$. Therefore, if we construct a continuous invertible path $t \mapsto c_t$ in $(1-p)A(1-p)$ from

$$c_0 = v^* r_3 (1-p) = (1-p)v^* z_1 u(1-p) \cdot (1-p)b_0(1-p)$$

to

$$c_1 = v^* r_4 (1-p) = (1-p)b_0(1-p),$$

the assignment $t \mapsto v(c_t + b_0 p)$ will define a continuous invertible path from r_3 to r_4. Now $\mathrm{RR}(A) = 0$ and $K_1(A) = 0$, so Lemma 2.4 of [17] implies $K_1\big((1-p)A(1-p)\big) = 0$. Moreover, $\mathrm{sr}\big((1-p)A(1-p)\big) = 1$ (see the proof of Lemma 3.4 of [31]), so that Theorem 2.10 of [32] implies that $U\big((1-p)A(1-p)\big)$ is connected. Therefore there is a continuous unitary path $t \mapsto w_t$ in $(1-p)A(1-p)$ with $w_0 = (1-p)v^* z_1 u(1-p)$ and $w_1 = 1-p$. The path $c_t = w_t(1-p)b_0(1-p)$ satisfies the conditions above.

It remains to estimate the distance from a. Since $vb_0 = r_4$, we have

$$\begin{aligned}\|v(c_t + b_0 p) - a\| &\leq \|v(c_t + b_0 p) - vb_0\| + \|r_4 - a\| = \|c_t - (1-p)b_0\| + \|r_4 - a\| \\ &\leq \|c_t\| + \|(1-p)b_0\| + \|r_4 - a\| = 2\|(1-p)b_0\| + \|r_4 - a\| \\ &< 2\rho + 2\delta \leq \varepsilon.\end{aligned}$$

We have now connected $r_0 = a$ to $r_5 = b$ by a continuous invertible path which lies in the ε-ball $B_\varepsilon(a)$. ■

Theorem 4.3 *Let A be a unital C^*-algebra with $K_1(A) = 0$, $\mathrm{sr}(A) = 1$, and $\mathrm{RR}(A) = 0$. Then $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$.*

Proof Let $a \in C([0,1]) \otimes A$, and let $\varepsilon > 0$. We have to approximate a within ε by an invertible element of $C([0,1]) \otimes A$. Scaling both a and ε, we may assume without loss of generality that $\|a\| \leq 1$.

Choose $\delta > 0$ as in the previous lemma for $\frac{\varepsilon}{3}$ in place of ε. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$\|a(t_j) - a(t_{j-1})\| < \frac{\delta}{3} \quad \text{and} \quad \|a(t) - a(t_{j-1})\| < \frac{\varepsilon}{3}$$

for $1 \leq j \leq n$ and $t \in [t_{j-1}, t_j]$. Using the fact that $\mathrm{sr}(A) = 1$, choose $c_0, c_1, \ldots, c_n \in \mathrm{inv}(A)$ such that

$$\|c_j - a(t_j)\| < \min\left(\frac{\varepsilon}{3}, \frac{\delta}{3}\right).$$

Then $\|c_j - c_{j-1}\| < \delta$. For each j, use the previous lemma to choose a continuous path $t \mapsto b(t) \in \mathrm{inv}(A)$, defined for $t \in [t_{j-1}, t_j]$, such that

$$b(t_{j-1}) = c_{j-1}, \quad b(t_j) = c_j, \quad \text{and} \quad \|b(t) - c_{j-1}\| < \frac{\varepsilon}{3}.$$

The two definitions at t_j (one from the j-th interval, one from the $(j+1)$-st interval) agree, so $t \mapsto b(t)$ is a continuous invertible path defined for $t \in [0, 1]$. Moreover, for $t \in [t_{j-1}, t_j]$ we have

$$\begin{aligned}\|b(t) - a(t)\| &\leq \|b(t) - c_{j-1}\| + \|c_{j-1} - a(t_{j-1})\| + \|a(t_{j-1}) - a(t)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$ ■

We now give several examples of simple separable unital C^*-algebras which satisfy the hypotheses of this theorem but are not AF. In particular, $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$ does not imply that A is AF, even if A is nuclear. We will see in the next section that $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$ does not even imply that $\mathrm{RR}(A) = 0$.

Example 4.4 Example 4.11 of [22] gives a simple separable unital nuclear C^*-algebra A satisfying $K_1(A) = 0$ and $\mathrm{RR}(A) = 0$. It also has $\mathrm{sr}(A) = 1$ (by Theorem 2.4 (3) of [22] or by [8]). It thus satisfies the hypotheses of Theorem 4.3. It is not AF because $K_0(A)$ contains torsion.

Example 4.5 Apply Proposition 9 or Theorem 11 of [7], starting with a UHF algebra. The result is a simple separable unital C^*-algebra A satisfying $K_1(A) = 0$, $\mathrm{RR}(A) = 0$, and $\mathrm{sr}(A) = 1$. The group $K_0(A)$ is even torsion-free, and A is even asymptotically homotopy equivalent to the original UHF algebra. However, A is not AF because A is not nuclear.

More examples of this type are contained in [29].

5 Lower Bounds on Rank

In this section, we determine to what extent the converse of Theorem 4.3, the main result of the previous section, is true. We see that if $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$, then indeed necessarily $\mathrm{sr}(A) = 1$ and $K_1(A) = 0$, but we show by example that it need not follow that $\mathrm{RR}(A) = 0$, even for simple A. We also prove two related results: for any nonzero C^*-algebra A, we have $\mathrm{RR}\big(C([0,1]) \otimes A\big) \geq 1$, and for any unital C^*-algebra A, we have $\mathrm{sr}\big(C([0,1]^2) \otimes A\big) \geq 2$.

Proposition 5.1 *Let A be a nonzero C^*-algebra. Then* $\mathrm{RR}\big(C([0,1]) \otimes A\big) \geq 1$.

Proof Suppose that $\mathrm{RR}\big(C([0,1]) \otimes A\big) = 0$. We first reduce to the unital case. Since A is a quotient of $C([0,1]) \otimes A$, it follows that A is a nonzero C^*-algebra with $\mathrm{RR}(A) = 0$. Therefore A contains a nonzero projection p. The algebra $C([0,1]) \otimes$

pAp is a corner of $C([0,1]) \otimes A$, so also has real rank zero. Replacing A by pAp, we may therefore assume A is unital.

Define $f \in C([0,1], A)_{\mathrm{sa}}$ by $f(t) = (2t-1) \cdot 1_A$ for $0 \leq t \leq 1$. By assumption, there is an invertible selfadjoint element $g \in C([0,1], A)$ such that $\|f - g\| < \frac{1}{2}$. Define $h\colon [0,1] \to [0,\infty)$ by $h(t) = \sup \operatorname{sp}\bigl(g(t)\bigr)$. Let $M = \|g\|$. Since the element $g + M \cdot 1_{C([0,1]) \otimes A} \in C([0,1]) \otimes A$ is positive, we can write

$$h(t) = \sup \operatorname{sp}\bigl(g(t) + M \cdot 1_A\bigr) - M = \|g(t) + M \cdot 1_A\| - M,$$

so that h is continuous. Clearly $h(0) < 0$ and $h(1) > 0$. Therefore there exists $t_0 \in (0,1)$ such that $h(t_0) = 0$. So $\operatorname{sp}\bigl(g(t_0)\bigr)$ contains 0, that is, $g(t_0)$ is not invertible. This contradicts our assumption. ■

Proposition 5.2 *Let A be any C^*-algebra. Suppose that* $\operatorname{sr}\bigl(C([0,1]) \otimes A\bigr) = 1$. *Then* $\operatorname{sr}(A) = 1$ *and* $K_1(A) = 0$.

Proof We have $\operatorname{sr}(A) = 1$ because A is a quotient of $C([0,1]) \otimes A$.

We prove that $K_1(A) = 0$ in the nonunital case. (The proof in the unital case is similar but easier.) First, note that $M_n(A)$ also satisfies the hypothesis, by Theorem 6.1 of [31]. Therefore it suffices to show that $U(A^+)$ is connected. We recall that $U_0(B)$ and $\operatorname{inv}_0(B)$ denote the identity components of $U(B)$ and $\operatorname{inv}(B)$ respectively.

So let $u \in A^+$ be unitary. Let $\lambda \in \mathbb{C}$ be its image under the standard map $\pi\colon A^+ \to \mathbb{C}$. To show that $u \in U_0(A^+)$, it suffices to show that $\lambda^{-1}u \in U_0(A^+)$. Accordingly, we assume that $\pi(u) = 1$. Define $f \in \bigl[C([0,1]) \otimes A\bigr]^+$ by $f(t) = t \cdot 1 + (1-t)u \in A^+$ for $t \in [0,1]$. Note that f really is in $\bigl[C([0,1]) \otimes A\bigr]^+$, because $\pi(u) = 1$. Use $\operatorname{sr}\bigl(C([0,1]) \otimes A\bigr) = 1$ to choose an invertible element $g \in \bigl[C([0,1]) \otimes A\bigr]^+$ such that $\|g - f\| < \frac{1}{2}$. Because $\|u - g(0)\| = \|f(0) - g(0)\| < 1$, there is a continuous path in $\operatorname{inv}(A^+)$ from u to $g(0)$. Similarly, there is a continuous path in $\operatorname{inv}(A^+)$ from 1 to $g(1)$. Combining these paths with the continuous path g in $\operatorname{inv}(A^+)$ from $g(0)$ to $g(1)$, we see that $u \in \operatorname{inv}_0(A^+)$. As is well known, this implies $u \in U_0(A^+)$. ■

Proposition 5.3 *Let A be a unital C^*-algebra. Then* $\operatorname{sr}\bigl(C([0,1]^2) \otimes A\bigr) \geq 2$.

Proof Suppose that $\operatorname{sr}\bigl(C([0,1]^2) \otimes A\bigr) = 1$. Then $\operatorname{sr}\bigl(C(S^1) \otimes C([0,1]) \otimes A\bigr) = 1$ by Lemma 1.14. So $\operatorname{sr}\bigl(C(S^1) \otimes A\bigr) = 1$ and $K_1\bigl(C(S^1) \otimes A\bigr) = 0$ by Proposition 5.2. Therefore $0 = K_1\bigl(C(S^1) \otimes A\bigr) \cong K_1(A) \oplus K_0(A)$, whence $K_0(A) = 0$. Since A is stably finite (because $\operatorname{sr}(A) \leq \operatorname{sr}\bigl(C(S^1) \otimes A\bigr) \leq 1$) and unital, this is a contradiction. ■

The same proof works as soon as $K \otimes A$ has a nontrivial projection, using the fact (Theorem 3.6 of [31]) that $\operatorname{sr}(K \otimes B) = 1$ if and only if $\operatorname{sr}(B) = 1$. We do not know what happens if A is stably projectionless.

We now prove some lemmas in preparation for our example of a C^*-algebra A such that $\operatorname{sr}\bigl(C([0,1]) \otimes A\bigr) = 1$ but $\operatorname{RR}(A) \neq 0$. The algebra A will be a direct limit of the form considered in [13], with the space involved being $[0,1]$.

Lemma 5.4 *Let X be a compact Hausdorff space, let $a \in C(X, M_n)$, and let $x_0 \in X$. Then there are a neighborhood U of x_0, an integer k with $0 \leq k \leq n$, elements $r, s \in \operatorname{inv}\big(C(\overline{U}, M_n)\big)$, and $b \in C(\overline{U}, M_k)$, such that*

$$r(a|_{\overline{U}})s = b \oplus 1_{n-k} \quad \textit{and} \quad b(0) = 0.$$

Proof Let $k = n - \operatorname{rank}\big(a(x_0)\big)$. Let $p = \operatorname{diag}(0, \ldots, 0, 1, \ldots, 1) \in M_n$, with 0 appearing k times on the diagonal and 1 appearing $n - k$ times. By standard row and column reduction, there exist invertible $r_0, s_0 \in M_n$ such that $r_0 a(x_0) s_0 = p$. Write the matrix $r_0 a(x) s_0$ in block form as

$$r_0 a(x) s_0 = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

where

$$a_{11}(x) = (1-p) r_0 a(x) s_0 (1-p) \in (1-p) M_n (1-p) \cong M_k$$

and

$$a_{22}(x) = p r_0 a(x) s_0 p \in p M_n p \cong M_{n-k}.$$

Then $a_{22}(x_0) = p$. Therefore there is a neighborhood U of x_0 such that $c(x) = a_{22}(x)^{-1}$ exists in $pM_n p$ for every $x \in \overline{U}$. For such x, define

$$r(x) = \begin{pmatrix} 1 & 0 \\ 0 & c(x) \end{pmatrix} \begin{pmatrix} 1 & -a_{12}(x)c(x) \\ 0 & 1 \end{pmatrix} r_0 \quad \text{and} \quad s(x) = s_0 \begin{pmatrix} 1 & 0 \\ -c(x)a_{21}(x) & 1 \end{pmatrix}.$$

A computation shows that these choices give $r(a|_{\overline{U}})s = b \oplus p$ with $b(x) = a_{11}(x) - a_{12}(x)c(x)a_{21}(x)$. Since $a_{11}(x_0) = a_{12}(x_0) = a_{21}(x_0) = 0$, we get $b(x_0) = 0$. ∎

We denote by $\omega(f)$ the winding number about 0 of a continuous function $f \colon S^1 \to \mathbb{C} \setminus \{0\}$. We use the same notation when f is defined on the boundary of a disk in $\mathbb{R}^2$, or on the boundary of a rectangle in $\mathbb{R}^2$. (We take such boundaries with the positive orientation.)

Lemma 5.5 *Let $a \in C([0,1]^2, M_n)$, let $x_0 \in [0,1]^2$, and suppose that there is a closed disk $D \subset [0,1]^2$ with $x_0 \in \operatorname{int}(D)$ (with respect to $\mathbb{R}^2$) such that $a(x)$ is invertible for $x \in D \setminus \{x_0\}$ and $\omega\big(\det(a)|_{\partial D}\big) = 0$. Then for all $\varepsilon > 0$ there is $b \in C([0,1]^2, M_n)$ such that*

$$b|_{[0,1]^2 \setminus D} = a|_{[0,1]^2 \setminus D}, \quad b|_D \in \operatorname{inv}\big(C(D, M_n)\big), \quad \textit{and} \quad \|b - a\| < \varepsilon.$$

Proof We first consider the case $a(x_0) = 0$. Choose a smaller closed disk D_0 such that

$$x_0 \in \operatorname{int}(D_0) \subset D_0 \subset D \quad \text{and} \quad \|a|_{D_0}\| < \frac{\varepsilon}{2}.$$

Then

$$\omega\big(\det(a)|_{\partial D_0}\big) = \omega\big(\det(a)|_{\partial D}\big) = 0$$

by homotopy invariance of the winding number. Moreover, as is well known, the map $c \mapsto \omega\big(\det(c)\big)$ defines an isomorphism $\pi_1\big(\mathrm{inv}(M_n)\big) \to \mathbb{Z}$. Using a homotopy from $a|_{\partial D_0}$ to a constant in $\mathrm{inv}(M_n)$, it is easy to find $b_0 \in \mathrm{inv}\big(C(D_0, M_n)\big)$ such that $b_0|_{\partial D_0} = a|_{\partial D_0}$.

We will paste together a and b_0, but we need to control the size of b_0. Choose a continuous function $g\colon D_0 \to \mathbb{R}$ such that $g(x) = \|a(x)\|$ for $x \in \partial D_0$ and

$$\inf_{y\in\partial D_0} \|a(y)\| \le g(x) \le \sup_{y\in\partial D_0} \|a(y)\|$$

for all $x \in D_0$. Then define

$$b(x) = \begin{cases} g(x)\|b_0(x)\|^{-1} b_0(x) & x \in D_0 \\ a(x) & x \notin D_0. \end{cases}$$

Note that for $x \in \partial D_0$ we have $\|b_0(x)\|^{-1} = \|a(x)\|^{-1} = g(x)^{-1}$, so that b is in fact continuous. Moreover, b is invertible and

$$\|a - b\| = \sup_{x\in D_0} \|a(x) - b(x)\| \le \|a|_{D_0}\| + \|b|_{D_0}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired.

Now consider the general case. By the previous lemma, there are a neighborhood U of x_0, an integer k, invertible elements $r, s \in C(\overline{U}, M_n)$, and $b_0 \in C(\overline{U}, M_k)$ such that

$$r(a|_{\overline{U}})s = b_0 \oplus 1_{n-k} \quad \text{and} \quad b_0(0) = 0.$$

If $k = 0$, then $a|_D$ is already invertible and there is nothing to prove. Otherwise, let $D_0 \subset D \cap U$ be a closed disk with $x_0 \in \mathrm{int}(D_0)$. We have

$$\omega\big(\det(b_0)|_{\partial D_0}\big) = \omega\big(\det(a)|_{\partial D_0}\big) = 0.$$

By the case of the proof already done (but applied to a disk rather than the unit square), there is an invertible element $b_1 : D_0 \to M_k$ such that

$$b_1|_{\partial D_0} = b_0|_{\partial D_0} \quad \text{and} \quad \|b_1 - b_0\| < \frac{\varepsilon}{\|r^{-1}\|\,\|s^{-1}\|}.$$

Define

$$b(x) = \begin{cases} r(x)^{-1}\big(b_1(x) \oplus 1_{n-k}\big)\, s(x)^{-1} & x \in D_0 \\ a(x) & x \notin D_0. \end{cases}$$

Note that

$$\|a - b\| \le \|r^{-1}\|\,\|b_1 - b_0\|\,\|s^{-1}\| < \varepsilon. \qquad \blacksquare$$

Lemma 5.6 *Let $f\colon [\alpha, \beta] \to \mathbb{C}$ be continuous, let $\gamma \in (\alpha, \beta)$ be a number such that $f(\gamma) = 0$ and $f(x) \neq 0$ for $x \in [\alpha, \beta] \setminus \{\gamma\}$. Let $n \in \mathbb{Z}$, let $\varepsilon > 0$, and let $y_0 \in (0, 1)$. Then there is a continuous function $g\colon [\alpha, \beta] \times [0, 1] \to \mathbb{C}$ such that:*

(1) $|g(x, y) - f(x)| < \varepsilon$ *for all* x *and* y.
(2) $g(\gamma, y_0) = 0$ *and* g *does not vanish at any other point of* $[\alpha, \beta] \times [0, 1]$.
(3) $g(\alpha, y) = f(\alpha)$ *and* $g(\beta, y) = f(\beta)$ *for all* $y \in [0, 1]$.
(4) *For any closed disk* $D \subset [\alpha, \beta] \times [0, 1]$ *containing* (γ, y_0) *in its interior, we have* $\omega(g|_{\partial D}) = n$.

Proof Define $h\colon [-1, 1]^2 \to \mathbb{C}$ by

$$h(x, y) = \frac{1}{2}(1 + y) - \frac{1}{2}(1 - y)e^{\pi i x}.$$

Then $|h(x, y)| \leq 1$ for all x and y, for $x = \pm 1$ we have $h(x, y) = 1$, the function h vanishes at $(0, 0)$ and nowhere else in $[-1, 1]^2$, and $\omega(h|_{\partial([-1,1]^2)}) = 1$.

Let $f_0\colon [\alpha, \beta] \to \mathbb{C} \setminus \{0\}$ be a continuous function such that

$$f_0(\alpha) = f(\alpha), \quad f_0(\beta) = f(\beta), \quad \text{and} \quad \|f_0 - f\| < \frac{\varepsilon}{5}.$$

Choose $\delta > 0$ such that $[\gamma - \delta, \gamma + \delta] \subset [\alpha, \beta]$ and $|f_0(x)| < \frac{2\varepsilon}{5}$ for $x \in [\gamma - \delta, \gamma + \delta]$. Define

$$g(x, y) = \begin{cases} f_0(x) & x \notin [\gamma - \delta, \gamma + \delta] \\ f_0(x)\big[h\big(\delta^{-1}(x - \gamma), y - y_0\big)\big]^n & x \in [\gamma - \delta, \gamma + \delta]. \end{cases}$$

Then parts (2) and (3) of the conclusion are immediate. We check part (1). For $x \notin [\gamma - \delta, \gamma + \delta]$, we have

$$|g(x, y) - f(x)| = |f_0(x) - f(x)| < \frac{\varepsilon}{5},$$

and for $x \in [\gamma - \delta, \gamma + \delta]$ we have

$$|g(x, y) - f(x)| \leq |f_0(x) - f(x)| + |f_0(x)| + \|h\|^n |f_0(x)| < \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} + \frac{2\varepsilon}{5} = \varepsilon.$$

It is immediate that $\omega(h^n|_{\partial([-1,1]^2)}) = n$, and part (4) of the conclusion of the lemma follows. ∎

Theorem 5.7 *Let* $A = \varinjlim A_n$ *be a direct limit of interval algebras of the following form. Let* $(y_0, y_1, \dots)$ *be a dense sequence in* $[0, 1]$*, let* $1 = k(0) < k(1) < \cdots$ *be integers such that* $k(n) \mid k(n+1)$ *for all* n*, let* $A_n = C([0, 1], M_{k(n)})$*, and let* $\varphi_{n,n+1}\colon A_n \to A_{n+1}$ *be the unital maps given by*

$$\varphi_{n,n+1}(a) = \operatorname{diag}\big(a, a, \dots, a, a(y_n)\big),$$

where $a(y_n)$ *stands for the constant function on* $[0, 1]$ *with that value. Then we have* $\operatorname{sr}\big(C([0, 1]) \otimes A\big) = 1$.

https://doi.org/10.4153/CJM-2001-039-8 Published online by Cambridge University Press

Proof Let

$$\varphi_{m,n} = \varphi_{n-1,n} \circ \varphi_{n-2,n-1} \circ \cdots \circ \varphi_{m,m+1} \colon A_m \to A_n.$$

By abuse of notation, we also write $\varphi_{m,n}$ for $\mathrm{id}_{C([0,1])} \otimes \varphi_{m,n}$. It suffices to show that for any m, any $a \in C([0,1]) \otimes A_m$, and any $\varepsilon > 0$, there is $n \geq m$ and $b \in \mathrm{inv}\big(C([0,1]) \otimes A_n\big)$ such that $\|b - \varphi_{m,n}(a)\| < \varepsilon$.

We first prove this under the following assumptions: there is a unique point $(\gamma_1, \gamma_2) \in [0,1]^2$ such that $a(\gamma_1, \gamma_2)$ is not invertible, this point is in $(0,1)^2$, and $\ker\big(a(\gamma_1, \gamma_2)\big)$ is one dimensional. Each tail $(y_m, y_{m+1}, \dots)$ of the sequence of the hypotheses is dense, and in particular

$$\gamma_2 \in \overline{\{y_m, y_{m+1}, \dots\}}.$$

By an arbitrarily small perturbation of a we can therefore also assume $\gamma_2 = y_n$ for some n, and $y_k \neq \gamma_2$ for $m \leq k < n$.

Let $a(-, \gamma)$ denote the function $(x, y) \mapsto a(x, \gamma)$. We can write

$$\varphi_{m,n+1}(a) = v \operatorname{diag}\big(a, \dots, a, d_1, d_2, \dots, d_l, a(-, \gamma_2)\big) v^*,$$

for some permutation matrix v and where the d_j all have the form $a(-, \gamma)$ with $\gamma \in \{y_m, y_{m+1}, \dots, y_{n-1}\}$. The number of occurrences of a is

$$L = \left(\frac{k(m+1)}{k(m)} - 1\right)\left(\frac{k(m+2)}{k(m+1)} - 1\right) \cdots \left(\frac{k(n+1)}{k(n)} - 1\right).$$

Since $\gamma_2 \notin \{y_m, y_{m+1}, \dots, y_{n-1}\}$, the elements d_j are all invertible, and so it suffices to approximate

$$\operatorname{diag}\big(a, \dots, a, a(-, \gamma_2)\big)$$

by an invertible element.

By Lemma 5.4, there are a neighborhood U of (γ_1, γ_2), an integer l, invertible elements $r, s \in C\big(\overline{U}, M_{k(m)}\big)$, and $f \in C(\overline{U}, M_l)$ such that

$$r(a|_{\overline{U}})s = f \oplus 1_{k(m)-l} \quad \text{and} \quad f(\gamma_1, \gamma_2) = 0.$$

We must have $l = \dim\Big(\ker\big(a(\gamma_1, \gamma_2)\big)\Big) = 1$, so f is just a function. Choose α and β with $\gamma_1 \in (\alpha, \beta) \subset [0,1]$ and $[\alpha, \beta] \times \{\gamma_2\} \subset U$. By Lemma 5.6 there is $g \colon [\alpha, \beta] \times [0,1]$ such that $g(x, y) = 0$ if and only if $(x, y) = (\gamma_1, \gamma_2)$, such that

$$g(\alpha, y) = f(\alpha, \gamma_2) \quad \text{and} \quad g(\beta, y) = f(\beta, \gamma_2)$$

for all $y \in [0,1]$, and such that

$$|g(x, y) - f(x, \gamma_2)| < \frac{\varepsilon}{2\|r^{-1}\| \, \|s^{-1}\|} \quad \text{and} \quad \omega(g|_{\partial D}) = -L\omega\big(\det(a)|_{\partial D}\big)$$

for $(x, y) \in [\alpha, \beta] \times [0,1]$ and any closed disk $D \subset [\alpha, \beta] \times [0,1]$ containing (γ_1, γ_2) in its interior.

Define

$$a_0(x,y)=\begin{cases} a(x,\gamma_2) & x\notin[\alpha,\beta]\\ r(x,\gamma_2)^{-1}\big(g(x,y)\oplus 1\big)\,s(x,\gamma_2)^{-1} & x\in[\alpha,\beta].\end{cases}$$

This defines a continuous function because $g(\alpha,y)=f(\alpha,\gamma_2)$, $g(\beta,y)=f(\beta,\gamma_2)$, and $r(x,\gamma_2)a(x,\gamma_2)s(x,\gamma_2)=f(x,\gamma_2)\oplus 1$ for $x\in[\alpha,\beta]$. We have $\|a_0-a(-,\gamma_2)\|<\frac{\varepsilon}{2}$ by the estimate on $|g(x,y)-f(x,\gamma_2)|$ above. Therefore

$$\big\|\operatorname{diag}\big(a,\dots,a,a(-,\gamma_2)\big)-\operatorname{diag}(a,\dots,a,a_0)\big\|<\frac{\varepsilon}{2}.$$

The element $b_0=\operatorname{diag}(a,\dots,a,a_0)$ fails to be invertible only at (γ_1,γ_2). Moreover, if D is a closed disk with

$$(\gamma_1,\gamma_2)\in\operatorname{int}(D)\subset D\subset[\alpha,\beta]\times[0,1],$$

then

$$\omega\big(\det(b_0)|_{\partial D}\big)=L\omega\big(\det(a)|_{\partial D}\big)+\omega\big(\det(a_0)|_{\partial D}\big)=L\omega\big(\det(a)|_{\partial D}\big)+\omega(g|_{\partial D})=0.$$

By Lemma 5.5, there is an invertible element b with $\|b-b_0\|<\frac{\varepsilon}{2}$. Then

$$\big\|b-\operatorname{diag}\big(a,\dots,a,a(-,\gamma_2)\big)\big\|<\varepsilon,$$

which shows that $\varphi_{m,n+1}(a)$ can be approximated to within ε by an invertible element.

We now consider the general case. Let $a\in C([0,1])\otimes A_n=C([0,1]^2,M_{k(n)})$. Choose a smooth ($C^\infty$) function $a_1\in C([0,1])\otimes A_n$ such that $\|a_1-a\|<\frac{\varepsilon}{6}$. We now follow the argument in the proof of Lemma 7.2 of [26]. As there, the subset $W\subset M_{k(n)}$ consisting of those matrices which have an eigenvalue of multiplicity greater than 1 is a finite union of submanifolds $W_1,\dots,W_k$ of codimension at least 2. Moreover, if we assume they have been numbered so that $\dim(W_1)\le\dim(W_2)\le\cdots\le\dim(W_k)$, then each union $W_1\cup\cdots\cup W_l$, for $l\le k$, is closed. Therefore an arbitrarily small perturbation of a_1, say a_2 with $\|a_2-a_1\|<\frac{\varepsilon}{6}$, gives a function which is transverse to all the submanifolds W_l.

The argument just cited depends only on the fact that W is the zero set of an algebraic function (the discriminant) of the entries of a matrix, the function not being identically zero. Using the determinant in place of the discriminant, we can also express the set $M_{k(n)}\setminus\operatorname{inv}(M_{k(n)})$ as a finite union of submanifolds of the same type. So there is a_3 with $\|a_3-a_2\|<\frac{\varepsilon}{6}$ which is transverse to all of these submanifolds. We also choose $\|a_3-a_2\|$ so small that a_3 is still transverse to the submanifolds $W_1,\dots,W_k$. Since the W_l all have codimension at least 2, this means there is a finite subset $G\subset[0,1]^2$ such that $a_3(x,y)$ has no repeated eigenvalues for $(x,y)\notin G$.

Choose a number

$$\rho\in\Big(-\frac{\varepsilon}{6},\frac{\varepsilon}{6}\Big)\setminus\bigcup_{(x,y)\in G}\operatorname{sp}\big(a_3(x,y)\big)$$

which is so small that $a_4 = a_3 - \rho \cdot 1$ is still transverse to the finitely many submanifolds making up the set $M_{k(n)} \setminus \operatorname{inv}(M_{k(n)})$. (The disallowed set $\bigcup_{(x,y)\in G} \operatorname{sp}\big(a_3(x,y)\big)$ is finite.) Then zero is never a multiple eigenvalue of $a_4(x,y)$. Moreover, there is (by transversality and because the relevant submanifolds have codimension at least 2) a finite set $F \subset [0,1]^2$ such that $a_4(x,y)$ is invertible for $(x,y) \notin F$.

Define $h_\delta \colon [0,1]^2 \to [0,1]^2$ by

$$h_\delta(x,y) = \left((1-\delta)\left(x - \frac{1}{2}\right) + \frac{1}{2}, (1-\delta)\left(y - \frac{1}{2}\right) + \frac{1}{2}\right).$$

This function contracts $[0,1]^2$ about its center $(\frac{1}{2}, \frac{1}{2})$ by a factor of $1-\delta$. Choose $\delta \geq 0$ so that the finite set F is disjoint from $h_\delta\big(\partial([0,1]^2)\big)$, and also so small that $a_5 = a_4 \circ h_\delta$ satisfies $\|a_5 - a_4\| < \frac{\varepsilon}{6}$.

If $F \cap h_\delta([0,1]^2) = \varnothing$, then a_5 is invertible and satisfies $\|a_5 - a\| < \frac{5\varepsilon}{6} < \varepsilon$, so we are done. Otherwise, write $h_\delta^{-1}(F) = \{z_1, \ldots, z_N\}$. Note that no z_j is in $\partial([0,1]^2)$. Choose disjoint closed disks $D_1, \ldots, D_N$ contained in $[0,1]^2$ with centers $z_1, \ldots, z_N$.

We now construct by induction elements

$$b_0, \ldots, b_N, c_1, \ldots, c_N \in C([0,1]^2, M_{n(k)})$$

satisfying the following properties:

(1) $b_0 = a_5$ and $c_k b_k = b_{k-1}$.

(2) $c_k(z)$ is invertible for $z \neq z_k$ and $\dim\Big(\ker\big(c_k(z_k)\big)\Big) = 1$.

(3) $b_k(z)$ is invertible for $z \notin \{z_{k+1}, \ldots, z_N\}$, and $\dim\Big(\ker\big(b_k(z_l)\big)\Big) = 1$ for $k+1 \leq l \leq N$.

Start by taking $b_0 = a_5$. Given b_{k-1}, define $c_k(z) = b_{k-1}(z)$ for $z \in D_k$. For $z \notin D_k$, let $\widetilde{z}$ be the unique point in the intersection of ∂D_k and the line segment from z to z_k. Then set $c_k(z) = b_{k-1}(\widetilde{z})$. Define

$$b_k(z) = \begin{cases} 1 & z \in D_k \\ c_k(z)^{-1} b_{k-1}(z) & z \notin D_k. \end{cases}$$

It is easy to see that these satisfy the required conditions.

Set $c = b_N$. Then c is invertible, and $a_5 = c_1 c_2 \cdots c_N c$. Set

$$M = (\|c\| + 1) \prod_{k=1}^{N} (\|c_k\| + 1).$$

It follows from the special case done at the beginning of the proof that there are $n \geq m$ and $d_1, \ldots, d_N \in \operatorname{inv}(A_n)$ such that

$$\|d_k - \varphi_{m,n}(c_k)\| < \min\left(1, \frac{\varepsilon}{6M}\right).$$

Then $b = d_1 d_2 \cdots d_N \varphi_{m,n}(c) \in A_n$ is invertible and satisfies $\|\varphi_{m,n}(a_5) - b\| < \frac{\varepsilon}{6}$. Therefore $\|\varphi_{m,n}(a) - b\| < \varepsilon$. ■

Example 5.8 By Theorem 9 of [13], there is a simple C^*-algebra A of the form considered in Theorem 5.7 such that $\mathrm{RR}(A) = 1$. (See Example 7.3 of [26] for an explicit example.) The theorem gives $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$.

Question 5.9 Let A be a simple direct limit of direct sums of homogeneous C^*-algebras, satisfying the conditions of the real rank one classification theorem of [12]. Suppose $K_1(A) = 0$. Does it follow that $\mathrm{sr}\big(C([0,1]) \otimes A\big) = 1$? If not, does it suffice to assume in addition that $K_0(A)$ is torsion free?

We ask this question because it follows from [12] that if a C^*-algebra B of the sort considered there has the same Elliott invariant (the scaled ordered group $K_0(B)$, the group $K_1(B)$, the set $T(B)$ of normalized traces on B, and the pairing between $K_0(B)$ and $T(B)$) as a C^*-algebra A as in Theorem 5.7, then $B \cong A$. Theorem 5.7 as stated applies to direct limits patterned after UHF Bratteli diagrams, but can certainly be generalized to suitable direct limits patterned after the Bratteli diagrams of simple AF algebras. We do not know whether such direct limits exhaust the Elliott invariants of direct limits with the appropriate dimension growth conditions, with real rank one, and with the K-theory of a simple AF algebra.

References

[1] C. A. Akemann and G. K. Pedersen, *Ideal perturbations of elements in C^*-algebras.* Math. Scand. **41**(1977), 117–139.

[2] C. A. Akemann and F. Shultz, *Perfect C^*-algebras.* Memoirs Amer. Math. Soc. **326**, 1985.

[3] W. Arveson, *Notes on extensions of C^*-algebras.* Duke Math. J. **44**(1977), 329–355.

[4] E. J. Beggs and D. E. Evans, *The real rank of algebras of matrix valued functions.* Internat. J. Math. **2**(1991), 131–137.

[5] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero.* J. Funct. Anal. **99**(1991), 131–149.

[6] J. Cuntz, *K-Theory for certain C^*-algebras.* Ann. of Math. **113**(1981), 181–197.

[7] M. Dădărlat, *Nonnuclear subalgebras of AF algebras.* Amer. J. Math. **122**(2000), 581–597.

[8] M. Dădărlat, G. Nagy, A. Némethi, and C. Pasnicu, *Reduction of topological stable rank in inductive limit of C^*-algebras.* Pacific J. Math. **153**(1992), 267–276.

[9] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology.* Princeton University Press, Princeton, 1952.

[10] N. Elhage Hassan, *Rangs stables de certaines extensions.* J. London Math. Soc. (2) **52**(1993), 605–624.

[11] ———, *Rang réel de certaines extensions.* Proc. Amer. Math. Soc. **123**(1995), 3067–3073.

[12] G. A. Elliott, G. Gong, and L. Li, *On the classification of simple inductive limit C^*-algebras, II: The isomorphism theorem.* Preprint.

[13] K. R. Goodearl, *Notes on a class of simple C^*-algebras with real rank zero.* Publ. Mat. (Barcelona) **36** (1992), 637–654.

[14] R. H. Herman and L. N. Vaserstein, *The stable range of C^*-algebras.* Invent. Math. **77**(1984), 553–555.

[15] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Volume II.* Academic Press, New York-London-Paris-San Diego-San Francisco-São Paulo-Sydney-Tokyo-Toronto, 1983.

[16] K. Kodaka and H. Osaka, *FS-property for C^*-algebras.* Proc. Amer. Math. Soc. **129**(2001), 999–1003.

[17] H. Lin, *Generalized Weyl-von Neumann Theorem (II).* Math. Scand. **77**(1995), 129–147.

[18] H. Lin and N. C. Phillips, *Classification of direct limits of even Cuntz-circle algebras.* Mem. Amer. Math. Soc. **565**, 1995.

[19] T. A. Loring, *C^*-algebras generated by stable relations.* J. Funct. Anal. **112**(1993), 159–203.
[20] A. T. Lundell and S. Weingram, *The Topology of* CW *Complexes.* Van Nostrand Reinhold Company, New York-Cincinnati-Toronto-London-Melbourne, 1969.
[21] S. Mardešič and L. R. Rubin, *Approximate inverse systems of compacta and covering dimension.* Pacific J. Math. **138**(1989), 129–144.
[22] G. J. Murphy and N. C. Phillips, *C^*-algebras with the approximate positive factorization property.* Trans. Amer. Math. Soc. **348**(1996), 2291–2306.
[23] H. Osaka, *Real rank of crossed products by connected compact groups.* Bull. London Math. Soc. **27**(1995), 257–264.
[24] ———, *Certain C^*-algebras with non-zero real rank and extremal richness.* Math. Scand. **85**(1999), 79–86.
[25] A. R. Pears, *Dimension Theory of General Spaces.* Cambridge University Press, Cambridge-London-New York-Melbourne, 1975.
[26] N. C. Phillips, *Reduction of exponential rank in direct limits of C^*-algebras.* Canadian J. Math. **46**(1994), 818–853.
[27] ———, *A classification theorem for nuclear purely infinite simple C^*-algebras.* Doc. Math. **5**(2000), 49–114 (electronic).
[28] ———, *Real and exponential rank of tensor products with $\mathcal{O}_\infty$.* J. Operator Theory, to appear.
[29] ———, *Many nonisomorphic nonnuclear simple C^*-algebras with the same Elliott invariant.* In preparation.
[30] L. S. Pontrjagin, *Sur une hypothèse fondamentale de la théorie de la dimension.* C. R. Acad. Sci. Paris Sér. A–B **190**(1930), 1105–1107.
[31] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras.* Proc. London Math. Soc. **46**(1983), 301–333.
[32] ———, *The homotopy groups of the unitary groups of non-commutative tori.* J. Operator Theory **17**(1987), 237–254.
[33] S. Zhang, *A property of purely infinite simple C^*-algebras.* Proc. Amer. Math. Soc. **109**(1990), 717–720.
[34] ———, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras, Part 1.* Pacific J. Math. **155**(1992), 169–197.
[35] ———, *On the homotopy type of the unitary group and the Grassmann space of purely infinite simple C^*-algebras.* *K*-Theory, to appear.

Department of Mathematics and Informatics
Chiba University
Yayoi-cho 1-33
Inage-ku, Chiba City 263-8522
Japan

Department of Mathematical Sciences
Ritsumeikan University
Kusatsu, Shiga, 525-8577
Japan

Department of Mathematics
University of Oregon
Eugene, Oregon 97403-1222
USA