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Noncommutative schemes

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Abstract. The main purpose of this work is to introduce noncommutative relative schemes and establish some of basic properties of schemes and scheme morphisms. In particular, we prove an analogue of the canonical bijection: $\mathrm{Hom}_{\mathrm{schemes}/k}((X,\mathcal{O}),\mathrm{Spec}(A)) \simeq \mathrm{Hom}_{k-alg}(A,\Gamma(X,\mathcal{O}))$. We define a noncommutative version of the Čech cohomology of an affine cover and show that the Čech cohomology can be used to compute higher direct images. This fact is applied here to compute cohomology of invertible sheaves on skew projective spaces and in [LR3] to study D-modules on quantum flag varieties.

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Introduction

The following fact might be regarded as the point of departure for this work:

THEOREM. *Any scheme can be reconstructed uniquely up to isomorphism from the category of quasi-coherent sheaves on this scheme.*

This theorem was proved for noetherian schemes by P. Gabriel in the early sixties (cf. [Gab], Ch.VI) and in the full generality only recently [R2] (for the reader's convenience, we sketch the reconstruction procedure of [R2] in the Appendix to this paper). The possibility to replace schemes by categories of quasi-coherent sheaves on them is an important fact of commutative algebraic geometry. But for noncommutative algebraic geometry it is a source of existence. Apparently, Yu. I. Manin was the first one who proposed to use the identifying spaces with categories of structure sheaves on them as a 'right' way to introduce objects of noncommutative algebraic geometry – 'noncommutative spaces' (cf. [M1, p. 83]). In particular, the projective spectrum of a noncommutative $\mathbb{Z}_+$-graded ring can be defined by imitating the Serre's description of the category of quasi-coherent sheaves on projective schemes ([A], [AZ], [M1], [V1], [R], [LR2]). But, in spite of the growing interest in noncommutative algebraic geometry, an adequate analog of the most important notion of commutative algebraic geometry – the notion of a *scheme* – had not been found (see [M2], p. 7).

One of the purposes of this work is to show that there is, in a sense, only one notion of a noncommutative scheme. Most of the constructions, assertions and examples of Sections 1–5, and Section C4 are essential to see this fact. A key tool here is the Barr–Beck theorem (cf. [ML], Ch. VI, 7). Recall that the 'descent'–type theorems in algebraic geometry are its simple consequences (cf. [Dl], pp. 132–134). The Barr–Beck's theorem garanties that the 'space' for which the covers and standard complexes make sense is, locally, modules over a monad. And then different exactness properties of this monad (right exact, having a right adjoint) distinguish respectively quasi-schemes (= the widest class of 'spaces' for which Čech complex computes derived functors) and schemes = the largest class of 'spaces' for which (locally) bimodules make sense (this last point should become clear in the process of reading Section C4).

The paper is organized as follows.

In Section 1 we introduce a 'geometrical' language: continuous morphisms, flat, coflat, and Zariski covers and associated cosimplicial complexes, the standard complex of a functor depending on a cover. We show that the standard complex of an exact functor is exact.

In the second section we prove that, for any coflat finite cover of an abelian category and for any adapted to this cover ('locally exact') functor, the corresponding standard complex is a resolution of the functor.

In Section 3 we consider Zariski covers and show that if the cover is 'semiseparated' (semiseparated affine covers are available on semiseparated schemes), the standard complex is homotopically equivalent to the Čech complex of the cover.

In Section 4 we define the category of *quasi-schemes* and *schemes* over a given category. Relative quasi-schemes are locally cohomologically trivial morphisms. More explicitly, quasi-schemes are defined as morphisms with locally exact direct image. They are the most natural class of 'spaces' to introduce after learning first properties of the standard complex of a cover. *Schemes* are defined as morphisms direct image of which have locally a right adjoint. Surprisingly, this general nonsense definition gives what one would like to expect of schemes. For instance, schemes over a commutative ring k (i.e. the base category is the category of k-modules) are locally categories of modules over k-algebras. The category of affine k-schemes is equivalent to the category dual to the category of k-algebras. And morphisms from an arbitrary k-scheme to an affine k-scheme are in bijective correspondence with morphisms of k-algebras of their global sections. Note by passing that Drinfeld's 'quantum spaces' over a commutative ring k [Dr] are nothing else but affine schemes over k.

In Section 5 we introduce noncommutative projective spectra and their cones and consider two important examples: skew projective spaces and quantized flag varieties.

In the second part of the paper, Complementary Facts and Examples, we study

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– Connections between some properties of flat covers and those of associated Zariski covers. Compatibility of standard complexes with certain localizations. Resolutions related to infinite covers.

– Standard resolutions of functors and, more specifically, resolutions of 'invertible sheaves'. As an example, we compute cohomology of invertible sheaves on a skew projective space getting direct analogs of the classical results [S] and their consequences including the Serre duality. In [LR3], the standard complex is used for studying cohomologies of invertible sheaves on quantized flag varieties.

In the appendix we recall what is the spectrum of an abelian category (introduced in [R1]) and explain how to reconstruct an arbitrary scheme from the category of quasi-coherent sheaves on the scheme.

1. Covers and associated standard complexes

Categories here are thought as categories of quasi-coherent sheaves on 'spaces' and are identified with the 'spaces'. Accordingly some of the functors could be upgraded to morphisms.

MORPHISMS 1.0. We define *a morphism* f from a category $\mathcal{A}$ to a category $\mathcal{B}$ as an isomorphness class of right exact functors from $\mathcal{B}$ to $\mathcal{A}$. Any functor $\mathcal{B} \to \mathcal{A}$ from f will be called *an inverse image functor of* f. And once we made a choice of an inverse image functor, we shall denote it by f^*. The composition of morphisms is natural: $f \circ g = [g^* \circ f^*]$. Here $[u]$ means 'all functors isomorphic to u'. Allowing only categories which are equivalent to 'small categories' with respect to some universum, we define this way a category which shall be denoted by $\mathcal{R}Cat$.

A morphism f is *continuous* if its inverse image functor f^* has a right adjoint called a *direct image functor of* f and denoted usually by f_*. We call a morphism f *flat* if it is continuous and its inverse image functor is exact. We call a continuous morphism f *coflat* if its direct image functor is exact. Finally, we call f *biflat* if it is flat and coflat.

A morphism $f\colon \mathcal{B} \to \mathcal{A}$ such that f^* is a localization (i.e. a universal functor making invertible all arrows of $\Sigma_f := \{s \in \mathrm{Hom} | f^*(s) \text{ is invertible}\}$) will be called by abuse of language a localization. We call a localization f a *flat localization* if f is a flat morphism. This means that f is a flat morphism having a fully faithful direct image functor (cf. [GZ], Proposition I.1.3).

COVERS 1.1. We call a set of flat morphisms $\{f_i\colon \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ a *flat cover of* $\mathcal{A}$ if any morphism s of $\mathcal{A}$ such that $f_i^*(s)$ is invertible for all $i \in J$ is invertible.

We call a flat cover $\{f_i\colon \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ a *Zariski cover* if each of f_i is a localization.

EXAMPLE 1.1.1. Let $\mathcal{A}$ be an abelian category. And let $\{\mathbb{S}_i | i \in J\}$ be a family of localizing subcategories of $\mathcal{A}$. Recall that a subcategory $\mathbb{S}$ is called *localizing* if

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it is thick and the localization $\mathcal{A} \to \mathcal{A}/\mathbb{S}$ at $\mathbb{S}$ has a right adjoint. Being an exact functor, a localization at $\mathbb{S}_i$ might be regarded as inverse image functor of a flat morphism $f_i: \mathcal{A}/\mathbb{S}_i \to \mathcal{A}$. The family $\{f_i | i \in J\}$ is a cover iff $\bigcap_{i\in J} \mathbb{S}_i = \mathbf{0}$. And any Zariski cover of $\mathcal{A}$ is of this form. ∎

1.2. *The standard cosimplicial resolution of a continuous morphism.* Fix a continuous morphism $f: \mathcal{B} \to \mathcal{A}$ with the inverse image functor f^* and a direct image functor f_*. Let η: $\mathrm{Id}_{\mathcal{A}} \to f_*f^*$ and $\epsilon: f^*f_* \to \mathrm{Id}_{\mathcal{B}}$ be adjunction arrows. Set $\mathfrak{G}_f := f_*f^*$ and $\mu := f_*\epsilon f^* : \mathfrak{G}_f^2 \to \mathfrak{G}_f$. *The standard cosimplicial resolution* $\mathfrak{CR}(f)$ *of the morphism* f is the standard cosimplicial resolution of the pair of adjoint functors (f^*, f_*); i.e. $\mathfrak{CR}(f)$ is the augmented cosimplicial object in $\mathrm{End}\mathcal{A}$ defined by

$$d_n^i = \mathfrak{G}_f^i \eta \mathfrak{G}_f^{n-i} : \mathfrak{G}_f^n \longrightarrow \mathfrak{G}_f^{n+1}, \quad s_n^i = \mathfrak{G}_f^i \mu \mathfrak{G}_f^{n-1-i} : \mathfrak{G}_f^{n+1} \longrightarrow \mathfrak{G}_f^n$$

with the augmentation morphism $\eta : \mathrm{Id}_{\mathcal{A}} \to \mathfrak{G}_f$.

1.3. *The standard cosimplicial resolution of a family of continuous morphisms.* Fix a family $\mathbf{f} = \{f_i: \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ of continuous morphisms. For each $i \in J$, denote by $\mathfrak{G}_i$ the composition $f_{i*} \circ f_i^*$ and by resp. η_i and ϵ_i adjunction arrows $\mathrm{Id}_{\mathcal{A}} \to \mathfrak{G}_i$ and $f_i^* f_{i*} \to \mathrm{Id}_{\mathcal{B}}$. For any positive integer n, let J^n denote the direct product of n copies of J. To this data, there corresponds a cosimplicial object

$$\mathfrak{C}(\mathfrak{F}) = \left(\mathrm{Id}_{\mathcal{A}} \xrightarrow{(\eta_i)} \prod_{\mathbf{i}\in J} \mathfrak{G}_i \underset{\mathfrak{G}_i\eta_j}{\overset{\eta_i\mathfrak{G}_j}{\rightrightarrows}} \prod_{\mathbf{i}\in J^2} \mathfrak{G}_{\mathbf{i}} \underset{}{\overset{}{\rightrightarrows\!\!\rightarrow}} \prod_{\mathbf{i}\in J^3} \mathfrak{G}_{\mathbf{i}} \ldots \right), \tag{1}$$

where, for any $\mathbf{i} = (i_1, \ldots, i_n) \in J^n$, $\mathfrak{G}_{\mathbf{i}} := \mathfrak{G}_{i_1} \circ \ldots \circ \mathfrak{G}_{i_n}$. We assume that all products in the diagram (1) exist.

Let now F be a functor from $\mathcal{A}$ to an additive category $\mathcal{B}$. And let

$$\mathfrak{G}(\mathfrak{F}, F) = \left(F \xrightarrow{(F\eta_i)} \prod_{i\in J} F \circ \mathfrak{G}_i \longrightarrow \prod_{\mathbf{i}\in J^2} F \circ \mathfrak{G}_{\mathbf{i}} \longrightarrow \prod_{\mathbf{i}\in J^3} F \circ \mathfrak{G}_{\mathbf{i}} \ldots \right) \tag{2}$$

be a cochain complex associated to the image $F \circ \mathfrak{C}(\mathfrak{F})$ of the cosimplicial object (1). We call $\mathfrak{C}(\mathfrak{F}, F)$ *the augmented standard complex of the functor* F *associated to the cover* $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$. The *standard complex of* F *with respect to* $\mathfrak{F}$ is the chain complex

$$\mathfrak{C}_+(\mathfrak{F}, F) = \left(\prod_{i\in J} F \circ \mathfrak{G}_i \longrightarrow \prod_{\mathbf{i}\in J^2} F \circ \mathfrak{G}_{\mathbf{i}} \longrightarrow \prod_{\mathbf{i}\in J^3} F \circ \mathfrak{G}_{\mathbf{i}} \ldots \right) \tag{3}$$

PROPOSITION 1.4. *Let* $\mathfrak{F} = \{f_i: \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ *be a finite flat cover of an abelian category* $\mathcal{A}$*. Then, for any exact additive functor* $F: \mathcal{A} \to \mathcal{B}$*, the standard complex* $\mathfrak{C}(\mathfrak{F}, F)$ *is exact.*

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Proof. Since $\mathfrak{C}(\mathfrak{F}, F) = F \circ \mathfrak{C}(\mathfrak{F}, \mathrm{Id}_{\mathcal{A}})$, it suffices to prove the assertion in the case $F = \mathrm{Id}_{\mathcal{A}}$.

(a) Suppose that card(J) = 1; i.e. the cover $\mathfrak{F}$ consists of one morphism f. The complex $f^* \circ \mathfrak{C}(f, \mathrm{Id}_{\mathcal{A}})$ is homotopically trivial, hence it is exact. This fact is in [Go], Appendix, Section 5. Since $\mathfrak{F} = \{f\}$ is a flat cover, the inverse image functor f^* is faithfully flat. Therefore the exactness of $f^* \circ \mathfrak{C}(f, \mathrm{Id}_{\mathcal{A}})$ implies the exactness of $\mathfrak{C}(f, \mathrm{Id}_{\mathcal{A}})$.

(b) Fix a family $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ of continuous morphisms. The family $\mathfrak{F}$ can be encoded in one morphism $\mathbf{f} : \prod_{i\in J} \mathcal{B}_i \to \mathcal{A}$ having the inverse image functor

$$\mathbf{f}^*: \mathcal{A} \longrightarrow \prod_{i\in J} \mathcal{B}_i, \quad X \longmapsto \prod_{i\in J} f_i^*(X). \tag{1}$$

The morphism $\mathbf{f}$ has a direct image functor: $\mathbf{f}_*(\prod_{i\in J} X_i) = \prod_{i\in J} f_{i*}(X_i)$. The adjunction arrow

$$\eta = \eta_{\mathbf{f}} : \mathrm{Id}_{\mathcal{A}} \longrightarrow \mathbf{f}_* \circ \mathbf{f}^* = \prod_{i\in J} f_{i*} \circ f_i^*$$

is determined by the adjunction arrows $\eta_i : \mathrm{Id}_{\mathcal{A}} \to \mathfrak{G}_i = f_{i*} \circ f_i^*$, $i \in$J. The adjunction arrow

$$\epsilon = \epsilon_{\mathbf{f}} : \mathbf{f}^* \circ \mathbf{f}_* \longrightarrow \mathrm{Id}_{\mathcal{B}_J},$$

where $\mathcal{B}_J := \prod_{i\in J} \mathcal{B}_i$, assigns to any $(X_i) \in Ob\mathcal{B}_J$ the composition of the projection

$$\mathbf{f}^* \circ \mathbf{f}_*(X_i) \longrightarrow (f_i^* \circ f_{i*}(X_i))$$

and the product $(\epsilon_i : f_i^* \circ f_{i*}(X_i) \to X_i)$ of adjunction morphisms ϵ_i. Note that

- The complex $\mathfrak{C}(\mathfrak{F}, \mathrm{Id}_{\mathcal{A}})$ of the family $\mathfrak{F}$ coincides with $\mathfrak{F}(\mathbf{f}, \mathrm{Id}_{\mathcal{A}})$.
- The family $\mathfrak{F}$ is a flat cover iff $\{\mathbf{f}\}$ is a flat cover, i.e. $\mathbf{f}^*$ is a faithfully flat functor.

Thus the assertion in the general case follows from (a). □

2. The standard complex of a cover and a resolution of locally exact functors

2.1. *Locally exact functors.* Fix a category $\mathcal{A}$ and a flat cover $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$. For any functor $F: \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is an additive category with products of $card(J)$ objects we have the chain complex $\mathfrak{C}(\mathfrak{F}, F)$. Therefore we have cohomology of F associated to the cover $\mathfrak{F}$. Suppose on the other hand that $\mathcal{A}$ is an abelian category with enough injectives. So that one can talk about derived functors $\mathfrak{R}^*F$ of the functor F. We are going to produce natural conditions on the

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cover $\mathfrak{F}$ and the functor F which garantee the isomorphism of $\mathfrak{R}^*F$ and the Čech cohomologies, $H^*\mathfrak{C}(\mathfrak{F}, F)$, of the functor F corresponding to the cover $\mathfrak{F}$.

Call a cover $\{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ *biflat* if the morphisms $f_i,\ i \in J$, are *biflat*, i.e. the direct image functors f_{i*} are also exact for all $i \in$ J. The property which we are going to use is that, for any $i \in J$, the composition $f_{i*} \circ f_i^*$ is an exact functor.

Let $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ be a flat cover. We say that a functor $F : \mathcal{A} \to \mathcal{C}$ is *adapted to* the cover $\mathfrak{F}$ if, for any $i \in J$, the composition $F \circ f_{i*}$ is exact.

We call a functor $F : \mathcal{A} \to \mathcal{C}$ *locally exact* if there exists a finite flat cover $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ such that F is adapted to $\mathfrak{F}$.

THEOREM 2.2. *Let $\mathcal{A}$ be an abelian category. And let $\mathfrak{F} = \{f_i: \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ be a finite biflat cover of $\mathcal{A}$. Suppose each category $\mathcal{B}_i$ has enough injectives. And let a functor $F: \mathcal{A} \to \mathcal{C}$ be adapted to $\mathfrak{F}$. Then the standard complex $\mathfrak{C}(\mathfrak{F}, F)$ of the functor F with respect to the cover $\mathfrak{F}$ is a resolution of the functor F.*

Proof. Let $\mathbf{f}^*$ denote the inverse image functor $\oplus_{i\in J} f_i^* : \mathcal{A} \to \oplus_{i\in J}\mathcal{B}_i = \mathcal{B}_J$ associated with the cover $\mathfrak{F}$. And let $\mathbf{f}_*$ denote a right adjoint to $\mathbf{f}^*$ (cf. the part (b) of the Proof of Proposition 1.4). Since the cover $\mathfrak{F}$ is biflat, the functor $\mathbf{G} := \mathbf{f}_* \circ \mathbf{f}^*$ is exact. This implies that the standard complex $\mathfrak{C}(\mathfrak{F}, \mathrm{Id}_{\mathcal{A}})$ of the cover $\mathfrak{F}$ provides a resolution

$$\mathrm{Id}_{\mathcal{A}} \longrightarrow \mathfrak{C}(\mathfrak{F}) = (\mathbf{G} \longrightarrow \mathbf{G}^2 \longrightarrow \cdots \longrightarrow \mathbf{G}^n \longrightarrow \cdots)$$

of the identical functor. To show that the standard complex $\mathfrak{C}(\mathfrak{F}, F) = F \circ \mathfrak{G}(\mathfrak{F})$ is a resolution of the functor F, it suffices to check that, for any $X \in Ob\mathcal{A}$, the object $\mathbf{G}^n(X)$ is F-acyclic (i.e. $R^pF(\mathbf{G}^n(X)) = 0$ if $p > 0$) for all $n \geqslant 1$. (cf. [Gr] Proposition 2.5.1 in and the following example).

Note that, since the functor $\mathbf{f}^*$ is exact, the functor $\mathbf{f}_*$ sends injectives into injectives. And since each category $\mathcal{B}_i$ has enough injectives, the product of the categories $\mathcal{B}_i$ has enough injectives too. Let $\mathfrak{I}(X)$ be an injective resolution of $\mathbf{f}^* \circ \mathbf{G}^n(X),\ n \geqslant 0$. Then, since the functor $\mathbf{f}_*$ is exact and sends injectives into injectives, $\mathbf{f}_*(\mathfrak{I}(X))$ is an injective resolution of $\mathbf{f}_* \circ \mathbf{f}^* \circ \mathbf{G}^n(X) = \mathbf{G}^{n+1}(X)$. Since the functor $F \circ \mathbf{f}_*$ is exact, the cohomology of the complex $F(\mathbf{f}_*(\mathfrak{I}(X))$ are zero in degrees $\geqslant 1$. This proves that the objects $\mathbf{G}^n(X)$ are F-acyclic for any $X \in Ob\mathcal{A}$ and all $n \geqslant 1$. □

3. Zariski covers

3.1. *First cancellations.* Let $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ be a Zariski cover; i.e. all inverse image functors f_i^* are flat localizations. This implies that the functors $\mathfrak{G}_i := f_{i*}f_i^*$ are idempotent. More explicitly, the morphisms $\mathfrak{G}_i\eta_i$ and $\eta_i\mathfrak{G}_i$ coincide and are isomorphisms for all $i \in J$. The latter allows to replace the standard cosimplicial complex $\mathfrak{C}(\mathfrak{F})$ of the cover $\mathfrak{F}$ by a more economic expression. Namely denote by J_n the subset of all elements $(i_1, \ldots, i_n)$ of $J^{\times n}$ (= the product of n

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copies of J) such that $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$. The complex $\mathfrak{C}(\mathfrak{F})$ is homotopically equivalent to the complex

$$\mathfrak{C}'(\mathfrak{F}) = \left(\mathrm{Id}_{\mathcal{A}} \xrightarrow{(\eta_i)} \prod_{i\in J} \mathfrak{G}_i \underset{\mathfrak{G}_i\eta_j}{\overset{\eta_i\mathfrak{G}_j}{\rightrightarrows}} \prod_{\mathbf{i}\in J_2} \mathfrak{G}_{\mathbf{i}} \underset{\longrightarrow}{\overset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \prod_{\mathbf{i}\in J_3} \mathfrak{G}_{\mathbf{i}} \dots \right), \tag{1}$$

where, for each $\mathbf{i} = (i_1, \dots, i_n) \in J_n$, $\mathfrak{G}_{\mathbf{i}} := \mathfrak{G}_{i_1} \circ \cdots \circ \mathfrak{G}_{i_n}$.

In fact, the canonical projection $\mathfrak{C}(\mathfrak{F}) \to \mathfrak{C}'(\mathfrak{F})$ is invertible in the homotopical category.

3.2. *Semiseparated Zariski covers.* Call a Zariski cover $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ *semiseparated* if $\mathfrak{G}_i \circ \mathfrak{G}_j \simeq \mathfrak{G}_j \circ \mathfrak{G}_i$ for all $i, j \in J$. Fix an order in J. Then $\mathfrak{C}(\mathfrak{F})$ is homotopically equivalent to the complex

$$C\mathfrak{C}(\mathfrak{F}) = \left(\mathrm{Id}_{\mathcal{A}} \xrightarrow{(\eta_i)} \prod_{i\in J} \mathfrak{G}_i \underset{\mathfrak{G}_i\eta_j}{\overset{\eta_i\mathfrak{G}_j}{\rightrightarrows}} \prod_{\mathbf{i}\in J_{2<}} \mathfrak{G}_{\mathbf{i}} \underset{\longrightarrow}{\overset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \prod_{\mathbf{i}\in J_{3<}} \mathfrak{G}_{\mathbf{i}} \dots \right), \tag{1}$$

where $J_{n<} := \{(i_1, \dots, i_n) \in J^{\times n} | i_1 < i_2 < \cdots < i_n\}$. The equivalence is given by the projection $\mathfrak{C}(\mathfrak{G}) \to C\mathfrak{G}(\mathfrak{F})$.

Moreover, if f_i^* is a localization at the class of arrows S_i, $i \in J$, then, for any $\mathbf{i} = (i_1, \dots, i_n) \in J^{\times n}$, $\mathfrak{G}_{\mathbf{i}} = f_{\mathbf{i}*} f_{\mathbf{i}}^*$, where $f_{\mathbf{i}}^*$ is a localization at (the saturation of) $\bigcup_{1\leqslant k\leqslant n} S_{i_k}$.

3.3. *Example: the standard complex of a cover of a scheme.* Let $\mathbf{X} = (X, \mathcal{O})$ be an arbitrary quasi-compact scheme. For any affine cover $\mathfrak{U}$ of $\mathbf{X}$, we have endofunctors $\{\mathfrak{G}_U = f_{U*} f_U^* | U \in \mathfrak{U}\}$ of the category $Q\mathrm{coh}_{\mathbf{X}}$ of quasi-coherent sheaves on $\mathbf{X}$. Note that $\mathfrak{G}_U \mathfrak{G}_{U'} \simeq \mathfrak{G}_{U'} \mathfrak{G}_U$ implies that $\mathfrak{G}_{U'} \mathfrak{G}_U$ is isomorphic to $\mathfrak{G}_{U\cap U'}$. Thus we have the following assertion:

PROPOSITION 3.3.1. *Let $\mathfrak{U}$ be any affine open semiseparated cover of a scheme $\mathbf{X}$; i.e. $\mathfrak{G}_{U'} \circ \mathfrak{G}_U \simeq \mathfrak{G}_U \circ \mathfrak{G}_{U'}$ for all $U, U' \in \mathfrak{U}$. Then the standard cosimplicial complex $\mathfrak{C}(\mathfrak{U})$ of the cover $\mathfrak{U}$ is equivalent to the complex*

$$C\mathfrak{C}(\mathfrak{F}) = \left(\mathrm{Id}_{\mathcal{A}} \xrightarrow{(\eta_i)} \prod_{i\in J} \mathfrak{G}_i \underset{\mathfrak{G}_i\eta_j}{\overset{\eta_i\mathfrak{G}_j}{\rightrightarrows}} \prod_{\mathbf{i}\in J_{2<}} \mathfrak{G}_{\mathbf{i}} \underset{\longrightarrow}{\overset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \prod_{\mathbf{i}\in J_{3<}} \mathfrak{G}_{\mathbf{i}} \dots \right), \tag{1}$$

where, for any $\mathbf{i} = (i_1, \dots, i_n)$, $\mathfrak{G}_{\mathbf{i}} := \mathfrak{G}_{U_{\mathbf{i}}}$ and $U_{\mathbf{i}} := \bigcap_{1\leqslant k\leqslant n} U_{i_k}$. In particular, for any additive functor $F : Q\mathrm{coh}_{\mathbf{X}} \to \mathcal{C}$, the standard chain complex $\mathfrak{C}(\mathfrak{U}, F)$ is homotopically equivalent to the Čech complex $C(\mathfrak{U}, F)$.

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Remark 3.3.2. Let $f : \mathbf{X} \to \mathbf{S}$ be a scheme morphism having a direct image functor f_* (for instance, f_* is the global section functor). Since $\mathbf{X}$ is quasi-compact, there exists a finite affine cover $\mathfrak{U}$ of $\mathbf{X}$ such that $f\mid_U$ is an affine morphism for any $U \in \mathfrak{U}$. Then the standard complex $\mathfrak{C}(\mathfrak{U}, f_*)$ corresponding to the cover $\mathfrak{U}$ is a resolution of the functor f_*. Therefore it can be used for computing higher direct images (= derived functors) of f_*.

If the localizations at different open sets of the cover $\mathfrak{U}$ commute (i.e. $\mathfrak{G}_{U'}\mathfrak{G}_U \simeq \mathfrak{G}_U\mathfrak{G}_{U'}$ for all $U, U' \in \mathfrak{U}$), the complex $\mathfrak{C}(\mathfrak{U}, f_*)$ is homotopically equivalent to the Čech complex, $C(\mathfrak{U}, f_*)$ of the cover $\mathfrak{U}$. One can show that the following conditions are equivalent:

(a) For any affine cover $\mathfrak{U}$ of a scheme $\mathbf{X}$, $\mathfrak{G}_{U'}\mathfrak{G}_U \simeq \mathfrak{G}_U\mathfrak{G}_{U'}$ for all $U, U' \in \mathfrak{U}$.
(b) The scheme $\mathbf{X}$ is separated.

In other words, the Čech complex is equivalent to the standard complex for any affine cover only if the scheme is separated. If the scheme $\mathbf{X}$ is not separated, the higher cohomology of the Čech complex $C(\mathfrak{U}, f_*)$ are not isomorphic, for a general affine cover $\mathfrak{U}$, to the corresponding derived functors of f_*. □

4. Quasi-schemes and schemes

4.1. *Relative quasi-schemes.* We call a continuous morphism $f\colon \mathcal{A} \to \mathcal{C}$ *almost affine* if f_* is an exact and faithful functor. A flat cover $\mathfrak{F} = \{f_i\colon \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ is called *almost affine* if each morphism f_i is almost affine.

We call a continuous morphism $f\colon \mathcal{A} \to \mathcal{C}$ a *quasi-scheme over* $\mathcal{C}$ if there exists an almost affine Zariski cover $\mathfrak{F} = \{u_i\colon \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ such that the direct image $f_* \circ u_{i*}$ of $f \circ u_i$ is exact and faithful (i.e. $f \circ u_i$ is almost affine) for all $i \in J$.

With any continuous morphism $f : \mathcal{A} \to \mathcal{C}$, we associate a monad $\mathbb{G}_f = (\mathfrak{G}_f, \mu)$ and a canonical functor $\mathfrak{f}_* : \mathcal{A} \to \mathbb{G}_f\text{–mod}$ such that f_* is the composition of $\mathfrak{f}_*$ and the forgetful functor $\mathbb{G}_f\text{–mod} \to \mathcal{C}$. Here $\mathfrak{G}_f := f_* \circ f^*$ and $\mu = f_*\epsilon f^*$; ϵ is an adjunction morphism $f^* \circ f_* \to \mathrm{Id}_{\mathcal{A}}$. The canonical functor $\mathfrak{f}_*$ assigns to any object M of $\mathcal{A}$ the $\mathbb{G}_f$-module $(f_*(M), f_*\epsilon)$.

If $f\colon \mathcal{A} \to \mathcal{C}$ is an almost affine quasi-scheme, it follows from the the Barr–Beck theorem (cf. [ML]) that $\mathfrak{f}_* : \mathcal{A} \to \mathbb{G}_f\text{–mod}$ is an equivalence of categories.

Note that, since f_* exact, the functor $\mathfrak{G}_f$ is right exact. Therefore, if $\mathcal{C}$ is an abelian category, the category $\mathbb{G}_f\text{–mod}$ of $\mathbb{G}_f$-modules is abelian too. Thus if $f : \mathcal{A} \to \mathcal{C}$ is almost affine and $\mathcal{C}$ is an abelian category, then $\mathcal{A}$ is abelian.

It follows that an arbitrary quasi-scheme is *locally* a category of modules over a right exact monad. This also implies that if $f : \mathcal{A} \to \mathcal{C}$ is a quasi-scheme and the category $\mathcal{C}$ is abelian, then $\mathcal{A}$ is abelian.

We call a continuous morphism $f : \mathcal{A} \to \mathcal{C}$ a *relative semiseparated quasi-scheme* if there exists a semiseparated biflat Zariski cover $\mathfrak{F} = \{u_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ adapted to f and such that $f_* \circ u_{i*}$ is faithful for all $i \in J$.

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Any almost affine morphism f is a relative semiseparated quasi-scheme, since f is adapted to the trivial cover $\{\mathrm{Id}_{\mathcal{A}}\}$.

We shall denote by $Cat_*/\mathcal{C}$ the full subcategory of $\mathcal{R}Cat/\mathcal{C}$ objects of which are continuous morphisms $\mathcal{A} \to \mathcal{C}$. We denote by **QSch**$/\mathcal{C}$ the *category of quasi-schemes over* $\mathcal{C}$ which is the full subcategory of $Cat_*/\mathcal{C}$ formed by quasi-schemes over $\mathcal{C}$. We single out the full subcategory **QSch**$_c/\mathcal{C}$ of *quasi-compact* quasi-schemes. *Quasi-compact* means that an almost affine cover in the definition of a quasi-scheme can be chosen finite. By 'technical reasons' (to avoid overloading with technicalities) and also because the known interesting examples of quasi-schemes are quasi-compact, we shall discuss mostly the category **QSch**$_c/\mathcal{C}$.

4.2. *Morphisms of quasi-schemes.* The main theorem about scheme morphisms says that, if $\mathbf{X} = (X, \mathcal{O}_X)$ is an arbitrary scheme and $\mathbf{Y} = (Y, \mathcal{O}_Y)$ is an affine scheme, then there is a natural isomorphism

$$\mathbf{Schemes}(\mathbf{X}, \mathbf{Y}) \longrightarrow \mathbf{Rings}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)). \tag{1}$$

The goal of this section is to establish an analog of this fact (actually a generalization) for quasi-schemes. To see better the nature of things, we begin with the category $Cat_*/\mathcal{C}$ of continuous morphisms to $\mathcal{C}$, where the desired fact is valid in a most naive form.

Denote by $\mathfrak{Mon}\mathcal{C}$ the category of monads in $\mathcal{C}$. Let $\mathfrak{L}$ denote the functor from $(\mathfrak{Mon}\mathcal{C})^o$ to the category $Cat_*/\mathcal{C}$ of continuous morphisms to $\mathcal{C}$ which assigns to any monad $\mathbb{F}$ the canonical continuous morphism $\mathbb{F} - mod \to \mathcal{C}$.

PROPOSITION 4.2.1 *The functor* $\mathfrak{L} : (\mathfrak{Mon}\mathcal{C})^o \to Cat_*/\mathcal{C}$ *is fully faithful and has a left adjoint.*

Proof. (a) We begin with the construction of a left adjoint functor to $\mathfrak{L}$.

Let h be a morphism from $f\colon \mathcal{A} \to \mathcal{C}$ to $g : \mathcal{B} \to \mathcal{C}$. After chosing g^* and h^*, we can take $f^* = h^* \circ g^*$ as an inverse image morphism of f. This way we have the equality $f_* \circ h^* \circ g^* = f_* \circ f^*$ which together with the composition

$$g_* \longrightarrow f_* \circ f^* \circ g_* = f_* \circ h^* \circ g^* \circ g_* \longrightarrow f_* \circ h^*$$

provides a morphism $g_* \circ g^* \to f_* \circ f^*$. We leave to a reader to check that this is a monad morphism and that we have defined the required functor. Moreover, there is a natural commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \stackrel{h}{\longrightarrow} & \mathcal{B} \\ {\scriptstyle \zeta_f}\downarrow & & \downarrow{\scriptstyle \zeta_g} \\ \mathbb{G}_f - \mathrm{mod} & \stackrel{\hat{}\mathfrak{L}(h)}{\longrightarrow} & \mathbb{G}_g - \mathrm{mod} \end{array} \tag{1}$$

Here the vertical arrows are canonical morphisms with direct image functors $\zeta_{f*} : X \longmapsto (f_*(X), f_*\epsilon(X))$, where ϵ is an adjunction arrow $f^*f_* \to \mathrm{Id}_{\mathcal{A}}$, and

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similarly ζ_{g*}; $\hat{}\mathfrak{L}(h)$ is the pull-back morphism determined by the monad morphism defined above. Note that (1) is a diagram of morphisms over $\mathcal{C}$. Clearly (ζ_f) is an adjunction morphism $\mathrm{Id}_{Cat_*/\mathcal{C}} \to \mathfrak{L} \circ \hat{}\mathfrak{L}$. The second adjunction morphism is identical. The latter fact implies that the functor $\mathfrak{L}$ is fully faithful. □

Let $\mathfrak{Monr}\mathcal{C}$ be the full subcategory of the category $\mathfrak{Mon}\mathcal{C}$ generated by right exact monads, i.e. monads (F, μ) such that F is a right exact functor.

LEMMA 4.2.2. *A monad $\mathbb{F} = (F, \mu)$ belongs to $\mathfrak{Monr}\mathcal{C}$ if and only if the canonical morphism $f : \mathbb{F} - mod \to \mathcal{C}$ is almost affine. In particular, $\mathfrak{L}$ induces a functor, $\mathcal{L}$, from the category $(\mathfrak{Monr}\mathcal{C})^o$ to the category* **QSch**$/\mathcal{C}$ *of quasi-schemes over $\mathcal{C}$.*

Proof. If f is almost affine (i.e. f_* is exact), then, clearly, $F = f_* \circ f^*$ is right exact. Conversely, suppose that F is right exact. Let $g, h : (M, m) \to (M', m')$ be arbitrary $\mathbb{F}$-module morphisms; and let $\mathfrak{e} : M' \to N$ be a coequalizer of the pair $f_*(g), f_*(h) : M \to M'$. Since $\mathfrak{e} \circ m' \circ Fg = \mathfrak{e} \circ m' \circ Fh$ and, because F is right exact, $F\mathfrak{e}$ is a coequalizer of the pair (Fg, Fh), there exists a unique arrow $\nu : F(N) \to N$ such that $\mathfrak{e} \circ m' = \nu \circ F\mathfrak{e}$. One can check that $\nu \circ F\nu = \nu \circ \mu$ and $\nu \circ F\eta(N) = \mathrm{Id}_N$, i.e. (N, ν) is an $\mathbb{F}$-module. The equality $\mathfrak{e} \circ m' = \nu \circ F\mathfrak{e}$ means that $\mathfrak{e}$ is an $\mathbb{F}$-module morphism $(M', m') \to (N, \nu)$. Clearly the module (N, ν) is an equalizer of the pair (g, h). This shows that the direct image functor f_* is right exact. Therefore it is exact. □

PROPOSITION 4.2.3. *The functor $\mathcal{L} : (\mathfrak{Monr}\mathcal{C})^o \to$* **QSch**$_c/\mathcal{C}$ *is fully faithful and has a left adjoint.*

Proof. Let $f : \mathcal{A} \to \mathcal{C}$ be a quasi-scheme; and let $\mathfrak{U} = \{u_i : \mathfrak{U}_i \to \mathcal{A} | i \in J\}$ be a coflat Zariski cover such that, for any $i \in J$, $f_* \circ u_{i*}$ is exact and faithful (i.e. $f \circ u_i : \mathfrak{U}_i \to \mathcal{C}$ is almost affine). Let **u** denote the morphism $\coprod_{i \in J} \mathfrak{U}_i \to \mathcal{A}$ corresponding to the cover $\mathfrak{U}$. Note that $f_* \circ \mathfrak{C}(\mathfrak{U}) \circ f^*$ is a complex in the category $Endr(\mathcal{C})$ of right exact functors $\mathcal{C} \to \mathcal{C}$. It follows from Proposition C4.3 that the functor

$$\mathfrak{G}'_f := H^0(f_* \circ \mathfrak{C}(\mathfrak{U}) \circ f^*) := \mathrm{Ker}(f_* \circ (\mathfrak{G}_{\mathbf{u}} \to \mathfrak{G}^2_{\mathbf{u}}) \circ f^*) : \mathcal{C} \to \mathcal{C}, \qquad (1)$$

where the kernel is taken in the category $Endr\mathcal{C}$, does not depend on the choice of the cover $\mathfrak{U}$. And $\mathfrak{G}'_f$ has a uniquely defined monad structure μ'. The morphism $f : \mathcal{A} \to \mathcal{C}$ decomposes uniquely into a continuous morphism $\zeta'_f : \mathcal{A} \to \mathbb{G}'_f -$mod and $\mathbb{G}_f -$ mod $\to \mathcal{C}$. Here $\mathbb{G}'_f := (\mathfrak{G}'_f, \mu')$. The map assigning to any quasi-scheme $f : \mathcal{A} \to \mathcal{C}$ the right exact monad $\mathbb{G}_f$ extends naturally to a functor which is a left adjoint to the functor $\mathcal{L}$. And $\zeta' = (\zeta'_f)$ is the adjunction arrow from $\mathrm{Id}_{\mathbf{QSch}/\mathcal{C}}$ to $\mathcal{L} \circ \hat{}\mathcal{L}$. The other adjunction arrow is identical. □

PROPOSITION 4.2.4. *The functor $\mathcal{L} : (\mathfrak{Monr}\mathcal{C})^o \to$* **QSch**$/\mathcal{C}$ *establishes an equivalence between the category $(\mathfrak{Monr}\mathcal{C})^o$ dual to the category of right exact monads and the category* **QAff**$/\mathcal{C}$ *of almost affine quasi-schemes over $\mathcal{C}$.*

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Proof. The assertion follows from Proposition 4.2.3 and the fact that any almost affine quasi-scheme $f\colon \mathcal{A} \to \mathcal{C}$ satisfies the conditions of the Barr–Beck theorem; hence the canonical morphism $\mathcal{A} \to \mathbb{G}_f - \mathrm{mod}$ is an equivalence (cf. the discussion in Section 4.1). □

4.3. *Relative schemes.* We call a continuous morphism $f\colon \mathcal{A} \to \mathcal{C}$ *affine* if its direct image functor f_* is faithful and has a right adjoint. Usually, we shall denote the right adjoint to f_* by $f_\dagger$.

A family of morphisms (in particular a cover) $\mathfrak{F} = \{u_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ will be called *affine* if each u_i is affine.

A continuous morphism $f : \mathcal{A} \to \mathcal{C}$ shall be called *a scheme over* $\mathcal{C}$ if there exists an *affine* Zariski cover $\mathfrak{U} = \{u_i\colon \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ such that $f \circ u_i$ is an affine morphism for all $i \in J$.

Clearly any scheme over $\mathcal{C}$ is a quasi-scheme over $\mathcal{C}$.

Any scheme morphism $\mathbf{X} \to \mathbf{Y}$ having a direct image functor defines a relative scheme $Q\mathrm{coh}_{\mathbf{X}} \to Q\mathrm{coh}_{\mathbf{Y}}$ in the sense of the definition above.

Denote by $\mathbf{Sch}/\mathcal{C}$ the full subcategory of $Cat_*/\mathcal{C}$ objects of which are schemes over $\mathcal{C}$. We single out the full subcategory $\mathbf{Sch}_c/\mathcal{C}$ of *quasi-compact* schemes, i.e. schemes $f\colon \mathcal{A} \to \mathcal{C}$ which have a finite Zariski cover $\mathfrak{U} = \{u_i\mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ such that $f \circ u_i$ is an affine morphism for all $i \in J$.

Finally, we denote by $\mathbf{Aff}/\mathcal{C}$ the full subcategory of $\mathbf{Sch}/\mathcal{C}$ objects of which are affine schemes over $\mathcal{C}$.

4.4. *The main theorem on scheme morphisms.* Denote by $\mathfrak{Mon}_c\mathcal{C}$ the full subcategory of $\mathfrak{Mon}\mathcal{C}$ objects of which are *continuous monads in* $\mathcal{C}$, i.e. monads $\mathbb{F} = (F, \mu)$ such that the functor F has a right adjoint. Clearly every continuous monad is right exact: $\mathfrak{Mon}_c\mathcal{C} \subseteq \mathfrak{Mon}_r\mathcal{C}$.

LEMMA 4.4.1. *A monad $\mathbb{F} = (F, \mu)$ in $\mathcal{C}$ is continuous if and only if the canonical morphism $f\colon \mathbb{F} - mod \to \mathcal{C}$ is affine.*

Proof. (a) One direction is trivial: for any affine morphism g the functor $g_* \circ g^*$ has a right adjoint by definition; and for the canonical morphism $f\colon \mathbb{F}-$ mod $\to \mathcal{C}$, $f_* \circ f^*$ = F.

(b) Suppose that $\mathbb{F} = (F, \mu)$ is continuous. Let $F\hat{}$ be a right adjoint to F; and let $\epsilon' : F \circ F\hat{} \to \mathrm{Id}_{\mathcal{C}}$, $\eta' : \mathrm{Id}_{\mathcal{C}} \to F\hat{} \circ F$ be adjunction arrows. Then

$$\nu := F\hat{}(\epsilon' \circ \mu F\hat{}) \circ \eta' F \circ F\hat{} : F \circ F\hat{} \to F\hat{} \tag{1}$$

is an action of F on $F\hat{}$ which satisfies the properties:

$$\nu \circ \eta F\hat{} = \mathrm{Id}_{F\hat{}} \quad \text{and} \quad \nu \circ \mu F\hat{} = \nu \circ F\nu. \tag{2}$$

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Here $\eta : \mathrm{Id}_{\mathcal{C}} \to F$ is the identity of the monad $\mathbb{F}$. In fact,

$$\begin{aligned}\nu \circ \eta F\hat{} &:= F\hat{}(\epsilon' \circ \mu F\hat{}) \circ \eta' F \circ F\hat{} \circ \eta F\hat{} = F\hat{}\epsilon' \circ (F\hat{}\mu \circ \eta' F \circ \eta)F\hat{} \\ &= F\hat{}\epsilon' \circ (F\hat{}\mu \circ F\hat{}F\eta \circ \eta')F\hat{} = F\hat{}\epsilon' \circ \eta' F\hat{} = id_{F\hat{}}.\end{aligned}$$

We leave the checking the associativity of the action to the reader.

The relations (2) imply that $(F\hat{}(M), \nu(M))$ is an $\mathbb{F}$-module for any $M \in Ob\mathcal{C}$. Clearly the map $f_\dagger$ assigning to any $M \in Ob\mathcal{C}$ the module $(F\hat{}(M), \nu(M))$ and to any arrow f of $\mathcal{C}$ the morphism $F\hat{}f$ of the corresponding modules is a functor. The compositions of the forgetful functor f_* and $f_\dagger$ are:

$$f_* \circ f_\dagger = F\hat{}, \quad f_\dagger \circ f_* : (M, m) \longmapsto (F\hat{}(M), \nu(M)).$$

There are canonical morphisms

$$\epsilon'' := \epsilon' \circ \eta F\hat{} : f_* \circ f_\dagger = F\hat{} \longrightarrow \mathrm{Id}_{\mathcal{C}} \tag{3}$$

and $\eta'' : \mathrm{Id}_{\mathbb{F}-\mathrm{mod}} \to f_\dagger \circ f_*$ defined by

$$\eta''(M, m) := F\hat{}m \circ \eta'(M) : (M, m) \longrightarrow (F\hat{}(M), \nu(M)). \tag{4}$$

We leave it to the reader to check that (4) is really an $\mathbb{F}$-module morphism and the functor morphisms ϵ'' and η'' are adjunction arrows for f_* and $f_\dagger$.

COROLLARY 4.4.2. *A morphism $f : \mathcal{A} \to \mathcal{C}$ is affine if and only if $\mathbb{G}_f = (f_* \circ f^*, \mu)$ is a continuous monad.*

Proof. **Only if** is trivial.

If: In fact, if f is almost affine, f_* is (cf. Proposition 4.2.4) equivalent to the forgetful functor $\mathfrak{f} : \mathbb{G}_f - \mathrm{mod} \to \mathcal{C}$, $\mathbb{G}_f = (f_* \circ f^*, \mu)$. By Lemma 4.4.1, the existence of a right adjoint to $\mathfrak{f}_*$ is equivalent to the existence of a right adjoint to $f_* \circ f^*$. □

Thus the functor $\mathfrak{L} : (\mathfrak{Mon}\mathcal{C})^o \to Cat_*/\mathcal{C}$ which assigns to any monad $\mathbb{F}$ the canonical morphism $\mathbb{F}- \mathrm{mod} \to \mathcal{C}$ induces a functor $\mathfrak{S}$ from $(\mathfrak{Mon}_c\mathcal{C})^o$ to $\mathbf{Sch}_c/\mathcal{C}$.

PROPOSITION 4.4.3. *The functor $\mathfrak{S} : (\mathfrak{Mon}_c\mathcal{C})^o \to \mathbf{Sch}_c/\mathcal{C}$ is fully faithful and has a left adjoint.*

Proof. Let $f : \mathcal{A} \to \mathcal{C}$ be a scheme; and let $\mathfrak{U} = \{u_i : \mathfrak{G}_i \to \mathcal{A} | i \in J\}$ be a coflat Zariski cover such that, for any $i \in J$, $f_* \circ u_{i*}$ is faithful and has a right adjoint (i.e. $f \circ u_i : \mathfrak{G}_i \to \mathcal{C}$ is affine). Let $\mathbf{u}$ denote the morphism $\prod_{i \in J} \mathfrak{U}_i \to \mathcal{A}$ corresponding to the cover $\mathfrak{U}$. Note that $f_* \circ \mathfrak{C}(\mathfrak{U}) \circ f^*$ is a complex in the category $\mathfrak{End}(\mathcal{C})$ of continuous functors $\mathcal{C} \to \mathcal{C}$. It follows from Proposition C4.3 that $f_* \circ \mathfrak{C}(\mathfrak{U}) \circ f^*$ is a resolution of the functor

$$\mathfrak{G}''_f := H^0(f_* \circ \mathfrak{C}(\mathfrak{U}) \circ f^*) := \mathrm{Ker}(f_* \circ (\mathfrak{G}_{\mathbf{u}} \longrightarrow \mathfrak{G}^2_{\mathbf{u}}) \circ f^*) : \mathcal{C} \longrightarrow \mathcal{C}, \tag{1}$$

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where the kernel is taken in the category $\mathfrak{End}\,\mathcal{C}$. In particular, $\mathfrak{G}''_f$ does not depend on the choice of the cover $\mathfrak{U}$. The functor $\mathfrak{G}''_f$ has a uniquely defined monad structure μ''. The morphism $f : \mathcal{A} \to \mathcal{C}$ decomposes uniquely into a continuous morphism $\zeta''_f : \mathcal{A} \to \mathbb{G}''_f-$ mod and $\mathbb{G}_f-$ mod $\to \mathcal{C}$. Here $\mathbb{G}''_f := (\mathfrak{G}''_f, \mu'')$. The map assigning to any scheme $f : \mathcal{A} \to \mathcal{C}$ the continuous monad $\mathbb{G}''_f$ extends naturally to a 'global section' functor $\Gamma : \mathbf{Sch}/\mathcal{C} \to (\mathfrak{Monc}\mathcal{C})^o$ which is a left adjoint to the 'localization' functor $\mathfrak{S}$. And $\zeta'' = (\zeta''_f)$ is the adjunction arrow from $\mathrm{Id}_{\mathbf{Sch}/\mathcal{C}}$ to $\mathfrak{S} \circ \hat{}\Gamma$. The other adjunction arrow is identical. □

COROLLARY 4.4.3.1. *The functor* $\mathfrak{S} : (\mathfrak{Monc}\mathcal{C})^o \to \mathbf{Sch}/\mathcal{C}$ *establishes an equivalence between the category* $(\mathfrak{Monc}\mathcal{C})^o$ *dual to the category of continuous monads and the category* $\mathbf{Aff}/\mathcal{C}$ *of affine schemes over* $\mathcal{C}$.

4.5. *Schemes over a category of modules.* Denote by Cat_* the category objects of which are pairs $(\mathcal{A}, \mathcal{O})$, where $\mathcal{A}$ is a category (equivalent to a small category and thought as the category of quasi-coherent sheaves on a scheme) and $\mathcal{O}$ is an object of $\mathcal{A}$ (thought as the structure sheaf). Morphisms from $(\mathcal{A}, \mathcal{O})$ to $(\mathcal{A}', \mathcal{O}')$ are pairs (f, ϕ), where f is a morphism from $\mathcal{A}$ to $\mathcal{A}'$ and ϕ is an isomorphism from $f^*(\mathcal{O}')$ to $\mathcal{O}$.

Let $\mathcal{C}$ be the category $k-$ mod of left modules over a ring k, and let $f: \mathcal{A} \to \mathcal{C}$ be a morphism. We can assign to the morphism f the pair $(\mathcal{A}, f^*(k))$. This correspondence provides a functor from the category $Cat^*/\mathcal{C}$ to the category Cat_*.

Suppose now that $f: \mathcal{A} \to \mathcal{C}$ is a continuous morphism. Then it is defined uniquely up to isomorphism by the object $\mathcal{O} = f^*(k)$.

In fact, we have functorial isomorphisms $\mathcal{A}(f^*(k), X) \simeq \mathcal{C}(k, f_*(X)) \simeq f_*(X)$ which shows that the direct image functor f_* of f is naturally isomorphic to the functor $X \longmapsto \mathcal{A}(f^*(k), X)$. Therefore the inverse image functor f^* (representing f) is defined uniquely up to isomorphism (being a left adjoint to the functor f_*) by the object $f^*(k)$. Note that since f^* respects colimits, there exists a coproduct of any set of copies of $\mathcal{O} = f^*(k)$.

Conversely, suppose that $(\mathcal{A}, \mathcal{O})$ is an object of the category Cat_* such that the category $\mathcal{A}$ is abelian and there exists a coproduct of any set of copies of $\mathcal{O}$. Then the functor $X \longmapsto \mathcal{A}(\mathcal{O}, X)$ from $\mathcal{A}$ to the category $K-$ mod, where $K = \mathcal{A}(\mathcal{O}, \mathcal{O})^o$, is a direct image of a continuous morphism from $\mathcal{A}$ to $K-$ mod ([BD], Proposition 6.6.23).

Now fix an additive category $\mathcal{A}$ and a continuous morphism $f: \mathcal{A} \to \mathcal{C} = k$ = mod. And set $\mathcal{O} = f^*(k)$. The functor f_* is faithful iff $\mathcal{O}$ is a generator of the category $\mathcal{A}$.

Since $f_* \simeq \mathcal{A}(\mathcal{O},)$, the morphism f is coflat iff $\mathcal{O}$ is a projective object.

Thus f is almost affine iff $\mathcal{O}$ is a projective generator. Finally, f is affine iff $\mathcal{O}$ is a projective generator of finite type.

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PROPOSITION 4.5.1. (a) *For any continuous morphism* $f : \mathcal{A} \to \mathcal{C} = k - \text{mod}$, *there is a canonical functor morphism*

$$\psi_f : \mathcal{A}(\mathcal{O}, \mathcal{O})^o \otimes_k \longrightarrow \mathcal{A}(\mathcal{O}, f^* -) \tag{1}$$

such that $\psi_f(V)$ *is an isomorphism for any free* k*-module* V *of finite type.*

(b) *If* f *is almost affine, then* $\psi_f(V)$ *is an isomorphism for any finitely presented* k*-module* V. *In particular, if* k *is left noetherian, then* $\psi_f(V)$ *is an isomorphism for any finitely generated* k*-module* V.

(c) *The morphism* ψ_f *is an isomorphism if and only if* f *is affine.*

(d) *The morphism* f *is affine if and only if the functor*

$$\mathcal{A}(\mathcal{O}, -) : \mathcal{A} \longrightarrow \mathcal{A}(\mathcal{O}, \mathcal{O})^o - \text{mod}$$

is an equivalence of categories.

Proof. (a) For any additive functor $F : k-$ mod $\to k -$ mod, the module $F(k)$ has a natural k-bimodule structure, and there is a canonical functor morphism $\psi_F : F(k) \otimes_k \to F$ (see for instance, [Bass], Ch.I). Recall that, for any k-module V, the morphism $\psi_F(V)$ is the image of Id_V with respect to the composition

$$\begin{array}{ccc} \text{Hom}_k(V, V) & \longrightarrow & \text{Hom}_k(V, \text{Hom}_k(k, V)) \\ & & \downarrow \\ \text{Hom}_k(F(k) \otimes_k V, F(V)) & \longleftarrow & \text{Hom}_k(V, \text{Hom}_k(F(k), F(V))) \end{array}$$

Since $\psi_F(k)$ is an isomorphism and the functor F is additive, $\psi_F(V)$ is an isomorphism for any free k-module V of finite rank.

(b) If the functor F is right exact, i.e. it preserves cokernels, $\psi_F(V)$ is an isomorphism for any finitely presented object V, since finitely presented objects are exactly cokernels of morphisms between free objects of finite rank.

(c) The morphism ψ_F is an isomorphism iff the functor F preserves arbitrary colimits (or, equivalently, has a right adjoint).

The assertions *(a)–(c)* of the lemma are just specializations of these facts for the functor $f_* \circ f^* \simeq \mathcal{A}(\mathcal{O}, f^* -)$.

(d) By Proposition 4.2.4, if $f : \mathcal{A} \to k -$ mod is almost affine, the canonical functor $\mathcal{A} \to \mathbb{G}_f -$ mod is an equivalence of categories. The assertion *(c)* implies that if (and only if) f is affine, the monad $\mathbb{G}_f$ is naturally isomorphic to the monad $(\mathcal{A}(\mathcal{O}, \mathcal{O}) \otimes_k, m)$, where m is induced by the multiplication in $\mathcal{A}(\mathcal{O}, \mathcal{O})^o$. The category $(\mathcal{A}(\mathcal{O}, \mathcal{O}), m) -$ mod is isomorphic to the category of left modules over $\mathcal{A}(\mathcal{O}, \mathcal{O})^o$. □

Remark 4.5.2. The analysis above shows that when the ring k is commutative, the category of affine schemes over $\mathcal{C} = k -$ mod is naturally equivalent to the category of affine schemes in the sense of M. Artin and J. J. Zhang [AZ]. □

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4.5.3. *General schemes over* k. Let $f\colon \mathcal{A} \to \mathcal{C}$ be an arbitrary quasi-scheme over $\mathcal{C} = k-$ mod; and let $\mathfrak{U} = \{u_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ be a Zariski cover adapted to f such that $f_* \circ u_{i*}$ is an exact, faithful functor for any $i \in J$. Or, in our new language, $f \circ u_i$ is an almost affine quasi-scheme for any i. By 4.1, this means that the category $\mathcal{B}_i$ is naturally equivalent to the category $\mathbb{G}_{f\circ u_i} -$ mod, where $\mathbb{G}_{f\circ u_i}$ is the monad in the category $k-$ mod (shortly k-monad) associated with the morphism $f \circ u_i$. If $f\colon \mathcal{A} \to \mathcal{C}$ is a scheme and $\mathfrak{U}$ is the corresponding affine cover, then $\mathcal{B}_i$ is isomorphic to the category of R_i-modules, where $R_i = \mathcal{B}_i(\mathcal{O}_i,\ \mathcal{O}_i)^o$, $\mathcal{O}_i$ is the 'structure sheaf' on $\mathcal{B}_i$: $\mathcal{O}_i = u_i^*(\mathcal{O})$.

Thus any scheme over $k-$ mod is locally the category of left modules over k-rings. Recall that a *k-ring* is an arbitrary ring morphism $k \to A$.

Any quasi-scheme over $k-$ mod is locally the category of left modules over a k-monad.

Examples of interest of relative schemes are noncommutative projective spaces and quantized flag varieties of semisimple Lie algebras. We discuss them in Section 5.

5. Noncommutative quasi-affine spaces and projective spectra

5.1. *Projective spectrum and a quasi-affine space related to a graded algebra.* Let k be a commutative ring, Γ a commutative directly ordered group. And let R be an associative Γ-graded k-algebra. For any $\gamma \in \Gamma$, set $R_{>\gamma} := \oplus_{\sigma>\gamma} R_\sigma$. For any R-module M and any $\gamma \in \Gamma$, denote by M_γ the subset of all elements of M annihilated by $R_{>\gamma}$. Denote by $\mathcal{T}_+$ the full subcategory of the category $R-$ mod generated by all R-modules M such that $M = \sup\{M_\gamma | \gamma \in \Gamma\}$. One can see that $\mathcal{T}_+$ is a Serre subcategory of the category $R-$ mod. The quotient category $\mathbf{Cone}_\Gamma(R) := R - \mathrm{mod}/\mathcal{T}_+$ is called *the quasi-affine space* (or *affine cone*) *of* R.

Let $\mathcal{F}$ be the natural functor $\mathfrak{gr}_\Gamma R - \mathrm{mod} \to R - \mathrm{mod}$. And let $\mathfrak{T}_+$ denote the preimage of $\mathcal{T}_+$ in $\mathfrak{gr}_\Gamma R - mod$. Since the functor $\mathcal{F}$ is exact, $\mathfrak{T}_+$ is a Serre subcategory of $\mathfrak{gr}_\Gamma R -$ mod. The quotient category $\mathbf{Proj}_\Gamma(R) = \mathfrak{gr}_\Gamma R - \mathrm{mod}/\mathfrak{T}_+$ is called *the projective spectrum of* R.

We have the following canonical continuous morphisms:

(a) The 'embedding' u : $\mathbf{Cone}_\Gamma(R) \to R -$ mod with a localization at $\mathcal{T}_+$ as an inverse image functor.

(b) The morphism π' : $\mathbf{Proj}_\Gamma(R) \to \mathfrak{gr}_\Gamma R -$ mod with a localization at $\mathfrak{T}_+$ as an inverse image functor.

(c) The morphism φ : $\mathfrak{gr}_\Gamma R -$ mod $\to R_0 -$ mod, with the direct image functor

$$\varphi_* : \mathfrak{gr}_\Gamma R - \mathrm{mod} \longrightarrow R_0 - \mathrm{mod}, \quad M = \oplus_{\gamma\in\Gamma} M_\gamma \longmapsto M_0. \tag{1}$$

Here 0 denotes the identity element of the group Γ.

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Note by passing that the inverse image functor $\varphi^*: V \longmapsto R \otimes_{R_0} V$ is fully faithful. This follows from the fact that the adjunction morphism

$$\eta : \mathrm{Id}_{R_0-\mathrm{mod}} \longrightarrow \varphi_* \circ \varphi^*, \quad \eta(V) : V \longrightarrow (R \otimes_{R_0} V)_0 = R_0 \otimes_{R_0} V$$

is an isomorphism.

(d) Set $\pi := \varphi \circ \pi' : \mathbf{Proj}_\Gamma(R) \to R_0 - \mathrm{mod}$. If $\Gamma = \mathbb{Z}$ and R is a commutative $\mathbb{Z}_+$-graded algebra, the morphism π defines the structure of a scheme over R_0. In the general case, when $\mathrm{rank}(\Gamma) \geqslant 2$, the direct image functor π_* of π is not locally faithful; hence π is not a structure of a scheme.

5.1.1. *Changing the grading.* Let R be a Γ-graded k-algebra; and let $\psi : \Gamma \to \Gamma'$ be a group epimorphism. Then R becomes a Γ'-graded algebra. So that we have the Γ'-cone of R, $\mathbf{Cone}_{\Gamma'}(R)$, and the Γ'-projective space $\mathbf{Proj}_{\Gamma'}(R)$. The natural functor $F : \mathfrak{gr}_\Gamma R - \mathrm{mod} \to \mathfrak{gr}_{\Gamma'} R - \mathrm{mod}$ induces exact and faithful functors $\mathbf{Cone}_\Gamma(R) \to \mathbf{Cone}_{\Gamma'}(R)$ and $\mathbf{Proj}_\Gamma(R) \to \mathbf{Proj}_{\Gamma'}(R)$ such that the diagram

$$\begin{array}{ccc} \mathbf{Cone}_\Gamma(R) & \longrightarrow & \mathbf{Cone}_{\Gamma'}(R) \\ \uparrow & & \uparrow \\ \mathbf{Proj}_\Gamma(R) & \longrightarrow & \mathbf{Proj}_{\Gamma'}(R) \end{array} \tag{1}$$

commutes.

The standard example: Γ is a free abelian group of finite rank (greater than one) with the lexicographic preorder corresponding to a choice of a free basis. The homomorphism $\psi : \Gamma \to \mathbb{Z}$ assigns to any element the sum of its components. If the algebra R is such that $R_\gamma = 0$ for $\gamma \ll 0$, then the functor $\mathbf{Cone}_\Gamma(R) \to \mathbf{Cone}_{\mathbb{Z}}(R)$ in the diagram (1) is an equivalence of categories. This is not the case, however, with the functor $\mathbf{Proj}_\Gamma(R) \to \mathbf{Proj}_{\mathbb{Z}}(R) = \mathbf{Proj}(R)$.

5.2. *Affine covers of projective spectra.* Let k and Γ be as in Section 5.1. Fix a Γ-graded associative k-algebra R.

LEMMA 5.2.1. *Let $\mathcal{S} = \{S_i | i \in J\}$ be a family of left homogeneous Ore subsets of the algebra R. And let, for each $i \in J$, $\mathbb{S}_i$ be the Serre subcategory of $R -$ mod generated by all modules M such that any element of M is annihilated by some element of S_i. And let $\mathcal{S}_i$ be the preimage of $\mathbb{S}_i$ in $\mathfrak{gr}_\Gamma R -$ mod.*

The following conditions are equivalent:

(a) *The Serre subcategories $\{\mathbb{S}_i | i \in J\}$ provide a cover of the 'quasi-affine space' $\mathbf{Cone}_\Gamma(R)$; i.e. $\bigcap_{i \in J} \mathbb{S}_i = \mathcal{T}_+$.*

(b) *The Serre subcategories $\{\mathcal{S}_i | i \in J\}$ provide a cover of the projective spectrum $\mathbf{Proj}_\Gamma(R)$; i.e. $\bigcap_{i \in J} \mathcal{S}_i = \mathfrak{T}_+$.*

(c) *The family of Ore sets $\mathcal{S} = \{S_i | i \in J\}$ has the properties:*

(i) *for any $\gamma \in \Gamma$ and any $i \in J$, $S_i \cap R_{>\gamma} \neq \emptyset$.*

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(ii) *if m is a left ideal in R such that $m \cap S_i \neq \emptyset$ for all $i \in J$, then $R_{>\gamma} \subseteq m$ for some $\gamma \in \Gamma$.*

Proof. (a) $\Rightarrow$ (c). The condition (i) is equivalent to the inclusion $\mathcal{T}_+ \subseteq \mathbb{S}_i$ for all $i \in J$. The condition (ii) says that $\mathcal{T}_+$ contains the intersection $\bigcap_{i\in J} \mathbb{S}_i$. Therefore $\mathcal{T}_+ = \bigcap_{i\in J} \mathbb{S}_i$. The implications (b) $\Rightarrow$ (c) are established the same way. □

Remark 5.2.1.1. In [VW], a $\mathbb{Z}_+$-graded noetherian ring R such that there exists a family of *left and right* Ore sets $\mathcal{S} = \{S_i | i \in J\}$ satisfying the equivalent conditions of Lemma 5.2.1 is called *schematic*. Quite a few algebras of interest are schematic. We refer to [VW] for examples. □

PROPOSITION 5.2.2. *Any family $\mathcal{S} = \{S_i | i \in J\}$ of left homogeneous Ore subsets of R satisfying the conditions* (i), (ii) *of Lemma 3.2.1 determines an affine cover $\{\mathcal{S}'_i | i \in J\}$ of $\mathbf{Proj}_\Gamma(R)$ adapted to the global section functor $\pi'_* : \mathbf{Proj}_\Gamma(R) \to \mathfrak{gr}_\Gamma R - \mathrm{mod}$ and, therefore, to the functor $\pi_* : \mathbf{Proj}_\Gamma(R) \to R_0 - \mathrm{mod}$* (cf. (b) and (d) in 5.1).

Proof. Here $\mathcal{S}'_i$ denote the image of the Serre category $\mathcal{S}_i$ in $\mathbf{Proj}_\Gamma(R)$.

The composition of π_* and the direct image of $u_i : \mathbf{Proj}_\Gamma(R)/\mathcal{S}'_i \to \mathbf{Proj}_\Gamma(R)$ equals to the composition

$$\pi'_* \circ u_{i*} = (\pi' \circ u_i)_* : \mathbf{Proj}_\Gamma(R)/\mathcal{S}'_i \longrightarrow \mathfrak{gr} R - \mathrm{mod} \tag{1}$$

and the functor $\varphi_* : \mathfrak{gr}_\Gamma R - \mathrm{mod} \to R_0 - \mathrm{mod}$ (cf. (1) in 5.1). Note that the category $\mathbf{Proj}_\Gamma(R)/\mathcal{S}'_i$ is naturally identified with the category $\mathfrak{gr}_\Gamma R - \mathrm{mod}/\mathcal{S}_i$; so that the functor (1) becomes a right adjoint to the localization

$$Q_i : \mathfrak{gr}_\Gamma R - \mathrm{mod} \longrightarrow \mathfrak{gr}_\Gamma R - \mathrm{mod}/\mathcal{S}_i. \tag{2}$$

Since (2) is a localization at a left Ore set S_i, the quotient category $\mathfrak{gr}_\Gamma R - \mathrm{mod}/\mathcal{S}_i$ is equivalent to the category $\mathfrak{gr} S_i^{-1} R - \mathrm{mod}$ of graded $S_i^{-1}R$-modules. Thus $\mathfrak{gr}_\Gamma R - \mathrm{mod}/\mathcal{S}_i$ can be replaced by $S_i^{-1}R - \mathrm{mod}$. And the localization Q_i can be identified with the tensoring $S_i^{-1}R \otimes_R$. Therefore a right adjoint to Q_i is an exact functor. Since the functor $\varphi_* : \mathfrak{gr}_\Gamma R - \mathrm{mod} \to R_0 - \mathrm{mod}$ is exact, we obtain the exactness of $\pi_* \circ u_{i*}$. Now the assertion follows from Proposition 2.2. □

Remark 5.3. Under the conditions of Proposition 5.2.2, the family of Ore sets $\mathcal{S} = \{S_i | i \in J\}$ determines an affine cover of $\mathbf{Cone}_\Gamma(R) := R - \mathrm{mod}/\mathcal{T}_+$ which is adapted to the direct image functor $u_* : \mathbf{Cone}_\Gamma(R) \to R - \mathrm{mod}$ (cf. 5.1). The covers of $\mathbf{Proj}_\Gamma(R)$ and $\mathbf{Cone}_\Gamma(R)$ defined by the family $\mathcal{S}$ are compatible with the natural (inverse image) functor $\mathbf{Proj}_\Gamma(R) \to \mathbf{Cone}_\Gamma(R)$; i.e. the diagram

$$\begin{array}{ccc} \mathbf{Proj}_\Gamma(R) & \longrightarrow & \mathbf{Proj}_\Gamma(R)/\mathcal{S}'_i \\ \downarrow & & \downarrow \\ \mathbf{Cone}_\Gamma(R) & \longrightarrow & \mathbf{Cone}_\Gamma(R)/\mathcal{S}'_i \end{array} \tag{3}$$

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is commutative for all $i \in J$. Here $\mathbb{S}'_i$ is the image of the Serre category $\mathbb{S}_i$ in $\mathbf{Cone}_\Gamma(R)$. And $\mathbf{Cone}_\Gamma(R)/\mathbb{S}'_i$ can be identified with $R - \mathrm{mod}/\mathbb{S}_i = S_i^{-1} R - \mathrm{mod}$. □

5.4. *Example: noncommutative skew projective spaces.* Let A be an arbitrary associative k-algebra. And let $\mathbf{q}$ denote a matrix $[q_{ij}]_{i,j\in J}$ with entrees in k such that $q_{ij}q_{ji} = 1$ for all $i, j \in J$. In particular, $q_{ii} = 1$ for all $i \in J$. To this data there corresponds a skew (or $\mathbf{q}-$)polynomial algebra $A_\mathbf{q}[\mathbf{x}]$, where $\mathbf{x}$ denotes the set of indeterminates $\{x_i | i \in J\}$. The defining relations are:

$$x_i x_j = q_{ij} x_j x_i \quad \text{for all} \quad i, j \in J, \tag{1}$$

$$x_i r = r x_i \quad \text{for all} \quad i \in J \quad \text{and} \quad r \in A. \tag{2}$$

Let $J = \{0, 1, \ldots, r\}$. Set $\Gamma := \mathbb{Z}^{r+1}$; and let $\gamma_i,\ i = 0, 1, \ldots, r$, denote the canonical generators of Γ. We provide Γ with the standard lexicographic preorder. Assigning to each x_i the parity γ_i, we turn the skew polynomial algebra $R := A_\mathbf{q}[\mathbf{x}]$ into a Γ-graded algebra with R_0 = A.

There is a natural choice of left (and right) homogeneous Ore subsets of the ring R : $S_i := \{x_i^n | n \geqslant 1\}$ for all $i \in J$. The family $\mathcal{S} = \{S_i | i \in J\}$ satisfies the conditions (i), (ii) of Lemma 5.2.1. Therefore $\mathcal{S}$ determines, by Proposition 5.2, affine covers of the spaces $\mathbf{Proj}_\Gamma(R)$ and $\mathbf{Cone}_\Gamma(R)$. These covers have all 'classical' properties:

(a) One can see that the category $\mathbf{Cone}_\Gamma(R)/\mathbb{S}'_i \approx A_\mathbf{q}[\mathbf{x}, x_i^{-1}] - \mathrm{mod}$.
(b) Let Γ_i denote the quotient group $\Gamma/\mathbb{Z}\gamma_i \simeq \mathbb{Z}^r$. We have:

$$\mathbf{Proj}_\Gamma(R)/\mathcal{S}'_i = \mathfrak{gr}_\Gamma R - \mathrm{mod}/\mathcal{S}_i \simeq \mathfrak{gr}_\Gamma((x_i)^{-1}R) - \mathrm{mod}, \tag{3}$$

where $(x_i)^{-1}R$ is the localization of the algebra R at the multiplicative set $(x_i)^{-1} := \{(x_i)^n | n \geqslant 0\}$. The category $\mathfrak{gr}_\Gamma(x_i)^{-1}R - \mathrm{mod}$ in (3) is naturally equivalent to the category $\mathfrak{gr}_{\Gamma_i} A_{\mathbf{q}_i}[\mathbf{x}/x_i] - \mathrm{mod}$ of left Γ_i-graded modules over the skew polynomial algebra $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$. Here $\mathbf{x}/x_i$ denotes $\{x_j/x_i | j \in J, j \neq i\}$, and $\mathbf{q}_i$ denotes the matrix $[q_{ni}q_{nm}q_{ni}^{-1}]_{n,m\in J-\{i\}}$ (cf. [R], Example I.7.2.2.4).

Note that $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$ is the Γ_i-component of the algebra $A_\mathbf{q}[\mathbf{x}, x_i^{-1}]$ of the 'functions on $\mathbf{Cone}_\Gamma(R)/\mathbb{S}''_i$.

(c) One can see that the category $\mathbf{Proj}_\Gamma(R)/\mathcal{S}'_i$ is naturally identified with the category $\mathfrak{gr}_{\Gamma_i} A_{\mathbf{q}_i}[\mathbf{x}/x_i] - \mathrm{mod}$ and $\mathbf{Cone}_\Gamma(R)/\mathbb{S}'_i$ with $\mathfrak{gr}_{\Gamma_i} A_\mathbf{q}[\mathbf{x}, x_i^{-1}] - \mathrm{mod}$. And the canonical functor $\mathbf{Proj}_\Gamma(R)/\mathcal{S}'_i \to \mathbf{Cone}_\Gamma(R)/\mathbb{S}'_i$ of Remark 5.3 is isomorphic to the tensoring by the algebra $A_\mathbf{q}[\mathbf{x}, x_i^{-1}]$ over its Γ_i-component $A_{\mathbf{q}_i}[\mathbf{x}/x_i] = A_\mathbf{q}[\mathbf{x}, x_i^{-1}]_0$.

(d) The composition of the Gabriel functors $\mathbf{G}_i := Q_i\hat{} \circ Q_i$, where Q_i is a localization at S_i, commute one with another. In other words, the canonical cover of $\mathbf{Proj}_\Gamma(R)$ is semiseparated. This implies that, for any subset J' of J, the composition of $\mathbf{G}_i,\ i \in J'$, is the Gabriel functor of the localization at the multiplicative

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set generated by $\{x_i | i \in J'\}$.

5.4.1. *The 'projective space'* $\mathbf{P}^r$**.** Let again $R = A_{\mathbf{q}}[\mathbf{x}]$, $\mathbf{x} = (x_0, x_1, \ldots, x_r)$. But take $\Gamma = \mathbb{Z}$ with the natural order; and set the parity of each x_i equal to 1. One can repeat with $\mathbf{Cone}_{\mathbb{Z}}(R)$ and $\mathbf{P}^r := \mathbf{Proj}_{\mathbb{Z}}(R)$ the same pattern as with $\mathbf{Cone}_{\Gamma}(R)$ and $\mathbf{P}^r_{\Gamma} := \mathbf{Proj}_{\Gamma}(R)$. Only this time the quotient groups Γ_i will be trivial, and we obtain a picture very similar to the classical one: $\mathbf{P}^r$ covered by $r+1$ affine spaces $A_{\mathbf{q}_i}[\mathbf{x}/x_i]-$ mod, $i = 0, 1, \ldots, r$. The details are left to the reader.

Note that the categories $\mathbf{P}^r$ and $\mathbf{P}^r_{\Gamma} := \mathbf{Proj}_{\Gamma}(R)$ are not equivalent if $r \geqslant 1$.

5.4.2. *A useful generalization.* Suppose we are given a k-algebra A and a matrix $\mathbf{q} = (q_{ij})_{0 \leqslant i,j \leqslant r}$ (as in 5.4), and a group homomorphism $\vartheta : \mathbb{Z}^{r+1} \to \mathrm{Aut}_k(A)$. Define $A_{\mathbf{q}}[\mathbf{x}, \vartheta]$ as the k-algebra generated by A and $\mathbf{x} = (x_0, x_1, \ldots, x_r)$ subject to the relations:

$$x_i x_j = q_{ij} x_j x_i, \quad x_i b = \vartheta_i(b) x_i \tag{1}$$

for all $0 \leqslant i, j \leqslant r$ and $b \in$ A. Here ϑ_i is the image with respect to ϑ of the canonical generator γ_i of $\Gamma = \mathbb{Z}^{r+1}$. The corresponding projective spaces shall be denoted by $\mathbf{P}^r_{\Gamma,\vartheta}$ and by $\mathbf{P}^r_{\vartheta}$.

5.5. *Flag varieties of quantized enveloping algebras.* Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{P}$ denote the group of integral weights of $\mathfrak{g}$, and let $\mathfrak{P}_+$ be the semigroup of nonnegative integral weights. Let $\mathcal{R} = \oplus_{\lambda \in \mathfrak{P}_+} \mathcal{R}_\lambda$, where $\mathcal{R}_\lambda$ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathfrak{P}$ -graded algebra with the multiplication determined by the projections $\mathcal{R}_\lambda \otimes \mathcal{R}_\nu \to \mathcal{R}_{\lambda+\nu}$, for all $\lambda, \nu \in \mathfrak{P}_+$. It is well known that the algebra $\mathcal{R}$ is isomorphic to the algebra of regular functions on the base affine space of $\mathfrak{g}$. Recall that $Y = G/U$, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The cone $\mathbf{Cone}(\mathcal{R})$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $\mathbf{Proj}(\mathcal{R}) = \mathbf{Proj}_{\mathbb{Z}}(\mathcal{R})$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic (say, $k = \mathbb{Q}(q)$) and $U_q(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathfrak{P}$-graded algebra $\mathcal{R} = \oplus_{\lambda \in \mathfrak{P}_+} \mathcal{R}_\lambda$ the same way as above. This time, however, the algebra $\mathcal{R}$ is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call the cone $\mathbf{Cone}(\mathcal{R}) = \mathbf{Cone}_{\mathbb{Z}}(\mathcal{R})$ *the quantum base affine space* and $\mathbf{Proj}(\mathcal{R}) = \mathbf{Proj}_{\mathbb{Z}}(\mathcal{R})$ the *quantum flag variety* of $\mathfrak{g}$.

5.5.1. *An affine cover of the flag variety.* Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathfrak{P}_+$, choose a nonzero w-extremal vector $e_{w\lambda}$

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generating the one dimensional vector space formed by the vectors of the weight $w\lambda$. Set $S_w := \{k^* e_{w\lambda} | \lambda \in \mathfrak{P}_+\}$. It follows from the Weyl character formula that $e_{w\lambda} e_{w\mu} \in k^* e_{w(\lambda+\mu)}$. Hence e_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in $\mathcal{R}$. The Ore sets $\{S_w | w \in W\}$ determine a locally affine cover of the quantum base affine space **Cone**$(\mathcal{R})$ and the quantum flag variety **Proj**$(\mathcal{R})$ of $\mathfrak{g}$. This cover enjoys properties similar to the properties (a)–(c) of the canonical cover of a 'projective space' and its cone (cf. 5.4). Namely, the quotient category **Cone**$(\mathcal{R})/\mathbb{S}_w$ is naturally equivalent to the category $S_w^{-1}\mathcal{R}-$mod, and **Proj**$(\mathcal{R})/\mathcal{S}_w$ is naturally equivalent to the category $(S_w^{-1}\mathcal{R})_0 -$ mod. But the analog of the property (d) in 5.4 does not hold: the multiplicative subset generated by S_w and $S_{w'}$ for different w and w' is not an Ore set in general. Which means that the situation is not analogous to the classical one: the canonical cover is not semiseparated. Still the standard complex allows to compute the cohomology of line bundles on the quantized flag variety by comparing them with the cohomology of the classical specialization. This is done in [LR3].

Complementary facts and examples

C1. *Flat covers and Zariski covers*

LEMMA C1.1. *Let $\mathcal{A}$ be a category with finite limits and colimits. Then any flat (i.e. exact and having a right adjoint) functor T: $\mathcal{A} \to \mathcal{B}$ is represented uniquely up to isomorphism as the composition $H \circ Q$, where Q is a flat localization, and H is a faithfully flat functor.*

Proof. The functor T is represented as the composition $T = H \circ Q$, where Q is the localization at $S = \{s \in \mathrm{Hom}\mathcal{A} \mid Ts \text{ is invertible}\}$. Since $T = H \circ Q$ has a right adjoint, $T^\wedge$, the functor H is left adjoint to the composition $Q \circ T^\wedge$; and the adjunction arrow $\epsilon : T \circ T^\wedge \to \mathrm{Id}$ can be also regarded as an adjunction morphism $H \circ (Q \circ T^\wedge) = H \circ H^\wedge \to \mathrm{Id}$. As for the second adjunction arrow, $\gamma : \mathrm{Id} \to H^\wedge \circ H = (Q \circ T^\wedge) \circ H$, it is uniquely defined by the equality $\gamma Q = Q\eta$ (cf. [GZ], Lemma I.1.3.1). Here η is the adjunction morphism $\mathrm{Id} \to T^\wedge \circ T$.

Let F denote the composition $T^\wedge \circ T$. Define the functor $G : \mathcal{A} \to \mathcal{A}$ as the equalizer of the pair $F\eta, \eta F : F \to F \circ F$. And let ν denote the canonical arrow $G \to F$.

Note that if $s \in \mathrm{HomS}$, then Gs is invertible. In fact, $s \in \mathrm{HomS}$ if and only if Ts is invertible. But, this implies that Fs and $F \circ F(s)$ are invertible. Hence Gs is invertible.

Therefore $G = L \circ Q$ for a uniquely defined functor L. We claim that the functor L is right adjoint to Q. In fact, since the adjunction arrow $\eta : \mathrm{Id} \to F = T^\wedge \circ T$ equalizes $(F\eta, \eta F)$, we have a canonical morphism $\delta : \mathrm{Id} \to L \circ Q = G$ uniquely defined by the equality: $\nu \circ \delta = \eta$.

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On the other hand, applying the localization Q to the arrows $F\eta, \eta F$, we obtain a pair of morphisms from

$$Q \circ F := Q \circ T\hat{} \circ T = (Q \circ T\hat{}) \circ H \circ Q = (H\hat{} \circ H) \circ Q$$

to

$$Q \circ F \circ F$$
$$= (Q \circ T\hat{}) \circ (H \circ Q) \circ F = (H\hat{} \circ H) \circ Q \circ F = (H\hat{} \circ H) \circ (H\hat{} \circ H) \circ \mathrm{Q}.$$

Since $\gamma Q = Q\eta$, we have:

$$Q\eta F = \gamma QF = \gamma \mathbf{H}Q, \quad \text{and} \quad QF\eta = H\hat{}HQ\eta = \mathbf{H}\gamma Q.$$

By Proposition I.3.4 in [GZ], the functor H is exact and faithful; therefore it is faithfully flat. By Lemma 1.3.1, the adjunction arrow $\gamma : \mathrm{Id} \to \mathbf{H}$ is the equalizer of the pair $\mathbf{H}\gamma, \gamma\mathbf{H}$. In particular,

$$\gamma Q = Q\eta : Q \longrightarrow \mathbf{H} \circ Q = Q \circ F$$

is the equalizer of $\gamma\mathbf{H}Q = Q\eta F$ and $\mathbf{H}\gamma Q = QF\eta$. Notice now that, by the same Proposition I.3.4 in [GZ], S admits left and right fractions. By Proposition I.3.1 in [GZ], the localization $Q\colon \mathcal{A} \to \mathcal{A}[S^{-1}]$ is an exact functor. In particular, it preserves equalizers. Thus, we have obtained: both the arrows, $Q\nu : Q \circ G \to QF$ and $Q\eta\colon Q \to QF$, are equalizers of the pair $Q\eta F, QF\eta$. Since $Q\eta = Q(\nu \circ \delta) = Q\nu \circ Q\delta$, this means that the arrow $Q\delta\colon Q \to Q \circ G = Q \circ L \circ Q$ is an isomorphism. By the universal property of the localization Q, there is a unique functor isomorphism $\sigma : Q \circ L \to \mathrm{Id}$ such that $\sigma Q = Q\delta^{-1}$. By the definition of σ, we have: $\sigma Q \circ Q\delta = \mathrm{id}_Q$.

Note that $(L\sigma \circ \delta L)Q \circ \delta = L\sigma Q \circ LQ\delta \circ \delta = \delta$. This implies, by the universal property of δ, that $(L\sigma \circ \delta L)Q = \mathrm{id}_{L\circ Q}$. Therefore, by the universal property of Q, we have: $L\sigma \circ \delta L = \mathrm{id}_L$. The equalities $\sigma Q \circ Q\delta = \mathrm{id}_Q,\ L\sigma \circ \delta L = \mathrm{id}_L$ show that the functor L is the right adjoint to the localization Q, and δ and σ are adjunction arrows. □

Let $\mathfrak{F} = \{f_i\colon \mathcal{B}_i \to \mathcal{A} | i \in J\}$ be a family of flat morphisms. By Lemma C1.1 each f_i is represented as a composition of a flat localization $q_i : \mathcal{A}_i \to \mathcal{A}$ and a faithfully flat morphism $h_i : \mathcal{B}_i \to \mathcal{A}_i$. Clearly $\mathfrak{F}$ is a cover iff $\{q_i : \mathcal{A}_i \to \mathcal{A} | i \in J\}$ is a cover.

Another application of Lemma C1.1 is the following proposition.

PROPOSITION C1.2. *Let $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} | i \in J\}$ be a finite family of flat morphisms to an additive category $\mathcal{A}$. And let q^* be a localization at $\Sigma_{\mathfrak{F}} := \{s \in$*

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$\mathrm{Hom}_{\mathcal{A}}|f_i^*(s)$ *is invertible for all* $i \in J\}$. *Then the standard complex*

$$0 \longrightarrow q^* \longrightarrow q^*\left(\prod_{i\in J}\mathfrak{G}_i\right) \longrightarrow q^*\left(\prod_{i,j\in J}\mathfrak{G}_i\mathfrak{G}_j\right) \longrightarrow \dots$$

$$\longrightarrow q^*\left(\prod_{\mathbf{i}\in J^{\times n}}\mathfrak{G}_{\mathbf{i}}\right) \longrightarrow \dots \tag{1}$$

is exact.

Proof. It suffices to prove the assertion in the case $card(J) = 1$ (cf. the argument of Proposition 1.4); i.e. when $\mathfrak{F} = \{f\}$ for a flat morphism $f\colon \mathcal{B} \to \mathcal{A}$. By Lemma C1.1, $f = q \circ h$, where $q : \mathcal{A}' \to \mathcal{A}$ is a flat localization (i.e. q^* is a flat localization) and $h : \mathcal{B} \to \mathcal{A}$ is a faithfully flat morphism. Since $q^* \circ q_* \simeq \mathrm{Id}_{\mathcal{A}'}$, we have canonical isomorphisms: $\mathfrak{G}_f^n \simeq q_* \circ \mathfrak{G}_h^n \circ q^*$. The complex (1) is isomorphic to

$$\mathfrak{C}(h, \mathrm{Id})q^* := (\mathrm{Id}_{\mathcal{A}'} \longrightarrow \mathfrak{G}_h \longrightarrow \mathfrak{G}_h^2 \longrightarrow \dots \longrightarrow \mathfrak{G}_h^n \longrightarrow \dots) \circ q^*. \tag{2}$$

Since the morphism h is faithfully flat, the complex $\mathfrak{C}(h, \mathrm{Id})$ is exact. Therefore the complex (2) is exact. □

C2. *Standard complex of a family of morphisms and localizations.* Here we shall discuss the compatibility of derived functors with certain localizations.

LEMMA C2.1. *Let* $f\colon \mathcal{B} \to \mathcal{A}$ *be continuous morphism; and let* $Q : \mathcal{A} \to \mathcal{A}'$ *be flat localization such that* f^* *factors through* Q : $f^* = f'^* \circ Q$.

(a) *Then* f'^* *is an inverse image functor of continuous morphism* $f' : \mathcal{B} \to \mathcal{A}'$.
(b) *If* $G_f := f_* \circ f^*$ *is an exact functor, then* $G_{f'} = f'_* \circ f'^*$ *is exact.*

In particular, if f *is a biflat localization, then* f' *is a biflat localization.*

Proof. (a) The statement is a consequence of Lemma I.1.3.1 in [GZ].

(b) Since $f^* = f'^* \circ Q$, $f_* \simeq Q\hat{} \circ f'_*$. So that $G_f \simeq Q\hat{} \circ G_{f'} \circ Q$; hence $Q \circ G_f \simeq G_{f'} \circ Q$. The latter isomorphism shows that the functor $Q \circ G_f$ factors through the localization Q. Since the functors Q and G_f are exact, $Q \circ G_f$ is exact and, therefore, $G_{f'}$ is exact by Propositions I.3.2 and I.3.4 in [GZ]. □

LEMMA C2.2. *Let* $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} | i \in J\}$ *be a set of biflat morphisms; and let* $Q : \mathcal{A} \to \mathcal{A}'$ *be a flat localization such that every* f_i *is compatible with* Q. *Then*

(a) *The induced localizations of* $\mathcal{A}'$, $\mathfrak{F}' = \{f'_i : \mathcal{B}'_i \to \mathcal{A}' | i \in J\}$, *are biflat.*
(b) *If* $Q(s)$ *is invertible for any arrow* s *of* $\mathcal{A}$ *such that* $f_i^*(s)$ *is invertible for all* $i \in J$, *then* $\mathfrak{F}$ *induces a biflat cover,* $\mathfrak{F}'$, *of* $\mathcal{A}'$.

Proof. The assertion follows from Lemma C2.1. □

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PROPOSITION C2.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $Q : \mathcal{A} \to \mathcal{A}'$ be a flat localization; and let $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} | i \in J\}$ be a finite set of biflat morphisms compatible with Q and such that* $\mathrm{Ker}(Q) = \bigcap_{i\in J} \mathrm{Ker}(f_i^*)$.

(i) *Then $\mathfrak{F}$ induces a biflat cover of $\mathcal{A}'$.*
(ii) *Suppose that $\mathcal{A}$ is an abelian category and each category $\mathcal{B}_i$ has enough injectives. Let a functor $F : \mathcal{A} \to \mathcal{C}$ be adapted to the family $\mathfrak{F}$. Then the standard complex $\mathfrak{C}(\mathfrak{F}, F)$ of the functor F with respect to the family $\mathfrak{F}$ is a resolution of the functor $F \circ Q\hat{}$.*

Proof. (i) Note that if the category $\mathcal{A}$ is abelian, the condition of the assertion *(b)* of Lemma C2.2 is equivalent to the inclusion: $\mathrm{Ker}(Q) \subseteq \bigcap_{i\in J} \mathrm{Ker}(f_i^*)$.

(ii) Recall that 'F is adapted to $\mathfrak{F}$ ' means that $F \circ f_{i*}$ is an exact functor for all $i \in J$. Since $f_{i*} = Q\hat{} \circ f_i'^*$ (in the notations of Lemma C2.1), F is adapted to $\mathfrak{F} = \{f_i | i \in J\}$ iff $F \circ Q\hat{}$ is adapted to $\mathfrak{F}' := \{f_i' | i \in J\}$. The statement follows now from Theorem 2.2. □

COROLLARY C2.4. *Suppose that the conditions of Proposition* 5.4.3 *hold. If the functor $F \circ Q\hat{}$ is exact, then $H^i(\mathfrak{C}(\mathfrak{F}, F)) = 0$ for all $i \geqslant 1$.*

C3. *A resolution related with an infinite cover.* Fix a family $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ of continuous morphisms. For each $i \in J$, denote by $\mathfrak{G}_i$ the composition $f_{i*} \circ f_i^*$ and by resp. η_i and ϵ_i adjunction arrows $\mathrm{Id}_{\mathcal{A}} \to \mathfrak{G}_i$ and $f_i^* \circ f_{i*} \to \mathrm{Id}_{\mathcal{B}}$. We can encode the family $\mathfrak{F}$ in one morphism $\mathbf{f}_J : \oplus_{i\in J}\mathcal{B}_i \to \mathcal{A}$ having the inverse image functor

$$\mathbf{f}_J^* : \mathcal{A} \longrightarrow \prod_{i\in J} \mathcal{B}_i, \quad X \longmapsto \prod_{i\in J} f_i^*(X). \tag{1}$$

LEMMMA C3.1. *Suppose that the category $\mathcal{A}$ has J-indexed products. Then the morphism $\mathbf{f}_J$ has a direct image functor: $\mathbf{f}_{J*}(\oplus_{i\in J} X_i) = \oplus_{i\in J} f_{i*}(X_i)$.*

Proof. Set for convenience $\mathcal{B}_J = \oplus_{i\in J}\mathcal{B}_i$. The adjunction arrow

$$\eta = \eta_{\mathbf{f}} : \mathrm{Id}_{\mathcal{A}} \longrightarrow \mathbf{f}_{J*}\mathbf{f}_J^* = \prod_{i\in J} f_{i*}f_i^*$$

is determined by the adjunction arrows $\eta_i, i \in J$. The adjunction arrow

$$\epsilon = \epsilon_{\mathbf{f}} : \mathbf{f}_J^*\mathbf{f}_{J*} \longrightarrow \mathrm{Id}_{\mathcal{B}_J}$$

assigns to any $(X_i) \in Ob\, \oplus_{i\in J} \mathcal{B}_i$ the composition of the natural projection

$$\mathbf{f}_J^*\mathbf{f}_{J*}(X_i) \longrightarrow (f_i^* f_{i*}(X_i))$$

and the product $(\epsilon_i : f_i^* f_{i*}(X_i) \to X_i)$ of adjunction morphisms ϵ_i. We leave the checking that η and ϵ are really adjunction arrows to a reader. □

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Note that if the family $\mathfrak{F}$ is biflat, i.e. the functors $f_{i*}f_i^*$ are exact for all $i \in J$, then the morphism $\mathbf{f}_J$ is biflat: the functor $\mathbf{f}_{J*}\mathbf{f}_J^* = \oplus_{i\in J} f_{i*}f_i^*$ is exact.

Note that even in the case of abelian categories, it is not true in general that the exactness of all $f_{i*}f_i^*,\ i \in J$, implies the exactness of $\mathbf{f}_{J*}\mathbf{f}_J^*$. Still we can use the standard resolution related to a family $\mathfrak{F}$ for a certain subcategory of $\mathcal{A}$ which, in the case when $\mathfrak{F}$ is a biflat cover could be thought as a full subcategory of sheaves with a compact support.

Denote by $\mathcal{A}_{\mathfrak{F}}$ the full subcategory of $\mathcal{A}$ generated by all objects X for which there exists a finite subset $J' = J'(X)$ of J with the following property:

(#) for any morphism $s : M \to L$ in $\mathcal{A}$ such that $f_i^*(s)$ is invertible for all $i \in J'$, the corresponding map $\mathcal{A}(s, X) : \mathcal{A}(L, X) \to \mathcal{A}(M, X)$ is bijective.

LEMMA C3.2. *Let $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ be a set* of *flat morphisms. Suppose that the categories $\mathcal{A}$ and $\mathcal{B}_i$ are abelian and the categories $\mathcal{B}_i$ have enough injectives. Then $\mathcal{A}_{\mathfrak{F}}$ is an abelian category with enough injectives.*

Sketch of proof. For any finite subset J' of J, denote by $\mathcal{A}_{J'}$ the full subcategory of $\mathcal{A}$ generated by $X \in Ob\mathcal{A}$ having the property (#). The category $\mathcal{A}_{\mathfrak{F}}$ is the union of the directed (with respect to $\subseteq$) set of subcategories $\mathcal{A}_{J'}$. Each of the subcategories $\mathcal{A}_{J'}$ is equivalent to the quotient category $\mathcal{A}/\mathbb{S}_{J'}$, where $\mathbb{S}_{J'}$ is the intersection of the kernels of functors $f_i^*,\ i \in J'$. In particular $\mathcal{A}_{J'}$ is an abelian category covered by subcategories $\mathcal{B}_i, i \in J'$. Since each $\mathcal{B}_i$ has enough injectives, $\mathcal{A}_{J'}$ has enough injectives, and these injectives are injectives of the category $\mathcal{A}$ (hence of $\mathcal{A}_{\mathfrak{F}}$) at the same time. □

Note that in general the category $\mathcal{A}_{\mathfrak{F}}$ does not have infinite direct sums.

THEOREM C3.3. *Let $\mathcal{A}$ be an abelian category. And let $\mathfrak{F} = \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J\}$ be an arbitrary set of biflat morphisms. Suppose each category $\mathcal{B}_i$ has enough injectives. And let $F : \mathcal{A} \to \mathcal{C}$ be a functor adapted to $\mathfrak{F}$. Then the standard complex $\mathfrak{C}(\mathfrak{F}, F)$ of the functor F with respect to the cover $\mathfrak{F}$ computes the values of the derived functors of the restriction of F to the subcategory $\mathcal{A}_{\mathfrak{F}}$.*

Sketch of proof. Fix a finite subset J' of J. The complex $\mathfrak{C}(\mathfrak{F}, F)$ restricted to the subcategory $\mathcal{A}_{J'}$ for some finite J' is (homotopically) equivalent to the complex $\mathfrak{C}(\mathfrak{F}_{J'}, F)$, where $\mathfrak{F}_{J'} := \{f_i : \mathcal{B}_i \to \mathcal{A} \mid i \in J'\}$. But $\mathfrak{F}_{J'}$ induces a finite biflat cover of the subcategory $\mathcal{A}_{J'}$. The assertion follows now from Theorem 2.2. □

C4. *Resolutions of functors.* Let $f\colon \mathcal{A} \to \mathcal{C}$ and $g\colon \mathcal{B} \to \mathcal{C}'$ be continuous morphisms. Suppose that the categories of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ are well defined (each of the categories in question is equivalent to a small category). The pair of morphisms f, g determines a continuous morphism $\Phi : \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ with an inverse and direct image functors resp. $\Phi^* : X \longmapsto g^* \circ X \circ f_*$ and

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$\Phi_* : Y \longmapsto g_* \circ Y \circ f^*$. Moreover, if f and g are localizations (i.e. f_* and g_* are fully faithful), then Φ is a localization too. This follows from the fact that the adjunction morphism $\Phi^* \circ \Phi_*(Y) := g^* \circ g_* \circ Y \circ f^* \circ f_* \to Y$ is an isomorphism for any Y if the adjunction arrows $f^* \circ f_* \to \mathrm{Id}_{\mathcal{C}}$ and $g^* \circ g_* \to \mathrm{Id}_{\mathcal{C}'}$ are isomorphisms.

LEMMA C4.1. (a) *Let* $\mathfrak{R} = (\ \mathrm{Id}_{\mathcal{B}} \to \mathfrak{R}_0 \to R_1 \to \dots)$ *be a resolution of the identical functor in* $\mathcal{B}$. *Then* $\tilde{}\mathfrak{R} : Y \longmapsto \mathfrak{R} \circ Y$ *is a resolution of* $\mathrm{Id}_{\mathrm{Fun}(\mathcal{A},\mathcal{B})}$.

(b) *If the resolution* $\mathfrak{R}$ *is adapted to* g; *i.e. the functors* $g_* \circ \mathfrak{R}_i$ *are exact for all* $i \geqslant 0$, *then the resolution* $\tilde{}\mathfrak{R}$ *is adapted to* $\Phi : \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}')$.

Proof. (a) The defining properties of $\mathfrak{R}$: the complex of endofunctors

$$0 \longrightarrow \mathrm{Id}_{\mathcal{B}} \longrightarrow \mathfrak{R}_0 \longrightarrow \mathfrak{R}_1 \longrightarrow \dots$$

is exact and each of the functors $\mathfrak{R}_i$ is exact, $i \geqslant 0$. Since the notions of a mono- and epimorphism and exactness of sequences for functors are defined object-wise, the functor $\tilde{}\mathfrak{R}$ has the same properties.

(b) By assumption $g_* \circ \mathfrak{R}_i$ is an exact functor for all $i \geqslant 0$. This implies that the functor $\Phi_* \circ \tilde{}\,\mathfrak{R}_i : \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}')$, $Y \longmapsto g_* \circ \mathfrak{R}_i \circ Y \circ f^*$, is exact for all $i \geqslant 0$. In particular, the resolution $\tilde{}\mathfrak{R}$ of the identical functor of $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ is adapted to the morphism Φ. □

C4.2. *Note.* Let $f : \mathcal{A} \to \mathcal{C}, g : \mathcal{B} \to \mathcal{C}$ be continuous morphisms. And let $\mathfrak{R}$ be a resolution of $\mathrm{Id}_{\mathcal{B}}$ such that $g_* \circ \mathfrak{R}_i$ is an exact functor for all $i \geqslant 0$.

(a) The functor $\Phi_* \circ \tilde{}\,\mathfrak{R}$ sends any right exact functor into a complex of right exact functors.

(b) If f is flat (i.e. f^* is exact), then $\Phi_* \circ \tilde{}\,\mathfrak{R}$ sends any (left) exact functor into a complex of (resp. left) exact functors. □

PROPOSITION C4.3. *Let* $f : \mathcal{A} \to \mathcal{C}, g : \mathcal{B} \to \mathcal{C}'$ *be continuous morphisms; and let* $\mathfrak{F}$ *be a biflat cover adapted to* g. *Let* $\mathfrak{R} = \mathfrak{G}(\mathfrak{F})$ *be the corresponding resolution of the identical functor* $\mathrm{Id}_{\mathcal{B}}$. *Then the resolution* $\tilde{}\mathfrak{R} : Y \longmapsto \mathfrak{G}(\mathfrak{F}) \circ Y$ *is adapted to the morphism*

$$\Phi : \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}'), \Phi_* : Y \longmapsto g_* \circ Y \circ f^*.$$

Proof. The assertion follows from Theorem 2.2 and Lemma C4.1. □

It follows from Proposition C4.3 that the derived functors of the direct image functor $\Phi_* : \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \to \mathrm{End}\mathcal{C}$ are isomorphic to the corresponding cohomology of the complex

$$\Phi_* \circ \tilde{}\,\mathfrak{G}(\mathfrak{F}) : Y \longmapsto g_* \circ \mathfrak{G}(\mathfrak{F}) \circ Y \circ f^*. \tag{1}$$

C5. *Cohomology of invertible sheaves on a skew projective space.* Return now to the setting of Section 3; i.e. fix a Γ-graded k-algebra R. For each $\gamma \in \Gamma$, we

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have an auto-equivalence ϑ_γ of the category $\mathfrak{gr}_\Gamma R-\text{mod}$ assigning to each graded module $M=\oplus_{\nu\in\Gamma}M_\nu$ the graded module $M(\gamma)$ defined by: $M(\gamma)_\nu:=M_{\gamma+\nu}$ for all $\nu\in\Gamma$. Clearly the 'torsion category' $\mathfrak{T}_+$ is invariant with respect to ϑ_γ for all $\gamma\in\Gamma$. Therefore ϑ_γ induces an auto-equivalence, $\mathcal{O}(\gamma)$, of the category $\mathbf{Proj}_\Gamma(R):=\mathfrak{gr}_\Gamma R-\text{mod}/T_+$. We call the auto-equivalences $\mathcal{O}(\gamma)$ *canonical line bundles* on $\mathbf{Proj}_\Gamma(R)$. One of the important problems is the computing cohomology of $\mathcal{O}(\gamma),\gamma\in\Gamma$.

We make these computations below, using the standard complex, for the skew projective space $\mathbf{P}^r_{\Gamma,k}$. The latter is by definition $\mathbf{Proj}_\Gamma(R)$, where $\Gamma=\mathbb{Z}^{r+1}$ and R is the algebra of skew polynomials in $r+1$ indeterminates (cf. Example 5.4).

Let R be the k-algebra of skew polynomials over a k-algebra A (cf. Example 5.4). We provide R with the canonical Γ-grading, $\Gamma=\mathbb{Z}^{r+1}$, assigning to any element of the algebra A the parity 0 and to any generator x_i the parity γ_i, where γ_i is the ith canonical generator of $\mathbb{Z}^{r+1}$.

PROPOSITION C5.1. (a) *The natural map* $R\to\oplus_{\gamma\in\Gamma}H^0(\mathcal{O}(\gamma))$ *is an isomorphism of the* Γ*-graded algebras.*

(b) $H^r(\mathcal{O}(\gamma))\simeq Ax^\gamma$ *if all the components of* γ *are negative, and* $H^r(\mathcal{O}(\gamma))=0$ *if some of the components of* γ *are nonnegative.*

(c) *There is a natural map* $H^0(\mathcal{O}(\gamma))\otimes_A H^r(\mathcal{O}(\nu)\to H^r(\mathcal{O}(\gamma+\nu))$ *for all* $\gamma,\nu\in\Gamma$ *which induces a perfect pairing*

$$H^0(\mathcal{O}(\gamma))\otimes_A H^r(\mathcal{O}(\mathbf{w}-\gamma)\to H^r(\mathcal{O}(\mathbf{w}))=Ax^{\mathbf{w}},$$

where $\mathbf{w}=(-1,-1,\dots,-1)$.

(b) $H^i(\mathcal{O}(\gamma))=0$ *for all* $\gamma\in\Gamma$ *if* $0<i<r$.

Proof. We shall use the standard (or ruther Čech) complex in an argument analogous to the proof of the Serre's theorem in [Ha] (Theorem III.5.1).

By Proposition C4.3 the cohomology of any sheaf $\mathcal{G}:\mathbf{Proj}_\Gamma(R)\to\mathbf{Proj}_\Gamma(R)$ can be computed as the cohomology of the standard complex

$$\tilde{\pi}_*\tilde{\mathfrak{C}}_+(\mathfrak{F})(\mathcal{G})):=\pi_*\mathfrak{C}_+(\mathfrak{F})\circ\mathcal{G}\circ\pi^*. \tag{1}$$

where $\mathfrak{F}$ is the canonical affine cover of $\mathbf{Proj}_\Gamma(R)$. Since the cover $\mathfrak{F}$ is semiseparated, the standard complex (1) is homotopically equivalent to the Čech complex

$$\tilde{\pi}_*\tilde{C}\mathfrak{G}_+(\mathfrak{F})(\mathcal{G}):=\pi_*\circ C\mathfrak{C}_+(\mathfrak{F})\circ\mathcal{G}\circ\pi^*. \tag{2}$$

The right hand side of the complex (2) is

$$\begin{aligned}\prod_{i\in J}\pi_*\circ\mathfrak{G}_{x_i}\circ\mathcal{G}\circ\pi^*\longrightarrow\prod_{\mathbf{i}\in J_{2<}}\pi_*\circ\mathfrak{G}_{x_{\mathbf{i}}}\circ\mathcal{G}\circ\pi^*\longrightarrow\cdots\\ \longrightarrow\prod_{\mathbf{i}\in J_{n<}}\pi_*\circ\mathfrak{G}_{x_{\mathbf{i}}}\circ\mathcal{G}\circ\pi^*\longrightarrow\dots\end{aligned} \tag{3}$$

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where, for each $\mathbf{i} = (i_1, \ldots, i_n)$, $x_{\mathbf{i}} := x_{i_1} \ldots x_{i_n}$; and $\mathfrak{G}_{x_{\mathbf{i}}} \simeq (x_{\mathbf{i}})^{-1}R\otimes_R$ regarded as an endofunctor of $\mathbf{Proj}_\Gamma(R)$.

Since the functor π^* is the composition of the tensoring $R\otimes_A$ and the localization $Q\colon \mathfrak{gr}_\Gamma R - \mathrm{mod} \to \mathbf{Proj}_\Gamma(R)$, the functor π_* is the composition of a right adjoint $Q\hat{}$ to the localization Q and the functor $\mathfrak{gr}_\Gamma R - \mathrm{mod} \to A - \mathrm{mod}$ assigning to any graded R-module its zero component, the functor

$$\pi_* \circ \mathfrak{G}_{x_{\mathbf{i}}} \circ \mathcal{G} \circ \pi^* : A - \mathrm{mod} \longrightarrow A - \mathrm{mod} \tag{4}$$

is isomorphic to the functor $(x_{\mathbf{i}})^{-1}\mathcal{G}(R(\gamma)_0\otimes_A$. In particular, if $\mathcal{G} = \mathcal{O}(\gamma)$ for some $\gamma \in \Gamma$, the functor (4) is isomorphic to $(x_{\mathbf{i}})^{-1}R(\gamma)_0\otimes_A$. Taking $\mathcal{G}$ equal the direct sum $\oplus_{\gamma\in\Gamma}\mathcal{O}(\gamma)$, we obtain that $\pi_* \circ \mathfrak{G}_{x_{\mathbf{i}}} \circ \mathcal{G} \circ \pi^* \simeq (x_{\mathbf{i}})^{-1}R\otimes_A$, where $(x_{\mathbf{i}})^{-1}R$ is provided with the natural Γ-grading. Set for convenience $R_{x_{\mathbf{i}}} := (x_{\mathbf{i}})^{-1}\mathrm{R}$. And let $\mathcal{G}$ denote $\oplus_{\gamma\in\Gamma}\mathcal{O}(\gamma)$. Then the Čech complex (3) is isomorphic to

$$C^{\cdot}(\mathfrak{F}, \mathcal{G})\otimes_A = \left(\prod_{i\in J} R_{x_i} \longrightarrow \prod_{\mathbf{i}\in J_{2<}} R_{x_{\mathbf{i}}} \longrightarrow \right.$$

$$\left. \cdots \longrightarrow \prod_{\mathbf{i}\in J_{n<}} R_{x_{\mathbf{i}}} \longrightarrow \cdots \longrightarrow R_{x_0\ldots x_r}\right) \otimes_A . \tag{5}$$

For any flat left A-module L, the cohomology of the complex $C^{\cdot}(\mathfrak{F}, R) \otimes_A L$ are isomorphic to $H^*(C^{\cdot}(\mathfrak{F}, R)) \otimes_A L$.

(a) One can see that the canonical morphism $R \to H^0(C^{\cdot}(\mathfrak{F}, R))$ is a monomorphism, since $\{x_i\}$ are not zero divisors (i.e. already $R \to R_{x_i}$ is a monomorphism for any i). Note that, for any $0 \leqslant i < m \leqslant r$, the sequence

$$0 \longrightarrow R \longrightarrow R_{x_i} \oplus R_{x_m} \longrightarrow R_{x_i x_m}$$

is exact. Therefore the sequence

$$0 \longrightarrow R \longrightarrow \prod_{i\in J} R_{x_i} \longrightarrow \prod_{0\leqslant i,m\leqslant r} R_{x_i x_m}$$

is exact.

(b) $H^r(C^{\cdot}(\mathfrak{F}, R))$ is the cokernel of

$$d_{r-1} : \prod_{0\leqslant k\leqslant r} R_{x_0\ldots x_{k-1}x_{k+1}\ldots x_r} \longrightarrow R_{x_0\ldots x_r} \tag{6}$$

Note that $R_{x_0\ldots x_r}$ is a free A-module with the basis $x^{\mathbf{i}}$, $\mathbf{i} \in \boldsymbol{\Gamma}$. The image of (6) is the free submodule of $R_{x_0\ldots x_r}$ generated by all $x^{\mathbf{i}}$, such that at least one of the components of $\mathbf{i} = (i_0, \ldots, i_r)$ is nonnegative. Therefore $H^r(C^{\cdot}(\mathfrak{F}, R))$ is a free

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A-module with the basis $x^{\mathbf{i}}$, where $\mathbf{i}$ runs through the set of elements of Γ with all components negative.

(c) By (a), $H^0(C^{\cdot}(\mathfrak{F},R)) = R = \prod_{\mathbf{i}\in\Gamma_{\geqslant 0}} Ax^{\mathbf{i}}$, where $\Gamma_{\geqslant 0}$ consists of all elements of Γ with nonnegative components. And, by (b), $H^r(C^{\cdot}(\mathfrak{F},R)) = \prod_{\mathbf{i}\in\Gamma_{<0}} Ax^{\mathbf{i}}$, where $\Gamma_{<0}$ is the set of elements of Γ with negative components. Therefore, we have a natural action

$$H^0(C^{\cdot}(\mathfrak{F},R)) \otimes_A H^r(C^{\cdot}(\mathfrak{F},R)) \longrightarrow H^r(C^{\cdot}(\mathfrak{F},R)), \tag{7}$$

determined by $x^{\mathbf{i}} \otimes x^{\mathbf{m}-\mathbf{i}} \longmapsto \chi(\mathbf{m})x^{\mathbf{m}}$, where $\chi(\mathbf{m}) = 1$ if $\mathbf{m} \in \boldsymbol{\Gamma}_{<0}$ and $\chi(\mathbf{m}) = 0$ if $\mathbf{m} \notin \boldsymbol{\Gamma}_{<0}$. Let $\mathbf{w}$ denote the element $(-1,-1,\ldots,-1)$. Then the map (7) determines a perfect pairing (Serre duality)

$$H^0(\mathcal{O}(\gamma)) \otimes_A H^r(\mathcal{O}(\mathbf{w}-\gamma) \longrightarrow H^r(\mathcal{O}(\mathbf{w})) = Ax^{\mathbf{w}}. \tag{8}$$

(d) $H^i(C^{\cdot}(\mathfrak{F},R)) = 0$ for $0 < i <$r. Localizing the complex $C^{\cdot}(\mathfrak{F},R)$ at (x_r), we get the complex $C^{\cdot}(\mathfrak{F}(x_r),R)$, where $\mathfrak{F}(x_r)$ is the canonical cover of the open subscheme $U(x_r) := (x_r)^{-1}$**Proj**(R) of **Proj**(R). Since $U(x_r) \simeq (R_{x_r})_0 - \mathrm{mod} \to A - \mathrm{mod}$ is affine, $H^i(C^{\cdot}(\mathfrak{F}(x_r),R) = 0$ for $i \geqslant 1$. Since the localization at (x_r) is an exact functor, $H^*(C^{\cdot}(\mathfrak{F}(x_r),R)) \simeq H^*(C^{\cdot}(\mathfrak{F},R))_{x_r}$ - the localization of $H^*(C^{\cdot}(\mathfrak{F},R)$ at (x_r)). Therefore the equality $H^i(C^{\cdot}(\mathfrak{F}(x_r),R)) = 0$ means that any element of $H^i(C^{\cdot}(\mathfrak{F},R))$ is annihilated by some power of x_r. It remains to show that, for any $0 < i < r$, the multiplication by x_r induces an injective map from $H^i(C^{\cdot}(\mathfrak{F},R))$ to itself.

The exact sequence of Γ-graded R-bimodules

$$0 \longrightarrow R(-\gamma_r) \xrightarrow{x_r} R \longrightarrow R/Rx_r \longrightarrow 0 \tag{9}$$

and the corresponding cohomological long exact sequence:

$$\cdots \longrightarrow H^i(\mathcal{G}(-\gamma_r)) \longrightarrow H^i(\mathcal{G}) \longrightarrow H^i(\mathcal{G}_{\mathcal{H}}) \longrightarrow H^{i+1}(\mathcal{G}(-\gamma_r)) \longrightarrow$$
$$\cdots \longrightarrow H^r(\mathcal{G}) \longrightarrow 0 \tag{10}$$

The quotient ring R/Rx_r is actually the Γ'-graded skew polynomial algebra, $\Gamma' = \mathbb{Z}^r$, with the Γ-grading induced by the projection $\Gamma = \mathbb{Z}^{r+1} \to \Gamma'$ sending γ_r to 0. Therefore, by the induction hypothesis, $H^i(\mathcal{G}_{\mathcal{H}}) = 0$ for $0 < i < r-1$ which implies that $x_r : H^i(\mathcal{G}(-\gamma_r)) \to H^i(\mathcal{G})$ is an isomorphism for $0 < i < r-1$. Since the multiplication by x_r is locally nilpotent, this means that $H^i(\mathcal{G}) = 0$ for $0 < i < r-1$.

Note that $H^i(\mathcal{G}(-\gamma_r)) \simeq H^i(\mathcal{G})$. Thus, for $i = 0$, we have an exact sequence

$$0 \longrightarrow H^0(\mathcal{G}(-\gamma_r))$$
$$= R(-\gamma_r) \longrightarrow H^0(\mathcal{G}) = R \longrightarrow H^0(\mathcal{G}_{\mathcal{H}}) = R/Rx_r \longrightarrow 0. \tag{11}$$

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So that $H^1(\mathcal{G})(-\gamma_r) = 0 = H^1(\mathcal{G})$. At the other end of the long exact sequence(10) we have the exact sequence

$$0 \longrightarrow H^{r-1}(\mathcal{G}_{\mathcal{H}}) \xrightarrow{\text{tial}_r} H^r(\mathcal{G}(-\gamma_r)) \xrightarrow{x_r} H^r(\mathcal{G}) \longrightarrow 0. \tag{12}$$

Indeed, $H^{r-1}(\mathcal{G}_{\mathcal{H}}) = \prod_{\mathbf{i}\in\Gamma'} Ax^{\mathbf{i}}$; $H^r(\mathcal{G}(-\gamma_r)) = \prod_{\mathbf{i}\in\Gamma_r} Ax^{\mathbf{i}}$, where Γ_r is the subset $\{\mathbf{i} = (i_0, \ldots, i_r) \in \Gamma | i_r = -1\}$; $H^r(\mathcal{G}) = \prod_{\mathbf{i}\in\Gamma} Ax^{\mathbf{i}}$. And the morphism tial_r is the dividing by x_r.

Therefore $H^{r-1}(\mathcal{G}) = 0$. □

For $\mathbf{Proj}_{\mathbb{Z}}(R)$, we have a direct analog of the classical result:

PROPOSITION C5.2. (a) *The natural map* $R \to \oplus_{n\in\mathbb{Z}} H^0(\mathcal{O}(n))$ *is an isomorphism of the* $\mathbb{Z}$*-graded algebras.*

(b) $H^r(\mathcal{O}(-r-1)) \simeq Ax^{-\mathbf{w}}$, *where* $\mathbf{w} = (-1, -1, \ldots, -1)$.

(c) *The natural map*

$$H^0(\mathcal{O}(n)) \otimes_A H^r(\mathcal{O}(-1-r-n) \longrightarrow H^r(\mathcal{O}(-r-1)) = Ax^{\mathbf{w}}$$

is a perfect pairing of free A*-modules of finite rank for all* $n \in \mathbb{Z}$.

(d) $H^i(\mathcal{O}(n)) = 0$ *for all* $n \in \mathbb{Z}$ *if* $0 < i <$r.

Proof. The assertions follow from the corresponding assertions of Proposition C5.1. The details are left to the reader. □

C5.3. *Generalizations:* Propositions C5.1 and C5.2 can be easily extended to the cases of skew projective spaces resp. $\mathbf{P}^r_{\Gamma,\vartheta}$ and $\mathbf{P}^r_{\vartheta}$ (cf. 5.4.2). We leave details to the reader.

Appendix: reconstruction of schemes

A.0. *Preliminaries on Spec.* Fix an abelian category $\mathcal{A}$ with the property.

(sup) For any ascending chain Ω of subobjects of an object M, the supremum of Ω exists; and for any subobject L of M it, the natural morphism

$$\sup\{X \bigcap L \mid X \in \Omega\} \longrightarrow (\sup \Omega) \bigcap L$$

is an isomorphism.

The categories with property (*sup*) are called otherwise *the categories with exact direct limits*. Recall three important examples of such categories:

(1) The category $R-$ mod of left modules over an associative ring R.
(2) The category of sheaves of R-modules on an arbitrary topological space.
(3) The category of quasi-coherent sheaves on a scheme.

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The first two are examples of *Grothendieck categories*. In particular, they have limits and colimits of small diagrams, a set of generators and enough injective objects. The category of quasi-coherent sheaves on a scheme is also a Grothendieck category, if the scheme in question is quasi-compact and quasi-separated. It is not known, however, if the category of quasi-coherent sheaves on an arbitrary scheme has enough injectives or even all limits ([TT], B.2). But, one can easily see that it has the property (sup).

In fact, the inclusion of the category Qcoh(**X**) on a scheme **X** into the category $\mathcal{O}_{\mathbf{X}}$-mod of $\mathcal{O}_{\mathbf{X}}$-modules is a fully faithful exact functor which reflects finite limits and all colimits. This implies that the category Qcoh(**X**) has all colimits and inherits the property (sup) from $\mathcal{O}_{\mathbf{X}}$-mod.

Recall that, for any two objects X, Y of $\mathcal{A}$, we write $X \succ Y$ if Y is a subquotient of a finite direct sum of copies of X (cf. Note 2.5.1). For any $X \in Ob\mathcal{A}$, denote by $\langle X\rangle$ the full subcategory of $\mathcal{A}$ such that $Ob\langle X\rangle = Ob\mathcal{A} - \{Y \in Ob\mathcal{A} \mid Y \succ X\}$. It is easy to check that $X \succ Y$ iff $\langle Y\rangle \subseteq \langle X\rangle$. This observation provides a convenient realization of the quotient of $(Ob\mathcal{A}, \succ)$ with respect to the equivalence relation induced by $\succ$: $X \approx Y$ if $X \succ Y \succ X$. Namely, $(Ob\mathcal{A}, \succ)/\approx$ is isomorphic to $(\{\langle X\rangle \mid X \in Ob\mathcal{A}\}, \supseteq)$.

Set Spec$\mathcal{A} = \{P \in Ob\mathcal{A} \mid P \neq 0$, and for any nonzero subobject X of $P, X \succ P\}$. *The spectrum*, **Spec**$\mathcal{A}$, *of the category* $\mathcal{A}$ is the preordered set of equivalence (with respect to $\succ$) classes of objects of Spec$\mathcal{A}$. The canonical realization of $(Ob\mathcal{A}, \succ)/\approx$ induces a canonical realization of **Spec**$\mathcal{A}$: (**Spec**$\mathcal{A} = \{\langle P\rangle \mid P \in$ Spec$\mathcal{A}\}, \supseteq)$.

PROPOSITION A.0.1. *For any* $P \in$ Spec$\mathcal{A}$, *the subcategory* $\langle P\rangle$ *is a Serre subcategory of* $\mathcal{A}$. *If* $\mathcal{A}$ *is a category with the property (sup), then the converse is true: if* X *is an object of* $\mathcal{A}$ *such that* $\langle X\rangle$ *is a Serre subcategory of* $\mathcal{A}$, *then* X *is equivalent (in the sense of* $\succ$*) to a* $P \in Spec\mathcal{A}$; *i.e.* $\langle X\rangle = \langle P\rangle$.

Proof. See Proposition 2.3.3 and 2.4.7 in [R]. □

A nonzero object X of a category $\mathcal{A}$ is called *quasifinal* if, for any nonzero object Y of $\mathcal{A}$, $Y \succ X$. The category $\mathcal{A}$ having a quasifinal objects is called *local*.

One can check that all simple objects of a local category (if any) are isomorphic to each other. In particular, the category of left modules over a commutative ring R is local iff the ring R is local.

PROPOSITION A.0.2. *For any* $P \in Spec\mathcal{A}$, *the quotient category* $\mathcal{A}/\langle P\rangle$ *is local.*

Proof. See Proposition 3.3.1 and Corollary 3.3.2 in [R]. □

PROPOSITION A.0.3. (a) *For any topologizing (i.e. full and closed with respect to taking direct sums and subquotients) subcategory* $\mathbb{T}$ *of* $\mathcal{A}$, *the inclusion functor* $\mathbb{T} \to \mathcal{A}$ *induces an embedding* **Spec**$\mathbb{T} \to$ **Spec**$\mathcal{A}$.

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(b) *For any exact localization* $Q : \mathcal{A} \to \mathcal{A}/\mathbb{S}$ *and for any* $P \in \mathrm{Spec}\mathcal{A}$, *either* $P \in Ob\mathbb{S}$, *or* $Q(P) \in \mathrm{Spec}\mathcal{A}/\mathbb{S}$; *hence* Q *induces an injective map from* **Spec**$\mathcal{A} - $ **Spec**$\mathbb{S}$ *to* **Spec**$\mathcal{A}/\mathbb{S}$.

A.0.4. *The support of an object.* For any $M \in Ob\mathcal{A}$, *the support of* M, **Supp**(M), consists of all $\langle P\rangle \in$ **Spec**$\mathcal{A}$ such that $M \notin Ob\langle P\rangle$.

A.0.5. *Localizations at subsets of the spectrum.* For any subset U of **Spec**$\mathcal{A}$, denote by $\langle\langle U\rangle\rangle$ the intersection $\bigcap_{\langle P\rangle\in U}\langle P\rangle$. Being the intersection of a set of Serre subcategories, $\langle\langle U\rangle\rangle$ is a Serre subcategory. *A localization at the subset* U is a localization at the Serre subcategory $\langle\langle U\rangle\rangle$.

A.0.6. *The topology* τ. We denote this way the strongest topology compatible with the preorder $\supseteq$ (recall that $\mathbf{P} \supseteq \mathbf{P}'$ means that $\mathbf{P}'$ is a specialization of $\mathbf{P}$). Its explicit description: the closure of a subset W of **Spec**$\mathcal{A}$ consists of all specializations of all points of W.

A.0.7. *The Zariski topology.* A subscheme $\mathbb{T}$ of an abelian category $\mathcal{A}$ (cf. C6.0) is *Zariski closed* or simply *closed* if it is a reflective subcategory of $\mathcal{A}$; i.e. the inclusion functor has a left adjoint. One can show that the family of sets **Spec**$\mathbb{T}$, where $\mathbb{T}$ runs through the class of closed subschemes of $\mathcal{A}$ can be regarded as a base of closed sets of a topology which is called the *Zariski topology* (cf. [R], III.6.3.1).

A.1. *A locally ringed space associated to a category.* Fix an abelian category $\mathcal{A}$. Suppose we have fixed also a topology $\mathcal{T}$ on **Spec**$\mathcal{A}$. Then we can associate to the pair $(\mathcal{A}, \mathcal{T})$ a ringed space $(X, \mathcal{O})$, where the underlying topological space X is (**Spec**$\mathcal{A}, \mathcal{T}$) and the 'structure' sheaf $\mathcal{O}$ is a sheaf associated to the presheaf $\mathfrak{O}$ which assigns to every open set U the center of the quotient category $\mathcal{A}/\langle\langle U\rangle\rangle$. Recall that the center of a category is the ring of endomorphisms of its identical functor.

We define a *strongly closed subscheme* as a closed subscheme $\mathbb{T}$ of $\mathcal{A}$ compatible with localization at points of **Spec**$\mathcal{A}$. The latter means that the canonical functor $\mathbb{T}/\mathbb{T} \cap \langle P\rangle \to \mathcal{A}/\langle P\rangle$ establishes an equivalence of $\mathbb{T}/\mathbb{T} \cap \langle P\rangle$ and a closed subscheme of $\mathcal{A}/\langle P\rangle$ for any $\langle P\rangle \in$ **Spec**$\mathcal{A}$. We define the *strong Zariski topology*, $\mathfrak{T}_3$, on **Spec**$\mathcal{A}$ as the weakest topology on **Spec**$\mathcal{A}$ such that the subset **Spec**$\mathbb{T}$ is closed for any strongly closed subscheme. Denote by $\mathcal{O}_{\mathcal{A}}$ the structure sheaf associated with the topology 3. One can show that $\mathcal{O}_{\mathcal{A}}$ is a sheaf of local rings.

THEOREM A.2. *Suppose that* $\mathcal{A}$ *is the category of quasi-coherent sheaves on a scheme* $\mathbf{X} = (X, \mathcal{O}_X)$. *Then the locally ringed space* $((\mathbf{Spec}\mathcal{A}, \mathfrak{T}_3), \mathcal{O}_{\mathcal{A}})$ *is isomorphic to* **X**.

The assertion is proved in [R2].

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