Can. J. Math., Vol. XXXI, No. 2, 1979, pp. 282–299

α-SPEEDABLE AND NON α-SPEEDABLE SETS

BARRY E. JACOBS

α-Recursion theory was invented simultaneously by Kripke [**15**] and Platek [**22**] and served to generalize the theories of Takeuti [**34**], Machover [**20**], Kreisel and Sacks [**14**] and others. Kripke (in [**16**]) derived machinery to construct an analogue to Kleene's T-predicate enabling him to assert that all of unrelativized ordinary recursion theory (as found in Kleene [**13**]) lifted to α-recursion theory. As a result, we were able to set down in [**8**] α-analogues to Blum's [**1**] well-studied axioms, thus, introducing the study of α-computational complexity theory.

Our first activities in this area paralleled those in the beginnings of α-recursion theory; namely, we demonstrated that major results of the ω-theory held at α. In [**9**], it was shown that the Blum-Rabin arbitrary complex partial recursive function theorem and Borodin's Gap phenomenon generalized. In [**11**] we lift to α the classical Blum Speed-up Theorem and the McCreight-Meyer-Moll Honesty theorem, and in [**10**] the McCreight-Meyer Union theorem. In all cases, constructions and proofs had to be revised to make up for deficiencies within many Σ_1 admissibles (e.g. lack of regularity).

In this paper we initiate the second phase of our study of α-complexity theory (again, following the pattern set by α-recursion theorists). Namely, we try to isolate the differences between the ω- and α-theories and, in particular, seek out theorems of ordinary complexity theory which are *false* in α-complexity theory. Our current work revolves around some recent results of Soare [**33**] which strongly link recursion theoretic and complexity oriented notions. Consequently, we bring α-complexity theory closer to other major areas of activity of α-recursion theory (i.e., α-degrees, lattices of α-r.e. sets, classes of generalized simple sets, etc.).

An outline of our paper is as follows: In § 1 we present the basic definitions of α-recursion theory and α-complexity theory. We prove in § 2 one of the α-analogues to Soare's index set characterization for nonspeedable r.e. sets and investigate how the property of nonregularity (of α-r.e. sets) affects such a characterization.

In § 3 we revisit Sacks' regular representative theorem (for α-r.e. α-degrees containing nonregular sets) and relate it to an α-analogue of Jockusch's notion of semirecursive set. In § 4, we use results of § 2 and § 3 to prove a generalization of a theorem by Marques and Soare's classifying those α-r.e. α-degrees which contain generalized speedable sets. We also present in this section a class

Received October 14, 1977. This research was supported by NSF Grant 76-07129 and City University Faculty Research Award RF-11429.

of Σ_1 admissibles whose only speedable sets are those of α-degree $\underset{\sim}{0}'$. Finally, we conclude in § 5 with a list of open problems for further research.

The author wishes to express sincere thanks to Gerald Sacks, Richard Shore, Robert Soare and Robert DiPaola for their help in this research.

1. Preliminaries. The basic definitions of α-recursion theory are defined in terms of levels L_α of Gödel's constructible universe and the usual Σ_n hierarchy of formulae. α is *admissible* if L_α satisfies the replacement axiom schema of ZF for Σ_1 formulae. Hence, we think of L_α as a model of weak set theory. Throughout this paper α is assumed to be admissible.

A set $A \subseteq \alpha$ is *α-recursively enumerable* (α-r.e.) if it has a Σ_1 definition over L_α. A partial function $f : \alpha \to \alpha$ is *α-partial recursive* if its graph is α-r.e. and is *α-recursive* if its domain is α. (Since there is a one-one α-recursive map of α onto L_α, it suffices to consider only subsets of α and functions on α). $A \subseteq \alpha$ is *α-recursive* if its characteristic function is, and *α-finite* if it is a member of L_α. Equivalently, $A \subseteq \alpha$ is α-finite if it is both α-recursive and bounded below α. $A \subseteq \alpha$ is *α-infinite* if it is not α-finite and *regular* if $A \cap \beta$ is α-finite for all $\beta < \alpha$.

The basic recursion theoretic fact about admissible ordinals is that one may perform Δ_1 ($= \alpha$-recursive) recursions in L_α to produce α-recursive functions. Thus, we can α-recursively Gödel number the α-finite sets $\{K_\gamma | \gamma < \alpha\}$ and the Σ_0/L_α formulae of two free variables $\phi_\epsilon(x, y)$. This gives us a Gödel numbering for the α-r.e. sets, $R_\epsilon = \{x | L_\alpha \models \exists y \phi_\epsilon(y, x)\}$ and a *standard simultaneous α-recursive enumeration* of these sets, $R_\epsilon^\sigma = \{x | (\exists y \in L_\sigma) \phi_\epsilon(x, y)\}$.

An analogue to Blum's [**1**] notion of computational complexity measure is given by the following.

Definition. An *α-complexity measure* Φ is an enumeration (in α) of the α-partial recursive functions $\{\phi_\epsilon | \epsilon < \alpha\}$ to which are associated the α-partial recursive *α-step counting functions* $\{\Phi_\epsilon | \epsilon < \alpha\}$ for which the following axioms hold:

(1) For all β, ϵ, $\phi_\epsilon(\beta)$ is defined if and only if $\Phi_\epsilon(\beta)$ is defined.
(2) The predicate

$$M(\epsilon, \beta, \gamma) = \begin{cases} 1 & \text{if } \Phi_\epsilon(\beta) = \gamma \\ 0 & \text{if } \Phi_\epsilon(\beta) \neq \gamma \end{cases}$$

is α-recursive.

We also assume that α-recursive versions of the S_n^m and Universal Function Theorems (Kripke [**15**]) hold for the enumeration $\{\phi_\epsilon | \epsilon < \alpha\}$. Implicit in this definition is the capability to retrieve, given any index ϵ, both the function ϕ_ϵ, and step-counter Φ_ϵ in the form of algorithms.

By the *Σ_n projectum of* α, we mean the least $\beta \leqq \alpha$ such that every Σ_n set below β is α-finite. For the case $n = 1$, the Σ_1 projectum is called simply the *projectum* and denoted by α^*. Equivalently, α^* is the projectum of α if and only if there exists one-one α-recursive mapping of α into α^*.

As in ordinary recursion theory, we say that A is *many-one* or *m-reducible to* B, $A \leqq_m B$, if there is an α-recursive f such that for all x, $x \in A$ if and only if $f(x) \in B$.

By $\{\epsilon\}_\sigma{}^C(\gamma)$ we mean that

$$(\exists\, \rho)(\exists\, \eta)\langle \gamma, \delta, \rho, \eta\rangle \in R_\epsilon{}^\sigma$$

and

$$K_\rho \subseteq C \cap \sigma$$

and $K_\eta \subseteq (\alpha - C) \cap \sigma$. (Here $\langle\, , \ldots, \rangle$ is some α-recursive coding of n-tuples.) We say that $\{\epsilon\}^C(\gamma) = \delta$ if for some σ, $\{\epsilon\}_\sigma{}^C(\gamma) = \delta$. This enables us to define the notion of *weakly α-recursive in* ($\leqq_{w\alpha}$) for a partial function f and a set C; namely, $f \leqq_{w\alpha} C$ if and only if $f = \{\epsilon\}^C$ for some ϵ. Of course, for a set $B \subseteq \alpha$, $B \leqq_{w\alpha} C$ if and only if the characteristic function of B is weakly α-recursive in C. Related to weak α-recursiveness are two key notions. The *recursive cofinality* of a set A (rcf A) is the least $\gamma \leqq \alpha$ such that there is an $f \leqq_{w\alpha} A$ with domain γ and range unbounded in α. A is *hyperregular* if and only if rcf $A = \alpha$, otherwise it is *nonhyperregular*.

Although weak reducibility is useful, for many α's $\leqq_{w\alpha}$ is not transitive. Consequently, we define *α-recursive in* ($\leqq_\alpha$) by saying that $B \leqq_\alpha C$ if and only if there is an ϵ such that for all α-finite K_γ

$$K_\gamma \subseteq B \leftrightarrow (\exists\, \rho)(\exists\, \eta)(\exists\, \sigma)(\langle \rho, \eta, \gamma, 1\rangle \in R_\epsilon{}^\sigma \quad \text{and}$$

$$K_\rho \subseteq C \quad \text{and} \quad K_\eta \subseteq \alpha - C) \quad \text{and}$$

$$K_\gamma \subseteq \alpha - B \leftrightarrow (\exists\, \rho)(\exists\, \eta)(\exists\, \sigma)(\langle \rho, \eta, \gamma, 0\rangle \in R_\epsilon{}^\sigma \quad \text{and}$$

$$K_\rho \subseteq C \quad \text{and} \quad U_\eta \subseteq \alpha - C).$$

Since $\leqq_\alpha$ is obviously transitive, and reflexive, it provides us with the notion of *α-degree*: deg $(A) = \{B | B \leqq_\alpha A \leqq_\alpha B\}$. We call an α-degree α-r.e., *regular*, *irregular*, *hyperregular* or *nonhyperregular* if it contains an α-r.e., regular, nonregular, hyperregular or nonhyperregular set, respectively. We remark that if an α-degree $\underset{\sim}{a}$ is (non)hyperregular then every set in $\underset{\sim}{a}$ is (non)hyperregular and that $\underset{\sim}{a}$ can be both regular and irregular.

A third analogue to Turing reducibility is that of α-calculability which is defined in terms of Kripke's [**15**] equation calculus (EC) very much like Kleene's for ordinary partial recursive functions. If $B \subseteq \alpha$ then the *diagram* of B, denoted $\triangle_B$, is

$$\{\underset{\sim}{g}(\underset{\sim}{\gamma}) = \underset{\sim}{1} | \gamma \in B\} \cup \{\underset{\sim}{g}(\underset{\sim}{\gamma}) = \underset{\sim}{0} | \gamma \notin B\}$$

(i.e., equations indicating membership facts about B). If E is a finite set of equations (see Kripke [**16**]) whose parameters are ordinals less than α, then $S^{E,B}$ is the set of all equations deducible from $E \cup \triangle_B$ in the Kripke EC in any number of steps. Then a partial function $f \subseteq \alpha \times \alpha$ is α-calculable ($\leqq_{c\alpha}$)

from B if for some E $f(\gamma) = \delta \leftrightarrow \underset{\sim}{f}(\gamma) = \underset{\sim}{\delta} \in S^{E,B}$ for all $\gamma, \delta < \alpha$. $A \leqq_{c\alpha} B$ if the characteristic function of A is α-calculable from B.

2. The $\leqq_{w\alpha}$-Soare Theorem. The notion of speedable ω-recursively enumerable set was first introduced by Blum and Marques [**2**] to extend Blum's [**1**] original definition from total to partial recursive functions. Soare [**33**] recently discovered a pure recursion theoretic characterization for the notion of nonspeedability. Namely, that nonspeedable ω-r.e. sets are precisely those sets A whose complements have *weak jumps* (i.e. $H_{\bar{A}} = \{\rho | R_\rho \cap \bar{A} \neq \emptyset\}$) Turing reducible to the complete r.e. set O'. Soare's result, which we generalize below to α, makes use of the well known limit lemma of Schoenfield [**27**]; i.e., that Δ_2^0 sets are exactly those Turing reducible to O'.

Let Φ be any α-computational complexity measure as defined in § 1. A natural analogue to the Blum-Marques notion of speedability in α-recursion theory is provided by the following.

2.1. *Definition.* An α-r.e. set $A \subseteq \alpha$ is *α-speedable* if for all α-r.e. indices ϵ of A and all α-recursive h, there exists an index τ for A where

$$A \cap \{\beta | \Phi_\epsilon(\beta) > h(\Phi_\tau(\beta), \beta)\}$$

is unbounded in α.

It is easily seen that for $\alpha = \omega$, the definition of α-speedable coincides with that of Blum and Marques. However, as is so often the situation, phenomena arising in α-recursion theory (i.e. nonregularity, nonhyperregularity) split concepts at the α-level which at ω are coexistent (e.g. see Lerman [**18**] for many analogues to maximal r.e.). Our work here involves the splitting of nonspeedability.

2.2. *Definition.* An α-r.e. set $A \subseteq \alpha$ is called *weakly* (*strongly*) *non α-speedable* if there exists an index ϵ of A and an α-recursive h such that for all $R_\tau = A$,

$$A \cap \{\beta | \Phi_\epsilon(\beta) > h(\Phi_\tau(\beta), \beta)\}$$

is bounded (α-finite) in α.

Since several distinct interpretations exist for the notion of Turing reducibility (i.e. $\leqq_{w\alpha}$, $\leqq_\alpha$, $\leqq_{c\alpha}$) there arise several analogues to Schoenfield's limit lemma. One such is provided for weak reducibility.

2.3. LEMMA. ($\leqq_{w\alpha}$ Limit) *For $S(x) \subseteq \alpha$, $S(x) \leqq_{w\alpha} O'$ if and only if there exists an α-recursive sequence $\{S_\sigma(x)\}$ where* $\lim_{\sigma \to \alpha} S_\sigma(x) = S(x)$.

Proof. ($\Leftarrow$) Suppose $\{S_\sigma(x)\}$ is α-recursive and $\lim_{\sigma \to \alpha} S_\sigma(x)$ exists and equals $S(x)$. Let A, $\bar{A}$ be defined as

$$\langle x, \tau \rangle \in A \leftrightarrow \forall_{\sigma \geqq \tau} S_\sigma(x) = S_\tau(x)$$

$$\langle x, \tau \rangle \in \bar{A} \leftrightarrow \exists_{\sigma \geqq \tau} S_\sigma(x) \neq S_\tau(x)$$

Since $\bar{A}$ is α-r.e. and O' is m-complete α-r.e. (cf. Shore [**30**]), there exists an α-recursive f such that

$$\langle x, \tau \rangle \in A \leftrightarrow f(\langle x, \tau \rangle) \notin O'.$$

Hence, for $y = 0, 1$

$$S(x) = y \Leftrightarrow \exists \tau \quad \langle x, \tau \rangle \in A \quad \text{and} \quad S_\tau(x) = y$$
$$\Leftrightarrow \exists \tau \quad f(\langle x, \tau \rangle) \notin O' \quad \text{and} \quad S_\tau(x) = y.$$

Define $R_\epsilon = \{ \langle x, y, \xi, \eta \rangle | K_\eta = \{f(\langle x, \tau \rangle)\}$ and $K_\xi = \emptyset$ and $S_\tau(x) = y\}$. Then it follows that

$$S(x) = y \Leftrightarrow \exists \xi, \eta \quad \langle x, y, \xi, \eta \rangle \in R_\epsilon \quad \text{and} \quad K_\xi \subseteq O' \quad \text{and} \quad K_\eta \subseteq \overline{O'}.$$

Hence, $S(x) = \{\epsilon\}^{O'}(x)$ and $S(x) \leqq_{w\alpha} O'$.

($\Rightarrow$) Assume $S(x) \leqq_{w\alpha} O'$ via ϵ. Then

$$S(x) = y \leftrightarrow \exists \xi, \eta \langle x, y, \xi, \eta \rangle \in R_\epsilon \quad \text{and} \quad K_\xi \subseteq O' \quad \text{and} \quad K_\eta \subseteq \bar{O}'.$$

Since O' is α-r.e., it can be approximated by an α-recursive $O_\sigma'(x)$ with $\lim_{\sigma \to \alpha} O_\sigma'(x) = O'(x)$. Define $M(\sigma, x)$ as

$$\{y | \exists \xi, \eta < \sigma \quad \langle x, y, \xi, \eta \rangle \in R_\epsilon^\sigma \quad \text{and} \quad K_\xi \subseteq O_\sigma' \quad \text{and} \quad K_\eta \subseteq \overline{O_\sigma'}\}.$$

Then take

$$S_\sigma(x) = \begin{cases} \mu y \;\; y \in M(\sigma, x) & \text{if } M(\sigma, x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $S_\sigma(x)$ is α-recursive. It follows from the fact that $S = \{\epsilon\}^{O'}$ and the admissibility of α that for all x, $\lim_{\sigma \to \alpha} S_\sigma(x)$ exists and equals $S(x)$.

We next prove one generalization to α of Soare's index set characterization for nonspeedable sets. Although we employ a somewhat less restrictive analogue to Turing reducibility, weak reducibility, we still require that sets possess some degree of well-behavedness; namely, that of regularity.

2.4. THEOREM. ($\leqq_{w\alpha}$-Soare) *Let A be a regular* α-r.e. *subset of* α. *Then A is weakly non α-speedable if and only if*

$$H_{\bar{A}} = \{\epsilon' | R_{\epsilon'} \cap \bar{A} \neq \emptyset\} \leqq_{w\alpha} O'.$$

(*i.e. $\bar{A}$ is $\leqq_{w\alpha}$-semilow*).

Proof. ($\Rightarrow$) Let $R_\epsilon = A$ and h satisfy the definition of weakly non α-speedable. Since A is α-r.e. we have

$$R_{f(\epsilon')} = R_{\epsilon'} \cup A$$

for α-recursive f. Define

$$H_{\bar{A}}^\sigma(\epsilon') = \begin{cases} 1 & \text{if } (\exists x)\{x \in R_{\epsilon'}^\sigma - R_\epsilon^\sigma \quad \text{and} \quad \Phi_\epsilon(x) > h(x, \Phi_{f(\epsilon')}(x))\} \\ 0 & \text{otherwise.} \end{cases}$$

https://doi.org/10.4153/CJM-1979-030-8 Published online by Cambridge University Press

We will show that $\{H_{\bar{A}}^{\sigma}(x)\}$ is α-recursive and $\lim_{\sigma\to\alpha} H_{\bar{A}}^{\sigma}(x) = H_{\bar{A}}(x)$; hence, by the $\leqq_{w\alpha}$-Limit Lemma, $H_{\bar{A}} \leqq_{w\alpha} O'$.

Suppose $x \in R_{\epsilon'} \cap \bar{A}$ (i.e. $R_{\epsilon'} \cap \bar{A} \neq \emptyset$). Then

$$x \in R_{\epsilon'}^{\sigma} - A^{\sigma} = R_{\epsilon'}^{\sigma} - R_{\epsilon}^{\sigma}$$

for all $\sigma > \sigma_0$. Also, since $R_{\epsilon} = A$, $\Phi_{\epsilon}(x) \uparrow$ and since $R_{\epsilon'} \subseteq R_{f(\epsilon')}$, $h(\Phi_{f(\epsilon')}(x), x) \downarrow$, thus $H_{\bar{A}}^{\sigma}(\epsilon) = 1$, for all $\sigma > \sigma_0$.

Suppose $R_{\epsilon'} \cap \bar{A} = \emptyset$. Then $R_{\epsilon'} \subseteq A$ and hence, $R_{f(\epsilon')} = R_{\epsilon'} \cup A = A$. By the weak non α-speedability of A, since $A = R_{f(\epsilon')}$,

$$R_{\epsilon} \cap \{\sigma | \Phi_{\epsilon}(x) > h(\Phi_{f(\epsilon')}(x), x)\}$$

is bounded by some $\sigma_0 < \alpha$. By the regularity of A, $A \cap \sigma_0$ is α-finite. By a standard property of α-r.e. sets, there will exist a stage σ' such that $A \cap \sigma_0 \subseteq R_{\epsilon}^{\sigma}$ $(= A^{\sigma})$ for all $\sigma > \sigma'$. Hence, $(A \cap \sigma_0) \cap (R_{\epsilon'}^{\sigma} - R_{\epsilon}^{\sigma}) = \emptyset$. Thus, for all $\sigma > \sigma'$, $H_{\bar{A}}^{\sigma}(\epsilon') = 0$.

($\Leftarrow$) Assume $\bar{A}$ $\leqq_{w\alpha}$-semilow; that is,

$$H_{\bar{A}} = \{\epsilon | R_{\epsilon} \cap \bar{A} \neq \emptyset\} \leqq_{w\alpha} O'.$$

By the $\leqq_{w\alpha}$-Limit Lemma there are $H_{\bar{A}}^{\sigma}(x)$ and $H_{\bar{A}}(x)$ where $\lim_{\sigma\to\alpha} H_{\bar{A}}^{\sigma} = H_{\bar{A}}$. Let $\epsilon < \alpha$ be such that $R_{\epsilon} = A$ and define $H(x, y, \epsilon')$ as follows:

1. If $\Phi_{\epsilon'}(x) \neq y$ set $H(x, y, \epsilon') = 0$;
2. If $\Phi_{\epsilon'}(x) = y$ let $t = (\mu\sigma \geqq x)[\Phi_{\epsilon}(x) = \sigma$ or $H_{\bar{A}}^{\sigma}(\epsilon') = 0]$
   a. If $\Phi_{\epsilon}(x) = t$ set $H(x, y, \epsilon') = t$
   b. Otherwise, set $H(x, y, \epsilon') = 0$.

Observe that t exists in 2. when $\Phi_{\epsilon'}(x) = y$. For then $x \in R_{\epsilon'}$, hence either $x \in R_{\epsilon} = A$ or else $x \in R_{\epsilon'} \cap \bar{A}$ and then $H_{\bar{A}}^{\sigma}(\epsilon') = 0$ for all but a bounded subset of α.

Define

$$h(x, y) = \sup \{H(x, y, \beta) | \beta \leqq x\}$$

to see that R_{ϵ} and h witness the weak non α-speedability of A. For suppose $R_{\epsilon'} = A$. Then since $R_{\epsilon'} \cap \bar{A} = \emptyset$, $H_{\bar{A}}^{\sigma}(\epsilon') = 1$ for $\sigma > \sigma_0$. Let $x \in A$ where $x > \max \{\sigma_0, \epsilon', \Phi_{\epsilon}(x)\}$, to see that

$$x \in A \cap \{z | \Phi_{\epsilon}(z) \leqq h(\Phi_{\epsilon'}(z), z)\},$$

for

$$h(x, \Phi_{\epsilon'}(x)) = \sup \{H(x, \Phi_{\epsilon'}(x), \beta) | \beta \leqq x\} \geqq H(x, \Phi_{\epsilon'}(x), \epsilon').$$

But $H_{\bar{A}}^{\sigma}(\epsilon') = 1$, since $x > \sigma_0$, $\Phi_{\epsilon}(\sigma)$, hence, is $\Phi_{\epsilon}(x)$.

Observe that the latter half of the proof makes no use of the regularity of A.

2.5. COROLLARY. *If A is an α*-r.e. *set where $\bar{A}$ is $\leqq_{w\alpha}$-semilow, then A is weakly non α-speedable.*

2.6. COROLLARY. *If A is a regular α-r.e. set where $H_{\bar{A}} \nleqq_{w\alpha} O'$ (i.e. $\bar{A}$ is non $\leqq_{w\alpha}$-semilow), then A is α-speedable.*

Remark. The key role played by A's regularity in the proof of Theorem 2.4 arises in the case $R_{\epsilon'} \cap \bar{A} = \emptyset$ or $R_{\epsilon'} \subseteq A$. Here $R_{f(\epsilon')} = R_{\epsilon'} \cup A = A$ and by weak non α-speedability

$$(*) \quad A \cap \{\sigma | \Phi_\epsilon(x) > h(\Phi_{f(\epsilon')}(x), x)\}$$

is bounded by some $\sigma_0 < \alpha$. Regularity of A ensures that *all* members of (*) will ultimately be generated from A, thus ensuring that $H_{\bar{A}}^{\sigma}(\epsilon') = 0$ for all σ past some σ'.

We next prove that for all nonregular α-r.e. A (whether α-speedable or not) that $H_{\bar{A}} \equiv_{c\alpha} O''$. From this one would suspect that the regularity condition in Theorem 2.4 may be a fundamental one.

2.7. *Definition.* For any α-r.e. indices ϵ, ϵ' for A and α-recursive h, we define the *(h, ϵ, ϵ')-speedup set of A* as $A \cap \{\beta | \Phi_\epsilon(\beta) > h(\Phi_{\epsilon'}(\sigma), \sigma)\}$ and denote such a set as $M(h, \epsilon, \epsilon')$.

The technique used in the following lemma was first employed by Simpson [**31**] and is similar to Spector's classical proof that every Π_1^1 subset of ω is hyperarithmetic in every $\Pi_1^1 - \Delta_1^1$ subset of ω.

2.8. LEMMA. *Let h be α-recursive and ϵ, ϵ' be two α-r.e. indices for A such that the speed-up set $M(h, \epsilon, \epsilon')$ is nonhyperregular. Then $O'' \leqq_{c\alpha} M(h, \epsilon, \epsilon')$.*

Proof. Let $M = M(h, \epsilon, \epsilon')$ be nonhyperregular and let f be weakly α-recursive in M mapping γ unboundedly into α. Since O'' is Σ_2 (cf. Shore [**29**])

$$\sigma \in O'' \leftrightarrow \exists \beta \forall \delta R(\sigma, \beta, \delta)$$

where R is α-recursive. This is equivalent to

$$\exists \beta' < \gamma \quad \forall \delta' < \gamma \quad \exists \beta < f(\beta') \quad \forall \delta < f(\delta') \quad R(\sigma, \beta, \delta)$$

since f maps γ unboundedly into α. The result follows from the definitions of weak α-reducibility and α-calculability.

We next see that the complement of any α-r.e. A with a nonhyperregular speedup set has weak jump at least as high as O''.

2.9. LEMMA. *Let h be α-recursive and ϵ, ϵ' be two indices for A such that $M(h, \epsilon, \epsilon')$ is nonhyperregular. Then $O'' \leqq_{c\alpha} H_{\bar{A}}$.*

Proof. Define

$$R_{f(z)} = \begin{cases} \{z\} & \text{if } \Phi_\epsilon(z) > h(\Phi_{\epsilon'}(z), z), \\ \phi & \text{otherwise.} \end{cases}$$

for α-recursive t. Then

$$z \in M(h, \epsilon, \epsilon') \leftrightarrow R_{f(z)} \subseteq A \leftrightarrow f(z) \in \bar{H}_{\bar{A}}$$

and $M(h, \epsilon, \epsilon')$ is m-reducible to $\bar{H}_{\bar{A}}$, thus certainly α-calculable in $H_{\bar{A}}$. The result follows immediately from Lemma 2.8.

The next series of results shows that every nonregular set possesses at least one nonhyperregular speedup set.

2.10. LEMMA. *For any* α-r.e. *index* ϵ' *for* A, *measure* Φ *and* α*-recursive* h, *there exists an index* ϵ^* *for* A *such that*

(*) $\sigma \in A \rightarrow \Phi_{\epsilon^*}(\sigma) > h(\Phi_{\epsilon'}(\sigma), \sigma)$.

In other words, for any index ϵ' for A there exists another index ϵ^* such that $\sigma \in A \leftrightarrow \sigma \in M(h, \epsilon^*, \epsilon')$.

Proof. Define the algorithm:

$$\phi(\epsilon, \sigma) = \begin{cases} 0 \text{ if } \phi_{\epsilon'}(\sigma)\downarrow \text{ then loop around until:} \\ \quad (\beta)\ \Phi_\epsilon(\sigma) > h(\Phi_{\epsilon'}(\sigma), \sigma);\ \text{(i.e., perform } \kappa \text{ steps of some} \\ \qquad \text{loop while periodically checking } (\beta)\text{. Once } (\beta) \text{ holds,} \\ \qquad \text{output 1)} \\ \uparrow \text{ otherwise} \end{cases}$$

By the α-s-m-n Theorem (Kripke [**15**]), $\phi(\epsilon, \sigma) = \phi_{S(\epsilon)}(\sigma)$ for some α-recursive S. By the α-recursion theorem (also Kripke [**15**]), there exists an ϵ^* such that $\phi_{S(\epsilon^*)}(\sigma) = \phi_{\epsilon^*}(\sigma)$.

To see $R_{\epsilon'} \subseteq R_{\epsilon^*}$ and condition (*), let $\sigma \in A = R_{\epsilon'}$. Then $\phi_{\epsilon'}(\sigma)\downarrow$ and consequently $\Phi_{\epsilon'}(\sigma)\downarrow$, hence $h(\Phi_{\epsilon'}(\sigma), \sigma)\downarrow$. If $\Phi_{\epsilon^*}(\sigma) \leqq h(\Phi_{\epsilon'}(\sigma), \sigma)$ then the algorithm "ϵ^*" would loop around till $\Phi_{\epsilon^*}(\sigma) > h(\Phi_{\epsilon'}(\sigma), \sigma)$. Since $h(\Phi_{\epsilon'}(\sigma), \sigma)\downarrow$, "$\epsilon^*$" must eventually come to a halt, thus making $\sigma \in R_{\epsilon^*}$.

To see $R_{\epsilon^*} \subseteq R_\epsilon$, observe that for any $\sigma \in R_{\epsilon^*}$, we must have $\phi_{\epsilon'}(\sigma)\downarrow$. Hence, $\sigma \in R_{\epsilon'}$ and $R_{\epsilon'} = R_{\epsilon^*} = A$.

2.11. COROLLARY. *Let* ϵ' *be an* α-r.e. *index for* A *and* h *any* α*-recursive function. Then there exists an index* ϵ^* *for* A *such that*

A nonregular $\leftrightarrow M(h, \epsilon^*, \epsilon')$ *nonregular.*

Proof. Let ϵ^* be the index of the previous lemma. Then for any γ,

$$\sigma \in A \cap \gamma \leftrightarrow \sigma \in M(h, \epsilon^*, \epsilon') \cap \gamma.$$

Hence, nonregularity of A is synonomous with nonregularity of $M(h, \epsilon^*, \epsilon')$.

2.12. THEOREM. *Let* A *be* α-r.e. *nonregular. Then* $O'' \leqq_{c\alpha} H_{\bar{A}}$.

Proof. By Corollary 2.11, for any index ϵ' for A and α-recursive h, there exists an index ϵ^* such that $M(h, \epsilon^*, \epsilon')$ is nonregular. Since $M(h, \epsilon^*, \epsilon')$ is α-r.e. it is also nonhyperregular; thus, by Lemma 2.9, $O'' \leqq_{c\alpha} H_{\bar{A}}$.

3. Sacks' Regular Representative and α-Retrogressive Sets. The first consideration of complexity properties of sets and degrees of unsolvability

https://doi.org/10.4153/CJM-1979-030-8 Published online by Cambridge University Press

was given by Marques [21] who showed that if an ω-r.e. Turing degree a contained a speedable set then it must be non-low (i.e. $\underset{\sim}{a}' >_T \underset{\sim}{O}'$). In [33], Soare exploits his semilow characterization to not only provide a simpler proof of Marques' result but to also prove the converse—that all high ω-r.e. Turing degrees contain speedable sets. In § 3 and § 4, we prove an analogue to the Marques-Soare theorem for α-r.e. α-degrees. The proof is different from the one provided by Soare due to the condition of regularity imposed in Theorem 2.4. However, the general structure for our argument is suggested by a remark made by Soare in [33] that indicates an alternative proof.

We begin by reviewing one of the most frequently used results ([17], [28], [32]) of α-recursion theory—the Sacks Regular Representative Theorem. Although an α-r.e. α-degree may contain nonregular α-r.e. sets, Sacks' theorem tells us that there must be at least one regular α-r.e. set.

3.1. THEOREM (Sacks [24]). *Every* α-r.e. *α-degree contains a regular* α-r.e. *set.*

Proof. (Simpson [31]). Let $\underset{\sim}{a}$ be an α-r.e. α-degree and B an α-r.e. member of $\underset{\sim}{a}$. If B is regular we are done; otherwise, let $\beta < \alpha$ be such that $B \cap \beta$ is α-infinite. Let $\alpha^* < \alpha$ be the Σ_1 projectum of α and g a one-one α-recursive projection of α onto a subset of α^*. Define $N = g[B \cap \beta]$ which is α-r.e. and by admissibility is an α-infinite subset of α^*.

3.2. *Claim.* $N \leqq_\alpha B$. Since B is α-r.e. all we need show is the clause for $K_\gamma \subseteq \bar{N}$. However,

$$K_\gamma \subseteq \bar{N} \leftrightarrow \forall \sigma \in K_\gamma (\sigma \in g[\beta] \rightarrow \sigma \notin g[B]).$$

By admissibility, $g[\beta]$ is α-finite, hence equal to some K. Let

$$R_{\epsilon_0} = \{ \langle \gamma, \xi, \eta \rangle | K_\eta = g^{-1}[K_\gamma \cap K] \quad \text{and} \quad K_\xi = \emptyset \}.$$

Then from the definitions

$$K_\gamma \subseteq \bar{N} \leftrightarrow (\exists \xi)(\exists \eta) \langle \gamma, \xi, \eta \rangle \in R_{\epsilon_0} \quad \text{and} \quad K_\xi \subseteq B \quad \text{and} \quad K_\eta \subseteq \bar{B}\}.$$

Since N is α-r.e. and α-infinite, let n be a one-one α-recursive function with range N. Let

$$B^* = \{ n(\eta) | K_\eta \cap B \neq \emptyset \}.$$

Clearly B^* is an α-r.e. subset of α^* and again by admissibility is α-infinite.

3.3. *Claim.* (X) $B^* \leqq_{w\alpha} X \leftrightarrow B^* \leqq_\alpha X$. Clearly $B^* \leqq_\alpha X$ implies $B^* \leqq_{w\alpha} X$. Hence, suppose $B^* \leqq_{w\alpha} X$. Since B^* is α-r.e. we only need deal with the negative clause of $B^* \leqq_\alpha X$. Then

$$K_\alpha \subseteq \overline{B^*} \Leftrightarrow \textstyle\bigcup_{\delta \in K_\alpha} K_\delta \subseteq \bar{B}.$$

Letting $g(x)$ be the α-recursive function where $K_{g(\alpha)} = \bigcup_{\delta \in K_\alpha} K_\delta$, then

$$\Leftrightarrow K_{g(\alpha)} \subseteq \bar{B}$$
$$\Leftrightarrow g(\alpha) \in \overline{B^*}.$$

Since $B^* \leqq_{w\alpha} X$, it easily follows that $B^* \leqq_\alpha X$.

Let f be a one-one α-recursive enumeration function for B^*. Define the *deficiency set of f* by

$$D_f = \{\gamma | (\exists \delta)_{\gamma<\delta} f(\delta) < f(\gamma)\}.$$

Clearly D_f is α-r.e.

3.4. *Claim.* D_f *is regular*. For any $\beta < \alpha$

$$D_f \cap \beta = \{\gamma < \beta | (\exists \delta)_{\gamma<\delta} f(\delta) < f(\gamma)\}.$$

However, for each β there will exist a $\tau_\beta < \alpha$ such that

$$D_f \cap \beta = \{\gamma < \beta | (\exists \delta)_{\gamma<\delta\leqq\tau_\beta} f(\delta) < f(\gamma)\}.$$

For if such a τ_β did not exist, then in searching for the various δ, one could develop a sequence $\delta_n (n < \omega)$ whereby $f(\delta_{n+1}) < f(\delta_n)$ for all n.

3.5. *Claim.* $B \leqq_\alpha D_f$. For each $\nu < \alpha^*$, define

$$p(\nu) = \mu\gamma[\nu < f(\gamma) \quad \text{and} \quad \gamma \in \bar{D}_f].$$

By an argument similar to the one used in proving the previous claim, $\bar{D}_f$ is unbounded and consequently p is total on α^*. Clearly p is weakly α-recursive in D_f and from the definitions,

$$\nu \in B^* \Leftrightarrow \nu \in f[p(\nu)].$$

Since $B^* \leqq_m p$ and $p \leqq_{w\alpha} D_f$, then $B^* \leqq_{w\alpha} D_f$. From Claim 2, $B^* \leqq_\alpha D_f$. But for $t(\gamma)$, $K_{t(\gamma)} = \{\gamma\}$, $B \leqq_m B^*$, hence $B \leqq_\alpha D_f$.

Finally,

3.6. *Claim.* $D_f \leqq_\alpha B$. Observe that f is increasing on $\bar{D}_f$, for if $\gamma \in \bar{D}_f$ then $(\forall\delta)_{\gamma<\delta} f(\gamma) \leqq f(\delta)$. Consequently,

$$\gamma \in \bar{D}_f \Leftrightarrow f(\gamma) - f[\gamma] \subseteq \overline{B^*}.$$

Now for any α-finite set K, let $K' = \bigcup_{\gamma\in K} (f(\gamma) - f[\gamma])$. Observe that K' is α-finite and

$$K \subseteq \bar{D}_f \Leftrightarrow K' \subseteq \overline{B^*}.$$

Further, let $K'' = \bigcup \{K_\eta | \eta \in n^{-1}[K' \cap N]\}$. Then from the definitions, $K'' \subseteq \bar{B}$.

In [**12**], Jockusch studies properties of various classes of simple sets and their relationships regarding several reducibility orderings. Fundamental to Jockusch's investigation is his notion of semirecursive set.

3.7. *Definition.* A set A is *α-semirecursive* if there is an α-recursive function f of two variables such that for all β, γ:

(i) $f(\beta, \gamma) = \beta$ or γ
(ii) $\beta \in A$ or $\gamma \in A$ implies $f(\beta, \gamma) \in A$.

f is called a *selector function* for A.

https://doi.org/10.4153/CJM-1979-030-8 Published online by Cambridge University Press

The following properties follow directly from the definition of α-semirecursive set.

3.8. *Corollary. A is α-semirecursive if and only if $\bar{A}$ is α-semirecursive.*

3.9. *Definition.* A set B is called *m-compressible* if $B \times B \leqq_m B$.

3.10. *Corollary. A α-semirecursive implies $\bar{A}$ m-compressible.*

Dekker's [**3**] classical proof of the existence of hypersimple ω-r.e. sets in every nonrecursive Turing degree generated the well studied notions of regressiveness and retracibility [**23**]. Since both of these entail the concept of an immediate successor, they need be altered for study over the ordinals.

3.11. *Definition.* A set A is called *α-retrospective* if and only if for all $\beta \in A$ the set $A \cap \beta$ is α-r.e. with α-r.e. index uniformly obtainable from β (i.e. $A \cap \beta = R_{t(\beta)}$, t α-recursive).

3.12. *Definition.* A set A is called *strongly α-retrospective* if and only if for all $\beta \in A$ the set $A \cap \beta$ is α-finite with α-finite index uniformly obtainable from β (i.e. $A \cap \beta = K_{s(\beta)}$, s α-recursive).

3.13. COROLLARY. *A strongly α-retrospective implies A α-retrospective.*

The following generalizes Jockusch's [**12**] observation that ω-r.e. sets with regressive complements are semirecursive.

3.14. THEOREM. *If a set $A \subseteq \alpha$ is α-*r.e. *and $\bar{A}$ is α-retrospective then A is α-semirecursive.*

Proof. We define our selector function $f(x, y)$ by the following construction.

Stage σ.

1. If $\beta \in A^\sigma$ then $f(\beta, \gamma) = \beta$
2. If $\gamma \in A^\sigma$ then $f(\beta, \gamma) = \gamma$
3. If $\beta \in R_{t(\gamma)}{}^\sigma$ then $f(\beta, \gamma) = \gamma$
4. If $\gamma \in R_{t(\beta)}{}^\sigma$ then $f(\beta, \gamma) = \beta$
5. Otherwise, go to stage $\sigma + 1$.

f is α-partial recursive. To see it is total observe that if β or $\gamma \in A$ then 1 or 2 above will ultimately cause an output. If $\beta \in \bar{A}$ and $\gamma \in \bar{A}$, then since $\bar{A}$ is α-retrospective, $\beta \in R_{t(\gamma)}$ or $\gamma \in R_{t(\beta)}$; hence, one will eventually be located.

We next see that if $f(\beta, \gamma) \in \bar{A}$ then both $\beta \in \bar{A}$ and $\gamma \in \bar{A}$. For if $f(\beta, \gamma) \in \bar{A}$, then certainly $f(\beta, \gamma)$ was obtained via steps 3 or 4. Suppose $f(\beta, \gamma)$ was obtained at step 3; then $f(\beta, \gamma) = \gamma$ and so $\gamma \in \bar{A}$. Hence, by α-retrospectiveness, $R_{t(\gamma)} \subseteq \bar{A}$ and by step 3, $\beta \in R_{t(\gamma)}$, hence $\beta \in \bar{A}$. Similarly, for step 4.

We tie together our notion of α-retrospective sets with Sacks' constructed regular representative, D_f.

3.15. LEMMA. *D_f is α-*r.e. *and $\bar{D}_f$ is strongly α-retrospective.*

https://doi.org/10.4153/CJM-1979-030-8 Published online by Cambridge University Press

Proof. Recall from the proof of Theorem 3.1 that $D_f = \{\gamma | \exists_{\gamma<\delta} f(\delta) < f(\gamma)\}$. Clearly, D_f is α-r.e. To show $\bar{D}_f$ is α-retrospective we define for all $\beta < \alpha$,

$$K(\beta) = \{\gamma < \beta | f(\gamma) < f(\beta) \quad \text{and} \quad (\forall \delta)\gamma < \delta < \beta \rightarrow f(\gamma) < f(\delta)\}.$$

Clearly, $K(\beta)$ is α-finite with index α-effectively attainable from β. Hence, all that remains is to show that for all $\beta \in D_f$, $\bar{K}(\beta) = \bar{D}_f \cap \beta$.

Let $\gamma \in K(\beta)$. Then $\gamma < \beta$ and $f(\gamma) < f(\beta)$. Since $\beta \in \bar{D}_f$, $f(\gamma) < f(\beta) < f(\delta)$ for all $\delta > \beta$. By definition of $K(\beta)$, $f(\gamma) < f(\delta)$ for $\delta : \gamma < \delta < \beta$. Hence, for all $\delta > \gamma$, $f(\delta) > f(\gamma)$, and $\gamma \in \bar{D}_f \cap \beta$. Conversely, let $\gamma \in \bar{D}_f \cap \beta$. Then $\gamma < \beta$ and $f(\gamma) < f(\beta)$, else γ would *not* be in $\bar{D}_f$. For the same reason $f(\delta) > f(\gamma)$, for all $\delta > \gamma$; in particular, for $\gamma < \delta < \beta$. Hence $\gamma \in K(\beta)$.

3.16. Corollary. *Every irregular* α-r.e. α*-degree contains a regular* α-r.e. A *where* A *is m-compressible.*

Proof. D_f is α-r.e. and regular. By Lemma 3.15, $\bar{D}_f$ is α-retrospective. Thus by Theorem 3.14, D_f is α-semirecursive, and, by Corollary 3.10, satisfies $\bar{D}_f \times \bar{D}_f \leqq_m \bar{D}_f$.

In the regular representative proof (Theorem 3.1) it is argued for all $\beta < \alpha$, $D_f \cap \beta$ is α-finite using a *noneffective* step. Sacks [**26**] questions whether this step may be effectivized. We provide a partial answer to this in the following.

3.17. Corollary. *Every* α-r.e. α*-degree* **a** *contains a regular set* A *such that for all* $\beta \in A$, $A \cap \beta$ *is* α*-finite (effectively).*

Proof. For $\beta \in A$, $A \cap \beta$ is the α-finite set $K(\beta)$.

4. α-Degrees and α-Speedable Sets. In this section we consider properties of sets with m-compressible complements, in particular, with regard to their weak jumps and generalized α-jumps. The results of this and the previous section are then used to prove an α-analogue to the Soare-Marques characterization. We conclude by displaying a class of admissible α's for which a phenomenon at ω fails to hold at α; namely, the existence of incomplete speedable sets.

First, a technical result telling us that for α-semirecursive sets, α-finite membership questions can be reduced to single questions.

4.1. Lemma. *Let* $A \subset \alpha$ *be such that* $\bar{A}$ *is m-compressible. Then there exists an* α*-recursive* f^* *such that for all* $\eta < \alpha$,

$$K_\eta \subseteq \bar{A} \leftrightarrow f^*(\eta) \in \bar{A}$$

Proof. Let f be an α-recursive m-reducibility map such that $\bar{A} \times \bar{A} \leqq_m \bar{A}$ via f. We define values $\{\beta_\sigma, \beta_\sigma^*\}$ via a construction below and then use these to define our function f^*.

Stage 0. Set $\beta_0^* = \beta_0 = \mu\beta[\beta \in K_\eta]$

Stage σ. Set $\beta_\sigma = \mu\beta[\beta \in K_\eta - \bigcup_{\tau<\sigma} \beta_\tau]$.

https://doi.org/10.4153/CJM-1979-030-8 Published online by Cambridge University Press

If $\beta_\sigma = \emptyset$ then set $\beta_\sigma^* = \bigcup_{\tau<\sigma} \beta_\tau^*$ and halt: Otherwise set $\beta_\sigma^* = f(\bigcup_{\tau<\sigma} \beta_\tau^*, \beta_\sigma)$ and proceed to stage $\sigma + 1$.

For each $\eta < \alpha$, there exists (by admissibility) a least stage σ_η *after which* all of K_η has been enumerated in increasing order. Further, such a σ_η is α-effectively obtainable from η. Hence, the function defined by $f^*(\eta) = \beta_{\sigma_\eta}{}^*$ is well defined and α-recursive.

Claim. $\forall\eta \quad K_\eta \subseteq \bar{A} \leftrightarrow f^*(\eta) \in \bar{A}$. For suppose $f^*(\eta) \in \bar{A}$ and β is the least member of $K_\eta \cap A$. Let $\sigma_\beta < \sigma_\eta$ be the stage of the construction at which β arises (i.e. equals β_{σ_β}). Then at stage σ_β, $f(\bigcup_{\tau<\sigma\beta} \beta_\tau^*, \beta) \in A$ making it impossible at any later stage σ' for $\beta_{\sigma'}{}^* \in \bar{A}$; in particular, $\sigma' = \sigma_\eta$.

Conversely, suppose $K_\eta \subseteq \bar{A}$. Then by a straightforward induction argument it is seen that for all $\sigma \leqq \sigma_\eta$, $\beta_\sigma^* \in \bar{A}$. Since $f^*(\eta) = \beta_{\sigma_\eta}{}^*$, our result follows.

As a preliminary to his regular representative proof, Sacks [**24**] shows that for any α-r.e. A there exists an α-r.e. B of the same α-degree as A such that for $X \subseteq \alpha$, $B \leqq_{w\alpha} X \leftrightarrow B \leqq_\alpha X$. In [**5**] Gill and Morris show that for each ω-r.e. set A there exists an ω-r.e. B, $B \equiv_T A$, where A is Turing complete if and only if B is effectively speedable (actually subcreative). In both cases, the sets B turn out to be a q-cylindrication of A.

4.2. *Definition.* For any set A, the *q-cylindrification* of A, denoted A^q, is defined as

$$A^q = \{\eta | K_\eta \cap A \neq \emptyset\}.$$

4.3. LEMMA. *For $A \subset \alpha$ with m-compressible complement,* $H_{\overline{A^q}} \leqq_m H_{\bar{A}}$.

Proof. For all $\epsilon < \alpha$

$$\begin{aligned} \epsilon \in H_{\overline{A^q}} &\leftrightarrow R_\epsilon \cap \overline{A^q} \neq 0 \\ &\leftrightarrow \exists\, \eta(\eta \in R_\epsilon \quad \text{and} \quad \eta \in \overline{A^q}) \\ &\leftrightarrow \exists\, \eta(\eta \in R_\epsilon \quad \text{and} \quad K_\eta \subseteq \bar{A}). \end{aligned}$$

Let f^* be the α-recursive function of Lemma 4.1,

$$\leftrightarrow \exists\, \eta \quad \eta \in R_\epsilon \quad \text{and} \quad f^*(\eta) \in \bar{A}.$$

For $R_{t(\epsilon)} = f^*[R_\epsilon]$ with t α-recursive,

$$\begin{aligned} &\leftrightarrow \exists\, \eta(f^*(\eta) \in R_{t(\epsilon)} \cap \bar{A}) \\ &\leftrightarrow R_{t(\epsilon)} \cap \bar{A} \neq \emptyset \\ &\leftrightarrow t(\epsilon) \in H_{\bar{A}}. \end{aligned}$$

Hence $H_{\overline{A^q}} \leqq_m H_{\bar{A}}$ via t.

In [**30**], R. Shore proposes a definition for α-jump operator (which is equivalent to that of Simpson [**32**]) and provides justification for his over several alternatives. The basis for his definition is a notion of relative α-recursive enumerability.

4.4. *Definition.* For any α-r.e. set of triples R_ϵ, we say R_ϵ *enumerates* x *relative to* A if $(\exists\, \xi, \eta)(\langle x, \xi, \eta \rangle \in R_\epsilon$ and $K_\xi \subseteq A$ and $K_\eta \subseteq \bar{A})$. We denote by $R_\epsilon{}^A$ the set of all x enumerated by R_ϵ relative to A. Thus $B \subseteq \alpha$ is *α-recursively enumerable* (α-r.e.) *in* A *if* $B = R_\epsilon{}^A$ for some $\epsilon < \alpha$.

4.5. *Definition.* For any set $A \subseteq \alpha$ the *α-jump of* A is the set

$$A = \{\langle x, \epsilon \rangle | x \in R_\epsilon{}^A\}.$$

Shore demonstrates that for A, $B \subseteq \alpha$, (i) B α-r.e. in A if and only if $B \leqq_{m\alpha} A'$, (ii) $B \equiv_\alpha A$ implies $B' \equiv_m A'$, as well as analogs to other usual properties of the jump.

4.6. Lemma. *For all* B, $H_B \leqq_m B'$.

Proof. As in [**33**] we employ the set $H^B = \{\epsilon | R_\epsilon{}^B \neq \emptyset\}$.

Claim. $H_B \leqq_m H^B$ *for all* B. Define α-recursive $t(\epsilon)$ by

$$R_{t(\epsilon)} = \{\langle x, \xi, \eta \rangle | K_\xi = \{x\} \quad \text{and} \quad K_\eta = \emptyset \quad \text{and} \quad x \in R_\epsilon\}.$$

Then

$$\begin{aligned} \epsilon \in H_B &\leftrightarrow B \cap R_\epsilon \neq \emptyset \\ &\leftrightarrow \exists\, x \quad x \in B \cap R_\epsilon \\ &\leftrightarrow \exists\, x \quad x \in R^B{}_{t(\epsilon)} \\ &\leftrightarrow t(\epsilon) \in H^B \end{aligned}$$

Claim. $H^B \leqq_m B'$ *for all* B. Define α-recursive $r(\epsilon)$ by

$$R_{r(\epsilon)} = \{\langle 1, \xi, \eta \rangle | \exists\, x \langle x, \xi, \eta \rangle \in R_\epsilon\}$$

and α-recursive $f(\beta)$ by $f(\beta) = \langle 1, r(\epsilon) \rangle$. Then

$$\begin{aligned} \epsilon \in H^B &\leftrightarrow R_\epsilon{}^B \neq \emptyset \\ &\leftrightarrow \exists\, \xi, \eta, x \ \langle x, \xi, \eta \rangle \in R_\epsilon \quad \text{and} \quad K_\xi \subseteq B \quad \text{and} \quad K_\eta \subseteq \bar{B} \\ &\leftrightarrow \exists\, \xi, \eta \ \langle 1, \xi, \eta \rangle \in R_{r(\epsilon)} \quad \text{and} \quad K_\xi \subseteq B \quad \text{and} \quad K_\eta \subseteq \bar{B} \\ &\leftrightarrow \langle 1, r(\epsilon) \rangle \in B' \\ &\leftrightarrow f(\epsilon) \in B'. \end{aligned}$$

Hence, from the Claims, it follows that for all B, $H_B \leqq_m B'$.

4.8. Corollary. *Let* $\leqq$ *be one of the four reducibilities* $(\leqq_m, \leqq_\alpha, \leqq_{w\alpha}, \leqq_{c\alpha})$ *and call* $B \subseteq \alpha$ $\leqq$*-low if* $B' \leqq O'$ *and* $\leqq$*-semilow if* $H_B \leqq O'$. *Then* B $\leqq$*-low implies* B $\leqq$*-semilow.*

Proof. By Lemma 4.6 $H_B \leqq_m B'$ for all $B \subseteq \alpha$. If B is $\leqq$-low, then $B' \leqq O'$; hence, $H_B \leqq O'$.

https://doi.org/10.4153/CJM-1979-030-8 Published online by Cambridge University Press

Hay [**7**] shows that there can be sets $A \subseteq \omega$ within the same Turing degree having Turing incomparable weak jumps. Our next result shows that this cannot be the case for α-semirecursive sets.

4.9. LEMMA. *For $A \subseteq \alpha$ with m-compressible complement, $A' \equiv_m H_{\bar{A}}$.*

Proof. Following Shore's definition of α-jump

$$\epsilon \in A' \leftrightarrow \epsilon_0 \in R_{\epsilon_1}{}^A \text{ where } \epsilon = \langle \epsilon_0, \epsilon_1 \rangle$$
$$\leftrightarrow \exists\, \xi, \eta[\langle \epsilon_0, \xi, \eta \rangle \in R_{\epsilon_1} \quad \text{and} \quad K_\xi \subseteq A \quad \text{and} \quad K_\eta \subseteq \bar{A}]$$
$$\leftrightarrow \exists\, \xi[\eta \in R_{S(\epsilon)} \quad \text{and} \quad K_\eta \subseteq \bar{A}]$$

where $R_{S(\epsilon)} = \{\eta | \exists\, \xi \quad \langle \epsilon_0, \xi, \eta \rangle \in R_{\epsilon_1} \quad \text{and} \quad K_\xi \subseteq A\}$ and $S(\epsilon)$ α-recursive,

$$\leftrightarrow \exists\, \eta[\eta \in R_{S(\epsilon)} \quad \text{and} \quad K_\eta \cap A = \emptyset]$$
$$\leftrightarrow \exists\, \eta[\eta \in R_{S(\epsilon)} \cap \overline{A^q}]$$
$$\leftrightarrow R_{S(\epsilon)} \cap \overline{A^q} \neq \emptyset$$
$$\leftrightarrow S(\epsilon) \in H_{\overline{A^q}}.$$

Hence, from the above, $A' \leqq_m H_{\overline{A^q}}$. By Lemma 4.3, $H_{\overline{A^q}} \leqq_m H_{\bar{A}}$, thus, $A' \leqq_m H_{\bar{A}}$. For the opposite direction we let $B = \bar{A}$ in Lemma 4.6.

Corollary 4.10 is the key step of Soare's [**33**] proof that high Turing degrees contain at least one speedable set (since all sets $A \subseteq \omega$ are regular).

4.10. COROLLARY. *For any $A \subseteq \alpha$, $(A^q)' \equiv_m H_{\overline{A^q}}$.*

Proof. For all β_1, β_2, let $K_{f(\beta_1, \beta_2)} = K_{\beta_1} \cup K_{\beta_2}$ for some α-recursive f. It follows that $\overline{A^q} \times \overline{A^q} \leqq_m \overline{A^q}$ via f, hence that $\overline{A^q}$ is m-compressible. The result follows from Lemma 4.9.

4.11. COROLLARY. *Every α-r.e. α-degree a contains a regular A with $A' \equiv_m H_{\bar{A}}$.*

Proof. If a is a regular α-r.e. α-degree, then by Sacks [**26**] (Chap. 25) the q-cylindrication of any $A \in a$ is also in a; thus by Corollary 4.10 A^q has the desired property. If a contains a nonregular member, Corollary 3.16 provides us with a regular α-r.e. member with m-compressible complement. The result, in this case, follows from Lemma 4.9.

By $B <_{w\alpha} A$ we mean, as usual, that $B \leqq_{w\alpha} A$ and $A \nleqq_{w\alpha} B$. In the case of α-degrees a, b, $a <_{w\alpha} b$ denotes that for all $A \in a$ there is $B \in b$ with $A <_{w\alpha} B$.

4.12. THEOREM. *An α-r.e. α-degree a contains an α-speedable set if and only if $O' <_{w\alpha} a'$.*

Proof ($\Rightarrow$) Suppose $O' <_{w\alpha} a'$ fails to hold. Thus there exists a $C' \in O'$ so that for all $A' \in a'$ either $C' \nleqq_{w\alpha} A'$ or else $A' \leqq_{w\alpha} C'$. Since $C' \equiv_\alpha O'$ it easily follows that for all $A \in a$, $C' \leqq_{w\alpha} A'$. By Corollary 4.8 each A is $\leqq_{w\alpha}$-semilow and by Corollary 2.5, A is weakly non α-speedable.

https://doi.org/10.4153/CJM-1979-030-8 Published online by Cambridge University Press

($\Leftarrow$) Suppose $O' <_{w\alpha} a'$. Then by Corollary 4.11 there exists a regular α-r.e. $A \in a$ where $A' \equiv_m H_A$. Then, $O' <_{w\alpha} H_{\bar{A}}$ and $\bar{A}$ is non $\leqq_{w\alpha}$-semilow; its α-speedability follows from Corollary 2.6.

The proof of the first half of Theorem 4.12 goes through if we replace the condition $O' <_{w\alpha} a'$ by $O' <_\alpha a'$. However, difficulty resides in the second part. Namely, if $O' <_\alpha a'$, then there would still be a regular $A \in a$ where $A' \equiv_m H_{\bar{A}}$, $O' <_\alpha A' \equiv_m H_{\bar{A}}$ and thus $H_{\bar{A}} \nleqq_\alpha O'$. However, this last condition *does not* necessarily imply $H_{\bar{A}} \nleqq_{w\alpha} O'$, (that is, non $\leqq_{w\alpha}$-semilowness of $\bar{A}$) since $\leqq_\alpha$ and $\leqq_{w\alpha}$ are distinct.

4.13. COROLLARY. *Let a be an α-*r.e. *irregular α-degree where $O' <_{w\alpha} a'$. Then Sacks' regular representative in a is α-speedable.*

4.14. COROLLARY. *Let a be an α-*r.e. *irregular α-degree. Then a contains an α-speedable set if and only if Sacks' regular representative in a is α-speedable.*

Shore [**30**] discovered an interesting pathology for admissible α when O' is the only existing nonhyperregular α-r.e. α-degree. Namely, that incomplete α-r.e. degrees (sets) may not be α-jumped over O'.

4.15. COROLLARY. *Let α be such that there is only one α-*r.e. *nonhyperregular α-degree (e.g. $\alpha = \aleph_\omega{}^L$). Then the only α-speedable sets are the complete α-*r.e. *ones.*

Proof. By Shore [**30**], for every α-r.e. $a <_\alpha O'$, $a' \equiv_\alpha O'$. Hence, every $\bar{A} \in \alpha$ is $\leqq_{w\alpha}$-low; and by Corollary 4.8 is $\leqq_{w\alpha}$-semilow. By Theorem 2.5 each A is weakly non α-speedable.

This phenomenon differs from ω since Sacks [**25**] shows the existence of high ω-r.e. degrees below O'. By Soare [**33**] such degrees must contain speedable sets.

5. Open Problems. In § 2 we proved an analogue to Soare's theorem for regular sets, A, and showed that for nonregular ones $O'' \leqq_{c\alpha} H_{\bar{A}}$. Since non α-calculability does *not* imply non weak α-reducibility, this fails to give a complete answer to whether regularity is essential in Theorem 2.4.

1. Do there exist weakly non-α-speedable nonregular α-r.e. sets which are *not* $\leqq_{w\alpha}$-semilow? If so, characterize those α for which they exist.

Weak α-reducibility is only one of the reducibilities studied in α-recursion theory. Consequently, three different variations of non α-speedability should exist.

2. What form does Soare's theorem take on when we use $\leqq_\alpha$-semilow for semilow?

3. The same as the above, but for $\leqq_{c\alpha}$-semilow.

As in the proof of Theorem 2.4, answers to the above questions probably require investigating analogues to Shoenfield's limit lemma.

Marques [**21**] proves that there exists a nonspeedable set in every r.e. Turing degree. His result was obtained by using as a lemma an analogue to Sacks' Splitting Theorem [**25**]; namely, that any nonrecursive r.e. set may be decomposed into two nonrecursive low nonspeedable sets. Shore [**28**] lifts Sacks' result via a non α-finite injury argument for regular α-r.e. sets and shows in [**29**] pathologies for nonregular and nonhyperregular ones.

4. Can every regular non α-recursive α-r.e. set be decomposed into two non α-recursive weakly (strongly) non α-speedable ones?

5. Can any irregular (nonhyperregular) α-r.e. α-degree be similarly decomposed?

6. Does every α-r.e. non α-recursive α-degree possess a weakly non α-speedable set? a strongly non α-speedable set?

Soare [**33**] exploits his semilow criterion to yield a simpler proof of Marques' result. His argument is based upon the observation that every r.e. Turing degree possesses an r.e. A where $H_{\bar{A}} \leqq O'$. This last result is a special case of Hay's [**6**] analogue to the Sacks jump theorem, where weak jump (H_A) replaces the usual jump (A').

7. Classify those admissible α in which every α-r.e. α-degree contains an α-r.e. A where

(a) $H_{\bar{A}} \leqq_{w\alpha} O'$ $\quad$ ($\bar{A}$ $\leqq_{w\alpha}$-semilow)

(b) $H_{\bar{A}} \leqq_{\alpha} O'$ $\quad$ ($\bar{A}$ $\leqq_{\alpha}$-semilow)

(c) $H_{\bar{A}} \leqq_{c\alpha} O'$ $\quad$ ($\bar{A}$ $\leqq_{c\alpha}$-semilow).

8. Characterize those admissibles for which Hay's general result holds.

An ω-speedable r.e. set is *effectively speedable* if not only arbitrarily faster algorithms exist, but they are effectively obtainable from any algorithm determining the speedup. It was shown by Blum and Marques [**2**] that effective speedability is equivalent to *subcreativity* (a slightly weaker form of creative set) and that there exists sets which are *speedable* but not *effectively speedable*. Interestingly, the only proven witnesses to the differences between these classes are the r-maximal sets (i.e., r-maximals are speedable but not effectively speedable). Since, in α-recursion theory, r-maximal sets do not exist for all α (Lerman and Simpson [**19**]) we ask:

9. Do there exist other sets which are α-speedable but not effectively α-speedable?

10. Classify those α for which α-speedable equals effectively α-speedable.

Bibliography

1. M. Blum, *A machine-independent theory of the complexity of recursive functions*, J. ACM *14* (1967), 322–336.

2. M. Blum and I. Marques, *On complexity properties of recursively enumerable sets*, J. Symbol Logic *38* (1973), 579–593.

3. J. C. E. Dekker, *A theorem on hypersimple sets*, Proc. Amer. Math. Soc. *5* (1954), 791–796.

4. G. C. Driscoll, *Metarecursively enumerable sets and their metadegrees*, J. Symb. Logic *33* (1968), 389–411.

5. J. Gill and P. Morris, *On subcreative sets and S-reducibility*, J. Symb. Logic *39* (1974), 669–677.
6. L. Hay, *The class of recursively enumerable subsets of a recursively enumerable set*, Pacific J. Math. *46* (1973), 167–183.
7. ——— *The halting problem relativized to complements*, Proc. AMS, *41* (1973), 583–587.
8. B. Jacobs, *α-computational complexity*, Ph.D. Thesis, Technical report IMM 408, Courant Inst., (1975).
9. ——— *On generalized computational complexity*, J. Symb. Logic, *42* (1977), 47–58.
10. ——— *The α-union theorem and generalized primitive recursion*, Trans. A.M.S., *237* (1978), 63–81.
11. ——— *The α-speedup and α-naming theorems*, to appear.
12. C. Jockusch, *Semirecursive sets and positive reducibility*, Trans. Amer. Math. Soc. *131* (1968), 420–436.
13. S. C. Kleene, *Introduction to metamathematics*, (North-Holland, Amsterdam, 1971).
14. G. Kreisel and G. E. Sacks, *Metarecursive sets*, J. Symb. Logic. *30*, (1965), 318–338.
15. S. Kripke, *Transfinite recursion on admissible ordinals, I, II* (abstracts), J. Symb. Logic, *29* (1964), 161–162.
16. ——— *The theory of transfinite recursions*, unpublished, class notes by A. Thomas Tymoczko.
17. M. Lerman, *Least upper bounds for minimal pairs of* α-r.e. *α-degrees*, J. Symb. Logic, *39* (1974), 49–56.
18. ——— *Maximal* α-r.e. *sets*, Trans. A.M.S. *188* (1974), 341–386.
19. M. Lerman and S. G. Simpson, *Maximal sets in α-recursion theory*, Israel J. Math. *14* (1972), 236–247.
20. M. Machover, *The theory of transfinite recursion*, Bull. Am. Math. Soc. *67* (1961), 575–578.
21. I. Marques, *On degrees of unsolvability and complexity properties*, J. Symb. Logic *40* (1975), 529–540.
22. R. Platek, *Foundations of recursion theory*, Ph.D. Thesis, Stanford (1966).
23. H. Rogers, *Theory of recursive functions and effective computability*, (McGraw-Hill, 1967).
24. G. E. Sacks, *Post's problem, admissible ordinals and regularity*, Trans. Amer. Math. Soc. *124* (1966), 1–24.
25. ——— *Degrees of unsolvability*, Ann. Math. Study 55 (Princeton, 1966).
26. ——— *Higher recursion theory*, Springer-Verlag (Berlin), to appear.
27. J. R. Shoenfield, *Degrees of unsolvability*, (North-Holland, Amsterdam, 1971).
28. R. Shore, *Splitting an α-recursively enumerable set*, Trans. A.M.S., *204* (1975), 67–78.
29. ——— *The irregular and nonhyperbolic* α-r.e. *degree*, Israel J. Math. *22* (1975), 28–41.
30. ——— *On the jump of an α-recursively enumerable set*, Trans. Amer. Math. Soc. *217* (1976), 351–363.
31. S. G. Simpson, *Admissible ordinals and recursion theory*, Ph.D. Thesis, Mass. Inst. Tech., 1971.
32. ——— *Degree theory on admissible ordinals*, in Generalized Recursion Theory, (1972), North-Holland, 165–193.
33. R. Soare, *Computational complexity, speedable and levelable sets*, J. Symb. Logic, *42* (1977), 545–563.
34. G. Takeuti, *On the recursive functions of ordinal numbers*, J. Math. Soc. Japan *12* (1960), 119–128.

University of Maryland,
College Park, Maryland