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# GENERALIZATIONS OF A WEIGHTED TRAPEZOIDAL INEQUALITY FOR MONOTONIC FUNCTIONS AND APPLICATIONS

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**Abstract**

In this paper we establish some generalizations of a weighted trapezoidal inequality for monotonic functions and give several applications for the $r$-moments, the expectation of a continuous random variable and the Beta and Gamma functions.

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## 1. Introduction

The *trapezoidal inequality* states that if $f''$ exists and is bounded on $(a, b)$, then

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a) + f(b)] \right| \le \frac{(b-a)^3}{12} \left\| f'' \right\|_\infty, \tag{1.1}$$

where $\|f''\|_\infty := \sup_{x\in(a,b)} |f''(x)| < \infty$.

Now if we assume that $I_n : a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a, b]$ and $f$ is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *trapezoidal quadrature rule* $A_T(f, I_n)$, having an error denoted by $R_T(f, I_n)$, where

$$A_T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})]\, l_i,.$$

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and the remainder $R_T(f, I_n)$ satisfies the estimation

$$|R_T(f, I_n)| \leq \frac{1}{12} \|f''\|_\infty \sum_{i=0}^{n-1} l_i^3,$$

with $l_i := x_{i+1} - x_i$ for $i = 0, 1, \dots, n-1$.

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2–8].

Recently, Cerone and Dragomir [3] proved the following two trapezoidal-type inequalities for monotonic functions:

THEOREM A. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$, then*

$$\begin{aligned}\left|\int_a^b f(t)\,dt - [(x-a)f(a) + (b-x)f(b)]\right| \\ &\leq (b-x)f(b) - (x-a)f(a) + \int_a^b \operatorname{sgn}(x-t)f(t)\,dt \\ &\leq (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \\ &\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][f(b) - f(a)], \qquad (1.2)\end{aligned}$$

*for all $x \in [a, b]$. The above inequalities are sharp.*

Let $I_n, l_i$ $(i = 0, 1, \dots, n-1)$ be as above and let $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, \dots, n-1)$ be intermediate points. Define the *generalized trapezoidal quadrature rule* by

$$T_P(f, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})].$$

We have the following result for the approximation of $\int_a^b f(x)\,dx$ in terms of $T_P$.

THEOREM B. *Let $f$ be defined as in Theorem A, then we have*

$$\int_a^b f(x)\,dx = T_P(f, I_n, \xi) + R_P(f, I_n, \xi).$$

*The remainder term $R_P(f, I_n, \xi)$ satisfies the inequalities*

$$\begin{aligned}|R_P(f, I_n, \xi)| &\leq \sum_{i=0}^{n-1} \left[(x_{i+1}-\xi_i)f(x_{i+1}) - (\xi_i - x_i)f(x_i)\right] + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(\xi_i - t)f(t)\,dt \\ &\leq \sum_{i=0}^{n-1} (\xi_i - x_i)[f(\xi_i) - f(x_i)] + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)[f(x_{i+1}) - f(\xi_i)]\end{aligned}$$

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$$
\begin{aligned}
&\le \sum_{i=0}^{n-1}\left[\frac{1}{2}l_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right][f(x_{i+1}) - f(x_i)] \\
&\le \left[\frac{1}{2}\nu(l) + \max_{i=0,1,\ldots,n-1}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right][f(b) - f(a)] \\
&\le \nu(l)[f(b) - f(a)], \qquad (1.3)
\end{aligned}
$$

*where* $\nu(l) := \max\{l_i \mid i = 0, 1, \ldots, n-1\}$.

In this paper we establish weighted generalizations of Theorems A and B and give several applications for the $r$-moments, the expectation of a continuous random variable and the Beta and Gamma functions.

## 2. Some integral inequalities

The following result shows a generalization of the weighted trapezoidal inequality.

THEOREM 1. *Let* $g : [a, b] \to \mathbb{R}$ *be nonnegative and continuous with* $g(t) > 0$ *on* $(a, b)$ *and let* $h : [a, b] \to \mathbb{R}$ *be differentiable such that* $h'(t) = g(t)$ *on* $[a, b]$.

(a) *If* $f : [a, b] \to \mathbb{R}$ *is a monotonic nondecreasing function on* $[a, b]$, *then*

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)\,dt - [(x-h(a))f(a) + (h(b)-x)f(b)]\right| \\
&\quad\le (h(b)-x)f(b) - (x-h(a))f(a) + \int_a^b \operatorname{sgn}\left(h^{-1}(x) - t\right) f(t)g(t)\,dt \\
&\quad\le (x-h(a))\left[f\left(h^{-1}(x)\right) - f(a)\right] + (h(b)-x)\left[f(b) - f\left(h^{-1}(x)\right)\right] \\
&\quad\le \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right][f(b) - f(a)] \qquad (2.1)
\end{aligned}
$$

*for all* $x \in [h(a), h(b)]$.

(b) *If* $f : [a, b] \to \mathbb{R}$ *is a monotonic nonincreasing function on* $[a, b]$, *then*

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)\,dt - [(x-h(a))f(a) + (h(b)-x)f(b)]\right| \\
&\quad\le (x-h(a))f(a) - (h(b)-x)f(b) + \int_a^b \operatorname{sgn}\left(t - h^{-1}(x)\right) f(t)g(t)\,dt \\
&\quad\le (x-h(a))\left[f(a) - f\left(h^{-1}(x)\right)\right] + (h(b)-x)\left[f\left(h^{-1}(x)\right) - f(b)\right] \\
&\quad\le \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right][f(a) - f(b)] \qquad (2.2)
\end{aligned}
$$

*for all* $x \in [h(a), h(b)]$.

*The above inequalities are sharp.*

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PROOF. (a) Let $x \in [h(a), h(b)]$. Using the integration by parts formula for the Riemann-Stieltjes integral, we have the following identity:

$$\int_a^b (x - h(t))\,df(t) = (x - h(t))f(t)\Big|_a^b + \int_a^b f(t)g(t)\,dt$$
$$= \int_a^b f(t)g(t)\,dt - [(x - h(a))f(a) + (h(b) - x)f(b)]. \quad (2.3)$$

It is well known [1, page 813] that if $\mu, \nu : [a, b] \to \mathbb{R}$ are such that $\mu$ is continuous on $[a, b]$ and $\nu$ is monotonic nondecreasing on $[a, b]$, then

$$\left|\int_a^b \mu(t)d\nu(t)\right| \le \int_a^b |\mu(t)|\,d\nu(t). \quad (2.4)$$

Now, using identity (2.3) and inequality (2.4), we have

$$\left|\int_a^b f(t)g(t)\,dt - [(x - h(a))\,f(a) + (h(b) - x)f(b)]\right|$$
$$\le \int_a^b |x - h(t)|\,df(t)$$
$$= \int_a^{h^{-1}(x)} (x - h(t))df(t) + \int_{h^{-1}(x)}^b (h(t) - x)df(t)$$
$$= (x - h(t))f(t)\Big|_a^{h^{-1}(x)} + \int_a^{h^{-1}(x)} f(t)g(t)\,dt$$
$$+ (h(t) - x)\Big|_{h^{-1}(x)}^b - \int_{h^{-1}(x)}^b f(t)g(t)\,dt$$
$$= (h(b) - x)f(b) - (x - h(a))f(a) + \int_a^b \operatorname{sgn}\left(h^{-1}(x) - t\right)f(t)g(t)\,dt \quad (2.5)$$

and the first inequalities in (2.1) are proved.

As $f$ is monotonic nondecreasing on $[a, b]$, we obtain

$$\int_a^{h^{-1}(x)} f(t)g(t)\,dt \le f\left(h^{-1}(x)\right)\int_a^{h^{-1}(x)} g(t)\,dt = (x - h(a))f\left(h^{-1}(x)\right)$$

and

$$\int_{h^{-1}(x)}^b f(t)g(t)\,dt \ge f\left(h^{-1}(x)\right)\int_{h^{-1}(x)}^b g(t)\,dt = (h(b) - x)f\left(h^{-1}(x)\right),$$

then

$$\int_a^b \operatorname{sgn}\left(h^{-1}(x) - t\right) f(t)g(t)\,dt \le (x - h(a))f\left(h^{-1}(x)\right) + (x - h(b))f\left(h^{-1}(x)\right).$$

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Therefore

$$\begin{aligned}(h(b)-x)f(b)-(x-h(a))f(a)+\int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right) f(t)g(t)\,dt \\ \le (h(b)-x)f(b)-(x-h(a))f(a) \\ +(x-h(a))f\left(h^{-1}(x)\right)+(x-h(b))f\left(h^{-1}(x)\right) \\ =(x-h(a))\left[f\left(h^{-1}(x)\right)-f(a)\right]+(h(b)-x)\left[f(b)-f\left(h^{-1}(x)\right)\right], \quad (2.6)\end{aligned}$$

which proves the second inequality in (2.1).

As $f$ is monotonic nondecreasing on $[a,b]$, we have $f(a)\le f\left(h^{-1}(x)\right)\le f(b)$ and

$$\begin{aligned}(x-h(a))\left[f\left(h^{-1}(x)\right)-f(a)\right]+(h(b)-x)\left[f(b)-f\left(h^{-1}(x)\right)\right] \\ \le \max\{x-h(a),h(b)-x\}\left[f\left(h^{-1}(x)\right)-f(a)+f(b)-f\left(h^{-1}(x)\right)\right] \\ =\left[\frac{h(b)-h(a)}{2}+\left|x-\frac{h(a)+h(b)}{2}\right|\right][f(b)-f(a)] \\ =\left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right][f(b)-f(a)]. \quad (2.7)\end{aligned}$$

Thus, by (2.5)–(2.7), we obtain (2.1).

Let $g(t)\equiv 1$, $h(t)=t$, $t\in[a,b]$;

$$f(t)=\begin{cases}0, & t\in[a,b),\\ 1, & t=b;\end{cases}$$

and $x=(a+b)/2$. Then a simple calculation reveals that

$$\begin{aligned}\left|\int_a^b f(t)g(t)\,dt-[(x-h(a))\,f(a)+(h(b)-x)f(b)]\right| \\ =(h(b)-x)f(b)-(x-h(a))f(a)+\int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right) f(t)g(t)\,dt \\ =(x-h(a))\left[f\left(h^{-1}(x)\right)-f(a)\right]+(h(b)-x)\left[f(b)-f\left(h^{-1}(x)\right)\right] \\ =\left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right][f(b)-f(a)] \\ =\frac{b-a}{2},\end{aligned}$$

which proves that the inequalities (2.1) are sharp.

(b) If $f$ is replaced by $-f$ in (a), then (2.2) is obtained from (2.1). This completes the proof. □

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REMARK 1. If we choose $g(t) \equiv 1$ and $h(t) = t$ on $[a, b]$, then inequalities (2.1) reduce to (1.2).

COROLLARY 1. *If we choose* $x = (h(a) + h(b))/2$, *then we get*

$$\begin{aligned}
&\left| \int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt \right| \\
&\quad \le \frac{1}{2}[f(b)-f(a)]\int_a^b g(t)\,dt \\
&\qquad + \int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right) - t\right) f(t)g(t)\,dt \\
&\quad \le \frac{1}{2}[f(b)-f(a)]\int_a^b g(t)\,dt,
\end{aligned} \tag{2.8}$$

*where* $f$ *and* $g$ *are defined as in* (a) *of Theorem* 1, *and*

$$\begin{aligned}
&\left| \int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt \right| \\
&\quad \le \frac{1}{2}[f(a)-f(b)]\int_a^b g(t)\,dt \\
&\qquad + \int_a^b \operatorname{sgn}\left(t - h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right) f(t)g(t)\,dt \\
&\quad \le \frac{1}{2}[f(a)-f(b)]\int_a^b g(t)\,dt,
\end{aligned} \tag{2.9}$$

*where* $f$ *and* $g$ *are defined as in* (b) *of Theorem* 1.

The inequalities (2.8) and (2.9) are the "*weighted trapezoid*" inequalities.

Note that the trapezoidal inequalities (2.8) and (2.9) are, in a sense, the best possible inequalities we can obtain from (2.1) and (2.2). Moreover, the constant $1/2$ is the best possible for both inequalities in (2.8) and (2.9), respectively.

REMARK 2. The following inequality is well known in the literature as the Fejér inequality (see for example [9]):

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt \le \int_a^b f(t)g(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt, \tag{2.10}$$

where $f : [a, b] \to \mathbb{R}$ is convex and $g : [a, b] \to \mathbb{R}$ is positive, integrable and symmetric to $(a + b)/2$.

Using the above results and (2.8)–(2.9), we obtain the following error bound of the second inequality in (2.10):

$$\begin{aligned}
0 &\le \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt \\
&\le \frac{1}{2}[f(b)-f(a)]\int_a^b g(t)\,dt \\
&\quad + \int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)-t\right) f(t)g(t)\,dt \\
&\le \frac{1}{2}[f(b)-f(a)]\int_a^b g(t)\,dt,
\end{aligned} \tag{2.11}$$

provided that $f$ is monotonic nondecreasing on $[a,b]$.

Also,

$$\begin{aligned}
0 &\le \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt \\
&\le \frac{1}{2}[f(a)-f(b)]\int_a^b g(t)\,dt + \int_a^b \operatorname{sgn}\left(t-h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right) f(t)g(t)\,dt \\
&\le \frac{1}{2}[f(a)-f(b)]\int_a^b g(t)\,dt,
\end{aligned} \tag{2.12}$$

provided that $f$ is monotonic nonincreasing on $[a,b]$.

## 3. Applications for the quadrature rules

Throughout this section, let $g$ and $h$ be defined as in Theorem 1.

Let $f:[a,b]\to\mathbb{R}$, and let $I_n: a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$ and $\xi_i \in [h(x_i), h(x_{i+1})]$ $(i=0,1,\ldots,n-1)$ be intermediate points. Put $l_i := h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t)\,dt$ and define the quadrature rule

$$T_P(f,g,h,I_n,\xi) := \sum_{i=0}^{n-1}\left[(\xi_i - h(x_i))\,f(x_i) + (h(x_{i+1})-\xi_i)\,f(x_{i+1})\right].$$

We have the following result concerning the approximation of $\int_a^b f(t)g(t)\,dt$ in terms of $T_P$.

THEOREM 2. *Let* $\nu(l) := \max\{l_i \mid i=0,1,\ldots,n-1\}$, $f$ *be defined as in Theorem* 1 *and let*

$$\int_a^b f(t)g(t)\,dt = T_P(f,g,h,I_n,\xi) + R_P(f,g,h,I_n,\xi). \tag{3.1}$$

(a) *If $f$ is monotonic nondecreasing on $[a, b]$, then*

$$
\begin{aligned}
&|R_P(f, g, h, I_n, \xi)| \\
&\quad \le \sum_{i=0}^{n-1} [(h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i)] \\
&\qquad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(h^{-1}(\xi_i) - t\right) f(t) g(t)\, dt \\
&\quad \le \sum_{i=0}^{n-1} (\xi_i - h(x_i)) \left[f\left(h^{-1}(\xi_i)\right) - f(x_i)\right] \\
&\qquad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) \left[f(x_{i+1}) - f\left(h^{-1}(\xi_i)\right)\right] \\
&\quad \le \sum_{i=0}^{n-1} \left[\frac{1}{2} l_i + \left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] [f(x_{i+1}) - f(x_i)] \\
&\quad \le \left[\frac{1}{2}\nu(l) + \max_{i=0,1,\ldots,n-1} \left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] [f(b) - f(a)] \\
&\quad \le \nu(l)[f(b) - f(a)].
\end{aligned} \tag{3.2}
$$

(b) *If $f$ is monotonic nonincreasing on $[a, b]$, then*

$$
\begin{aligned}
&|R_P(f, g, h, I_n, \xi)| \\
&\quad \le \sum_{i=0}^{n-1} [(\xi_i - h(x_i)) f(x_i) - (h(x_{i+1}) - \xi_i) f(x_{i+1})] \\
&\qquad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(t - h^{-1}(\xi_i)\right) f(t) g(t)\, dt \\
&\quad \le \sum_{i=0}^{n-1} (\xi_i - h(x_i)) \left[f(x_i) - f\left(h^{-1}(\xi_i)\right)\right] \\
&\qquad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) \left[f\left(h^{-1}(\xi_i)\right) - f(x_{i+1})\right] \\
&\quad \le \sum_{i=0}^{n-1} \left[\frac{1}{2} l_i + \left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] [f(x_i) - f(x_{i+1})] \\
&\quad \le \left[\frac{1}{2}\nu(l) + \max_{i=0,1,\ldots,n-1} \left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] [f(a) - f(b)] \\
&\quad \le \nu(l)[f(a) - f(b)].
\end{aligned}
$$

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PROOF. (a) Apply Theorem 1 on the intervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ to get

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}} f(t)g(t)\,dt - [(\xi_i - h(x_i))\, f(x_i) + (h(x_{i+1}) - \xi_i)\, f(x_{i+1})]\right| \\
&\quad \le (h(x_{i+1}) - \xi_i)\, f(x_{i+1}) - (\xi_i - h(x_i))\, f(x_i) \\
&\qquad + \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(h^{-1}(\xi_i) - t\right) f(t)g(t)\,dt \\
&\quad \le (\xi_i - h(x_i))\left[f\left(h^{-1}(\xi_i)\right) - f(x_i)\right] \\
&\qquad + (h(x_{i+1}) - \xi_i)\left[f(x_{i+1}) - f\left(h^{-1}(\xi_i)\right)\right] \\
&\quad \le \left[\frac{1}{2}l_i + \left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right][f(x_{i+1}) - f(x_i)]
\end{aligned}$$

for all $i \in \{0, 1, \ldots, n-1\}$.

Using this and the generalized triangle inequality, we have

$$\begin{aligned}
&|R_P(f, g, h, I_n, \xi)| \\
&\quad \le \sum_{i=0}^{n-1} \left|\int_{x_i}^{x_{i+1}} f(t)g(t)\,dt - [(\xi_i - h(x_i))\, f(x_i) + (h(x_{i+1}) - \xi_i)\, f(x_{i+1})]\right| \\
&\quad \le \sum_{i=0}^{n-1} [(h(x_{i+1}) - \xi_i)\, f(x_{i+1}) - (\xi_i - h(x_i))\, f(x_i)] \\
&\qquad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(h^{-1}(\xi_i) - t\right) f(t)g(t)\,dt \\
&\quad \le \sum_{i=0}^{n-1} (\xi_i - h(x_i))\left[f\left(h^{-1}(\xi_i)\right) - f(x_i)\right] \\
&\qquad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i)\left[f(x_{i+1}) - f\left(h^{-1}(\xi_i)\right)\right] \\
&\quad \le \sum_{i=0}^{n-1} \left[\frac{1}{2}l_i + \left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right][f(x_{i+1}) - f(x_i)] \\
&\quad \le \left[\frac{1}{2}\nu(l) + \max_{i=0,1,\ldots,n-1}\left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right][f(b) - f(a)]. \qquad (3.3)
\end{aligned}$$

Next, observe that

$$\left|\xi_i - \frac{h(x_i) + h(x_{i+1})}{2}\right| \le \frac{1}{2}l_i \quad (i = 0, 1, \ldots, n-1); \qquad (3.4)$$

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and then

$$\max_{i=0,1,\ldots,n-1}\left|\xi_i - \frac{h(x_i)+h(x_{i+1})}{2}\right| \le \frac{1}{2}\nu(l). \tag{3.5}$$

Thus, by (3.3)–(3.5), we obtain (3.2).

(b) The proof is similar to that for (a) and we omit the details. □

REMARK 3. If we choose $g(t) \equiv 1$, $h(t) = t$ on $[a, b]$, then the inequalities (3.2) reduce to (1.3).

Now, let $\xi_i = (h(x_i) + h(x_{i+1}))/2$ $(i = 0, 1, \ldots, n-1)$ and let $T_{PW}(f, g, h, I_n)$ and $R_P(f, g, h, I_n)$ be defined as

$$T_{PW}(f, g, h, I_n) = T_P(f, g, h, I_n, \xi) = \frac{1}{2}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]\int_{x_i}^{x_{i+1}} g(t)\,dt$$

and

$$\begin{aligned} R_{PW}(f, g, h, I_n) &= R_P(f, g, h, I_n, \xi) \\ &= \int_a^b f(t)g(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]\int_{x_i}^{x_{i+1}} g(t)\,dt. \end{aligned}$$

If we consider the *weighted trapezoidal formula* $T_{PW}(f, g, h, I_n)$, then we have the following corollary.

COROLLARY 2. *Let $f$, $g$, $h$ be defined as in Theorem 2 and $\xi_i = (h(x_i)+h(x_{i+1}))/2$ $(i = 0, 1, \ldots, n-1)$. Then*

$$\int_a^b f(t)g(t)\,dt = T_{PW}(f, g, h, I_n) + R_{PW}(f, g, h, I_n)$$

*where the remainder satisfies the following estimates:*

(a) *If $f$ is monotonic nondecreasing on $[a, b]$, then*

$$\begin{aligned} |R_{PW}(f, g, h, I_n)| &\le \frac{1}{2}\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} g(t)\,dt\right)[f(x_{i+1}) - f(x_i)] \\ &\quad + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(h^{-1}\left(\frac{h(x_i)+h(x_{i+1})}{2}\right) - t\right) f(t)g(t)\,dt \\ &\le \frac{1}{2}\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} g(t)\,dt\right)[f(x_{i+1}) - f(x_i)] \\ &\le \frac{\nu(l)}{2}[f(b) - f(a)]. \end{aligned} \tag{3.6}$$

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(b) *If $f$ is monotonic nonincreasing on* $[a, b]$, *then*

$$\begin{aligned}
|R_{PW}(f, g, h, I_n)| &\leq \frac{1}{2}\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} g(t)\,dt\right)[f(x_i) - f(x_{i+1})] \\
&\quad + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(t - h^{-1}\left(\frac{h(x_i) + h(x_{i+1})}{2}\right)\right) f(t)g(t)\,dt \\
&\leq \frac{1}{2}\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} g(t)\,dt\right)[f(x_i) - f(x_{i+1})] \\
&\leq \frac{\nu(l)}{2}[f(a) - f(b)]. \qquad (3.7)
\end{aligned}$$

REMARK 4. In Corollary 2, suppose $f$ is monotonic on $[a, b]$,

$$x_i = h^{-1}\left[h(a) + \frac{i(h(b) - h(a))}{n}\right] \quad (i = 0, 1, \dots, n),$$

and

$$l_i := h(x_{i+1}) - h(x_i) = \frac{h(b) - h(a)}{n} = \frac{1}{n}\int_a^b g(t)\,dt. \quad (i = 0, 1, \dots, n-1).$$

If we want to approximate the integral $\int_a^b f(t)g(t)\,dt$ by $T_{PW}(f, g, h, I_n)$ with an accuracy greater than $\varepsilon > 0$, we need at least $n_\varepsilon \in \mathbb{N}$ points for the partition $I_n$, where

$$n_\varepsilon := \left[\frac{1}{2\varepsilon}\int_a^b g(t)\ dt \cdot |f(b) - f(a)|\right] + 1$$

and $[r]$ denotes the Gaussian integer of $r$ $(r \in \mathbb{R})$.

## 4. Some inequalities for random variables

Throughout this section, let $0 < a < b$ , $r \in \mathbb{R}$, and let $X$ be a continuous random variable having the continuous probability density function $g : [a, b] \to \mathbb{R}$ with $g(t) > 0$ on $(a, b)$, $h : [a, b] \to \mathbb{R}$ with $h'(t) = g(t)$ for $t \in (a, b)$ and the $r$-moment $E_r(X) := \int_a^b t^r g(t)\,dt$, which is assumed to be finite.

THEOREM 3. *We have the inequalities*

$$\begin{aligned}
\left|E_r(X) - \frac{a^r + b^r}{2}\right| &\leq \frac{1}{2}(b^r - a^r) + \int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{1}{2}\right) - t\right) t^r g(t)\,dt \\
&\leq \frac{1}{2}(b^r - a^r) \quad \textit{as } r \geq 0 \qquad (4.1)
\end{aligned}$$

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*and*

$$\left|E_r(X) - \frac{a^r + b^r}{2}\right| \le \frac{1}{2}(a^r - b^r) + \int_a^b \operatorname{sgn}\left(t - h^{-1}\left(\frac{1}{2}\right)\right) t^r g(t)\,dt$$
$$\le \frac{1}{2}(a^r - b^r) \quad as \ r < 0, \tag{4.2}$$

*respectively.*

PROOF. If we put $f(t) = t^r$ ($t \in [a, b]$), $h(t) = \int_a^t g(x)\,dx$ ($t \in [a, b]$) and $x = (h(a) + h(b))/2 = 1/2$ in Corollary 1, then we obtain (4.1) and (4.2). This completes the proof. □

The following corollary is a special case of Theorem 3.

COROLLARY 3. *The inequalities*

$$\left|E(X) - \frac{a+b}{2}\right| \le \frac{b-a}{2} + \int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{1}{2}\right) - t\right) t g(t)\,dt \le \frac{b-a}{2}$$

*hold, where $E(X)$ is the expectation of the random variable $X$.*

## 5. Inequalities for the Beta and Gamma functions

The following two functions are well known in the literature as the *Beta function*

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x > 0,\ y > 0$$

and the *Gamma function*,

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\,dt, \quad x > 0,$$

respectively.

The following inequality is an application of Theorem 1 for the Beta function.

THEOREM 4. *Let $p, q > 0$. Then we have the inequality*

$$|B(p+1, q+1) - x| \le x + \int_a^b \operatorname{sgn}\left[t - ((p+1)x)^{1/(p+1)}\right] t^p (1-t)^q\,dt$$
$$\le x + \left(\frac{1}{p+1} - 2x\right)\left[1 - ((p+1)x)^{1/(p+1)}\right]^q$$
$$\le \frac{1}{2(p+1)} + \left|x - \frac{1}{2(p+1)}\right| \tag{5.1}$$

*for all $x \in \left[0, 1/(p+1)\right]$.*

PROOF. If we put $a = 0, b = 1$, $f(t) = (1-t)^q$, $g(t) = t^p$ and $h(t) = t^{p+1}/(p+1)$ ($t \in [0, 1]$) in Theorem 1, we obtain inequality (5.1) for all $x \in [0, 1/(p+1)]$. □

The following remark is an application of Theorem 4 for the Gamma function.

REMARK 5. As $B(p+1, q+1) = \Gamma(p+1)\Gamma(q+1)/\Gamma(p+q+2)$, inequality (5.1) is equivalent to

$$\begin{aligned}\left|\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} - x\right| &\le x + \int_a^b \operatorname{sgn}\left[t - ((p+1)x)^{1/(p+1)}\right] t^p (1-t)^q \, dt \\ &\le x + \left(\frac{1}{p+1} - 2x\right)\left[1 - ((p+1)x)^{1/(p+1)}\right]^q \\ &\le \frac{1}{2(p+1)} + \left|x - \frac{1}{2(p+1)}\right|,\end{aligned}$$

that is,

$$\begin{aligned}&|(p+1)\Gamma(p+1)\Gamma(q+1) - x(p+1)\Gamma(p+q+2)| \\ &\quad\le \left[x + \int_a^b \operatorname{sgn}\left[t - ((p+1)x)^{1/(p+1)}\right] t^p (1-t)^q \, dt\right](p+1)\Gamma(p+q+2) \\ &\quad\le \left[x + \left(\frac{1}{p+1} - 2x\right)\left[1 - ((p+1)x)^{1/(p+1)}\right]^q\right](p+1)\Gamma(p+q+2) \\ &\quad\le \left[\frac{1}{2} + \left|x(p+1) - \frac{1}{2}\right|\right]\Gamma(p+q+2)\end{aligned}$$

and as $(p+1)\Gamma(p+1) = \Gamma(p+2)$, we get

$$\begin{aligned}&|\Gamma(p+2)\Gamma(q+1) - x(p+1)\Gamma(p+q+2)| \\ &\quad\le \left[x + \int_a^b \operatorname{sgn}\left[t - ((p+1)x)^{1/(p+1)}\right] t^p (1-t)^q \, dt\right](p+1)\Gamma(p+q+2) \\ &\quad\le \left[x + \left(\frac{1}{p+1} - 2x\right)\left[1 - ((p+1)x)^{1/(p+1)}\right]^q\right](p+1)\Gamma(p+q+2) \\ &\quad\le \left[\frac{1}{2} + \left|x(p+1) - \frac{1}{2}\right|\right]\Gamma(p+q+2)\end{aligned}$$

for any $x \in [0, 1/(p+1)]$.

## References

[1] T. M. Apostol, *Mathematical Analysis*, 2nd ed. (Addison-Wesley, Reading, Mass.-London-Don Mills, Ont., 1975).

https://doi.org/10.1017/S1446181100003217 Published online by Cambridge University Press

[2] N. S. Barnett, S. S. Dragomir and C. E. M. Pearce, "A quasi-trapezoid inequality for double integrals", *ANZIAM J.* **44** (2003) 355–364.

[3] P. Cerone and S. S. Dragomir, "Trapezoidal type rules from an inequalities point of view", in *Handbook of Analytic-Computational Methods in Applied Mathematics* (ed. G. Anastassiou), (CRC Press, New York, 2000) 65–134.

[4] P. Cerone, S. S. Dragomir and C. E. M. Pearce, "A generalized trapezoid inequality for functions of bounded variation", *Turkish J. Math.* **24** (2000) 147–163.

[5] S. S. Dragomir, "On the trapezoid quadrature formula for Lipschitzian mappings and applications", *Tamkang J. Math.* **30** (1999) 133–138.

[6] S. S. Dragomir, "On the trapezoid quadrature formula and applications", *Kragujevac J. Math.* **23** (2001) 25–36.

[7] S. S. Dragomir, P. Cerone and A. Sofo, "Some remarks on the trapezoid rule in numerical integration", *Indian J. Pure Appl. Math.* **31** (2000) 475–494.

[8] S. S. Dragomir and T. C. Peachey, "New estimation of the remainder in the trapezoidal formula with applications", *Stud. Univ. Babeş-Bolyai Math. (Cluj)* **45** (2000) 31–42.

[9] L. Fejér, "Uberdie Fourierreihen, II", *Math. Natur. Ungar. Akad. Wiss.* **24** (1906) 369–390.