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# COINCIDENCE POINTS OF FUNCTION PAIRS BASED ON COMPACTNESS PROPERTIES

M. VÄTH

*University of Würzburg, Department of Mathematics, Am Hubland, D-97074 Würzburg, Germany*
*e-mail: vaeth@mathematik.uni-wuerzburg.de*

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**Abstract.** Existence theorems for the equation $F(x) = \varphi(x)$ are proved when $F$ is a function with "good" surjective properties and $\varphi$ satisfies certain compactness conditions on countable subsets of the space. Also results for certain homotopic perturbations of the equation are obtained. The results lead to various fixed point theorems of Darbo type for $F = id$, but they are also applicable if $F$ acts between different spaces. Also the inclusions $F(x) \in \varphi(x)$ (resp. $F(x) \subseteq \varphi(x)$) for multivalued functions $\varphi$ (resp. $F$ and $\varphi$) are studied. There are some connections with the theory of 0-epi maps.

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**1. Some coincidence theorems.** Let $X$ be a metric space, $\Omega \subseteq X$, and $Y$ be a metrizable locally convex space. Let $F{:}\overline{\Omega} \to Y$ be a "good" map in the sense that $F(x) = \varphi(x)$ has a solution $x$ in $\Omega$ for all continuous maps $\varphi{:}\overline{\Omega} \to Y$ that vanish on the boundary $B$ of $\Omega$ and with compact conv $\varphi(\overline{\Omega})$. (We shall see later that in the case $X = Y$, the map $F = id - x_0$ with $x_0 \in \Omega$ has this property; more generally, this is true for maps $F$ with nonzero degree $\deg(F, \Omega, 0)$.)

The following coincidence theorem states that then the equation $F(x) = \varphi(x)$ also has a solution $x$ in $\Omega$ if $\varphi$ is not necessarily compact but if certain compactness assumptions on countable sets are satisfied. Moreover, $\varphi$ need not vanish on the boundary. It suffices that $\varphi$ is homotopic to a map vanishing on the boundary via an homotopy $H$ which is admissible in the sense that $H$ has no coincidence points with $F$ on the boundary (and that $H$ satisfies certain compactness assumptions).

Thus, in a sense the following result may be interpreted as the homotopy invariance of a coincidence index. Note, however, that under the assumptions of the theorem no coincidence index needs to be defined: except for the compactness assumptions, we do not suppose any topological properties of the considered maps. $F$ need not even be continuous. In this sense, the result is a homotopic analogue to the homologic results in [**23**].

The importance of this result becomes clear in connection with 0-epi maps [**11**, **14**]: this is the class of maps with a zero which is stable under "admissible" compact homotopic perturbations, but in general not under noncompact perturbations. To attack this "flaw", in [**22**] the notion of $(0, k)$-epi maps was introduced. This has stronger stability properties. However, the problem was essentially left open in [**22**] as to how to verify that a given map is $(0, k)$-epi. Our result implies that any 0-epi map $F$ which is $k$-proper in the sense of [**22**] is also $(0, k_0)$-epi for each $k_0 < k$, even

$(0, k^-)$-epi. We discuss this more fully in Section 2. The following Theorem 1.1 is even more powerful, since it contains simultaneously the stability of such maps under certain homotopies: the combination of these two results leads to slightly weaker assumptions.

The author is aware that the result is rather technical, and so it is necessary to demonstrate its usefulness by some examples. However, we can give only some very simple applications in Section 2. More "natural" applications are rather complex and are given in the forthcoming papers [**12**,**21**,**26**]. In particular, the result in its general technical form is used in [**12**,**26**] as the main tool for a generalization of the two Hopf theorems (see e.g. [**28**]) on the connection between degree theory and homotopy theory for (countable) condensing maps.

For $F = id$, our result becomes of course a fixed point theorem. In fact, our coincidence result covers most known fixed point theorems involving compactness conditions like Darbo's fixed point theorem, as we shall see. But we may consider a much larger class of functions $F$. An important feature of the result is that we do not even need $X = Y$. (Important examples are monotone maps $F{:}X \to X^*$ and differential operators $F{:}C^1 \to C$ [**11**].)

For general functions $F$, one may consider Theorem 1.1 as a fixed point theorem for the multivalued map $F^{-1} \circ H(1, \cdot)$.

The result may also be considered as an extension of the coincidence theorem [**10**, Theorem 4.2.1]. (See Corollary 1.1 below.) However, our proof is rather different. An important advantage of the following Theorem 1.1 compared to [**10**, Theorem 4.2.1] is that we do not require that $F$ is onto, not even $F(\overline{\Omega}) \supseteq \varphi(\overline{\Omega})$; this is an unnatural requirement for 0-epi maps. In a sense, Theorem 1.1 has a similar meaning for 0-epi maps as [**10**, Theorem 4.2.1] has for so-called stably solvable maps. (For the latter, see [**2**,**10**].)

We think that our result reveals the deeper reason for all known fixed point and coincidence theorems dealing with maps which are noncompact but still have good compactness properties. This result is also the main technical tool needed for a natural definition of a spectrum for nonlinear operators [**21**,**24**].

We use the notation $F^{-1}(M) = \{x \in D : F(x) \in M\}$ for any $M \subseteq Y$, even if $M$ is not necessarily contained in the range of $F$. This convention implies in particular that the inclusion (2) below is always satisfied for the choice $U = D$ (which will be the "typical" choice for applications).

THEOREM 1. *Let $D$ be a metric space, $Y$ a metrizable locally convex space, and $B, O \subseteq D$ with $B$ closed ($B = \emptyset$ is not excluded). Let $H : [0,1] \times D \to Y$ be continuous, and $F : D \to Y$ be arbitrary. Assume that there is some $V \subseteq Y$ (possibly empty) with compact $\overline{\operatorname{conv}} V$, some $\lambda_0 \in [0,1]$ and some $U \subseteq D$ such that the following properties hold.*

1. *For any continuous $\varphi : D \to \overline{\operatorname{conv}}(H([0,1] \times (O \cap U)) \cup V)$ that satisfies $\varphi(x) = H(\lambda_0, x)$ for each $x \in B$ and with compact $\overline{\operatorname{conv}}\varphi(D)$, the equation $F(x) = \varphi(x)$ has a solution $x$ in $O$.*
2. *The union $S$ of all coincidence sets $\{x \in O \cap U : F(x) = H(\lambda, x)\}$ with $\lambda_0 \le \lambda < 1$ satisfies $\overline{S} \cap B = \emptyset$.*

   *This is satisfied if either $\lambda_0 = 1$, or if for each $x \in B \cap (\overline{O \cap U})$ the function $F$ is continuous at $x$ and satisfies $F(x) \neq H(\lambda, x)$ for all $\lambda \in [\lambda_0, 1]$.*
3. *There is some $A \subseteq U$ with $A \cap O \neq \emptyset$ and*

$$F(A) \cup H(\{\lambda_0\} \times B) \subseteq \overline{\mathrm{conv}}(H([0,1] \times (\overline{A \cap O})) \cup V). \tag{1}$$

4. *The set $U$ satisfies*

$$F^{-1}(\overline{\mathrm{conv}}(H([0,1] \times (U \cap O)) \cup V)) \subseteq U. \tag{2}$$

5. *For any subset $U_0 \subseteq U$ with*

$$F^{-1}(\overline{\mathrm{conv}}(H([0,1] \times (U_0 \cap O)) \cup V)) = U_0 \tag{3}$$

*and any countable $C \subseteq O \cap U_0$ the relation*

$$\overline{\mathrm{conv}(H([0,1] \times C) \cup V) \cap F(O)} \subseteq \overline{F(C)} \subseteq \overline{\overline{\mathrm{conv}}(H([0,1] \times C) \cup V) \cap F(O)} \tag{4}$$

*implies that $\overline{\mathrm{conv}}H([0,1] \times C)$ is compact.*

*Then the equation $F(x) = H(1, x)$ has a solution $x$ in $O \cap U$.*

Although the conditions appear very technical, they have a rather natural meaning, at least if $O = \Omega$, $B = \partial\Omega$, $D = \overline{\Omega}$, $F$ is continuous, and $H(\lambda_0, \cdot)|_B = 0$: then the condition 1 is a weakening of the assumption that $F$ be 0-epi [**11**,**14**] (see also Section 2), and condition 2 means that the homotopy $H$ is admissible in the sense that it has no coincidence points with $F$ on the boundary $\partial\Omega$. The conditions 3 and 4 are then satisfied with the choice $U = D$ and $V = \{0\}$ (choose $A = F^{-1}(\{0\})$), so that the compactness condition 5 is the only essential assumption in this situation. Note that this condition is satisfied if, roughly speaking, $H$ is "more compact than $F$ is proper". In this connection, it is interesting that it suffices to consider countable sets $C$, since for example in spaces of vector functions usually better "compactness estimates" are available on such sets. (See the remarks in Section 2.) Note that if certain a priori estimates are known, the conditions 3 and 4 may be satisfied also for smaller sets $U$ and $V$, and then the compactness requirement 5 becomes weaker. For example, if $0 \in H([0,1] \times F^{-1}(\{0\}))$, one might even choose $V = \emptyset$.

Before we attack the proof of Theorem 1.1, let us mention that if condition 1 of Theorem 1.1 is satisfied, one usually has that the set $F(O)$ is open in $Y$ or at least open in some subset $K \subseteq Y$ with $K \supseteq \overline{\mathrm{conv}}(H([0,1] \times (O \cap U)) \cup V) \cup F(O)$. (Typically, $K$ is a cone inducing some order in $Y$.) In this case, we may replace the strange relation (4) by a more natural equality.

Proposition 1.1. *Assume in the situation of Theorem 1.1 that the set $\overline{\mathrm{conv}}(H([0,1] \times (O \cap U)) \cup V) \setminus F(O)$ is closed. Then for any $C \subseteq O \cap U$ the relation* (4) *is equivalent to the equality*

$$\overline{F(C)} = \overline{\mathrm{conv}(H([0,1] \times C) \cup V) \cap F(O)}. \tag{5}$$

*Proof*. Putting $K = \overline{\mathrm{conv}}(H([0,1] \times (O \cap U)) \cup V)$, we have

$$\overline{\overline{A} \cap F(O)} = \overline{A \cap F(O)} \qquad (A \subseteq K). \tag{6}$$

Indeed, since $A \subseteq (A \cap F(O)) \cup (K \setminus F(O))$ and $K \setminus F(O)$ is closed, we have $\overline{A} \subseteq (\overline{A \cap F(O)}) \cup (K \setminus F(O))$ which implies that $\overline{A} \cap F(O) \subseteq \overline{A \cap F(O)}$, and (6) follows.

We get the statement for the particular choice $A = \mathrm{conv}(H([0, 1] \times C) \cup V)$. $\square$

Unless explicitly stated, we shall not make use of the axiom of choice. Instead, we require a weaker form, the so-called axiom of dependent choices throughout which allows countable (recursive or nonrecursive) choices (see e.g. [**15**]). This is the natural setting for applications and has the advantage that we are forced to give "countable constructive" proofs.

For this reason, we may not apply Dugundji's extension theorem [**7**]. (See also [**8**, Chap. IX, Theorem 6.1].) Nevertheless, we can use the following result.

LEMMA 1.1. *Let $Y$ be a metrizable locally convex space. If $K \subseteq Y$ is nonempty, compact, and convex, then there exists a retraction $\rho$ from $Y$ onto $K$, that is $\rho : Y \to K$ is continuous with $\rho(x) = x$ for $x \in K$.*

*Proof*. Using [**19**, Theorem 1.24], we may assume that $Y$ is equipped with a translation invariant metric. Then the completion $\overline{Y}$ of $Y$ is a locally convex Fréchet space. We may consider $K$ as a compact and convex subset of $\overline{Y}$. Since $K$ is separable, the constructive extension theorem from [**25**] implies that we may extend the identity of $K$ to a continuous map $R : \overline{Y} \to K$. Now, let $\rho = R|_Y$. $\square$

LEMMA 1.2. *If $A, B \subseteq Y$ are convex and compact, then $\overline{\mathrm{conv}}(A \cup B) = conv(A \cup B)$ and $\overline{\mathrm{conv}}(A + B) = A + B$ are compact.*

*In particular, if $\overline{\mathrm{conv}}\, V$ and $\overline{\mathrm{conv}}\, H([0, 1] \times C)$ are compact, it follows that $\overline{\mathrm{conv}}(H([0, 1] \times C) \cup V)$ is compact.*

*Proof*. To see that $C = (A \cup B)$ is compact, observe that the convexity of $A$ and $B$ imply that $C = g(A \times B \times [0, 1])$ where $g(a, b, \lambda) = \lambda a + (1 - \lambda)b$ is continuous. The proof of the convexity of $A + B$ is straightforward, and the compactness follows from $A + B = h(A \times B)$ where $h(a, b) = a + b$. The last statement follows from the first for $A = \overline{\mathrm{conv}}\, V$ and $B = \overline{\mathrm{conv}}\, H([0, 1] \times C)$, since $\overline{\mathrm{conv}}(H([0, 1] \times C) \cup V) \subseteq \overline{\mathrm{conv}}(A \cup B)$.

We emphasize that Lemma 1.2 is not a consequence of Mazur's Theorem [**9**, V.2, Theorem 6], because we do not require that $Y$ be complete.

*Proof of Theorem 1.1.* The statement in 2 will be proved in a more general setting in the proof of Theorem 1.4. Note that the continuity of $H(\lambda, \cdot)$ implies that $H(\{\lambda\} \times \overline{E}) \subseteq \overline{H(\{\lambda\} \times E)}$ for any $E \subseteq D$. Taking the union over all $\lambda \in [0, 1]$, we find that $H([0, 1] \times \overline{E}) \subseteq \overline{H([0, 1] \times E)}$. Hence

$$\begin{aligned}\overline{\mathrm{conv}}(H([0, 1] \times E) \cup V) = \overline{\mathrm{conv}}(\overline{H([0, 1] \times E)} \cup V) = \overline{\mathrm{conv}}(\overline{H([0, 1] \times \overline{E})} \cup V)\\ = \overline{\mathrm{conv}}(H([0, 1] \times \overline{E}) \cup V) \qquad (E \subseteq D). \qquad (7)\end{aligned}$$

In particular, (2) is equivalent to

$$F^{-1}(\overline{\mathrm{conv}}(H([0, 1] \times (\overline{U \cap O})) \cup V)) \subseteq U. \qquad (8)$$

Let $\mathfrak{U}$ denote the system of all sets $U \subseteq D$ that contain $A$ and satisfy (8). By assumption, $\mathfrak{U} \neq \emptyset$, and so $U_0 = \bigcap \mathfrak{U}$ is defined. Observe that $U_0 \in \mathfrak{U}$ since for each $U \in \mathfrak{U}$ we have, in view of $U_0 \subseteq U$, that

$$F^{-1}(\overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V))\subseteq F^{-1}(\overline{\text{conv}}(H([0,1]\times(\overline{U\cap O}))\cup V))\subseteq U. \quad (9)$$

Also the set $U_1 = F^{-1}(\overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V))$ belongs to $\mathfrak{U}$. Indeed, by (9) we have $U_1 \subseteq U_0$, and so $F^{-1}(\overline{\text{conv}}(H([0,1]\times(\overline{U_1\cap O}))\cup V))\subseteq F^{-1}(\overline{\text{conv}}\,(H([0,1]\times(\overline{U_0\cap O}))\cup V)) = U_1$. Moreover (1) implies, in view of $U_0 \supseteq A$, that $F(A)\subseteq \overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V) = U_1$, and so $A\subseteq U_1$. Since $U_1\in\mathfrak{U}$ and $U_1\subseteq U_0$, the definition of $U_0$ thus implies $U_0 = U_1$; that is,

$$F^{-1}(\overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V)) = U_0 \quad (10)$$

holds. By (7), this is equivalent to (3).

We claim that the set $\overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V)$ is compact. Otherwise it contains a sequence $\{y_n\}$ without a convergent subsequence. Since each $y_n$ is the limit of a sequence of (finite) convex combinations from $Z = H([0,1]\times(U_0\cap O))$, we find a countable subset $Z_0\subseteq Z$ with $y_n\in\overline{\text{conv}}Z_0$. In particular, we find a countable set $T\subseteq\overline{U_0\cap O}$ with $y_1, y_2, \ldots \in \overline{\text{conv}}(H([0,1]\times T)\cup V)$. We find some countable $C_1\subseteq U_0\cap O$ with $\overline{C}_1\supseteq T$, and then have $y_1, y_2, \ldots\in\overline{\text{conv}}(H([0,1]\times\overline{C}_1)\cup V)$. Then we may define recursively a sequence of countable sets $C_n\subseteq U_0\cap O$ satisfying the inclusions

$$C_n\subseteq C_{n+1}, \quad (11)$$

$$F(C_n)\subseteq\overline{\text{conv}}(H([0,1]\times\overline{C}_{n+1})\cup V), \quad (12)$$

and

$$\overline{\text{conv}}(H([0,1]\times\overline{C}_n)\cup V)\cap F(O)\subseteq\overline{F}(C_{n+1}) \quad (13).$$

Indeed, if $C_n$ is already defined, we have $C_n\subseteq U_0\subseteq F^{-1}(\overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V))$ by (10), and so $F(C_n)\subseteq\overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V)$. Hence, any of the countably many elements from $F(C_n)$ may be approximated by a sequence of (finite) convex combinations of elements from $H([0,1]\times(\overline{U_0\cap O}))\cup V$. Hence we can find some countable $E_n\subseteq\overline{U_0\cap O}$ with $F(C_n)\subseteq\overline{\text{conv}}(H([0,1]\times E_n)\cup V)$. Choose some countable $A_n\subseteq U_0\cap O$ with $\overline{A}_n\supseteq E_n$. Then any $C_{n+1}\supseteq A_n$ satisfies (12). Concerning (13), observe that $H_n = \overline{\text{conv}}(H([0,1]\times\overline{C}_n)\cup V)$ satisfies $F^{-1}(H_n)\subseteq F^{-1}(\overline{\text{conv}}(H([0,1]\times(\overline{U_0\cap O}))\cup V))\subseteq U_0$ by (10), and so $H_n\cap F(O)\subseteq F(U_0\cap O)$. Since $H_n$ is separable, we thus find a countable set $B_n\subseteq U_0\cap O$ such that $F(B_n)$ is dense in $H_n\cap F(O)$. Hence, we may choose $C_{n+1} = A_n\cup B_n\cup C_n$.

Put $C = \bigcup C_n$. Any $x\in(H([0,1]\times C)\cup V)$ is the convex combination of finitely many elements of $H([0,1]\times C)\cup V$. By (11), this means that we can find some $n$ such that $x$ is the convex combination of finitely many elements of $H([0,1]\times C_n)\cup V$. Hence $x\in(H([0,1]\times C_n)\cup V)$. If additionally $x\in F(O)$, then we have by (13) that $x\in\overline{F(C_{n+1})}\subseteq\overline{F(C)}$. Hence,

$$\text{conv}(H([0,1]\times C)\cup V)\cap F(O)\subseteq\overline{F(C)},$$

which means that $C$ satisfies the first inclusion of (4). For any $x\in F(C)$, we have by (12) that $x\in\overline{\text{conv}}(H([0,1]\times\overline{C}_n)\cup V)$ for some $n$, and so we have $F(C)\subseteq\overline{\text{conv}}(H([0,1]\times\,\overline{C})\cup V)$ which implies that

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$$\overline{F(C)} = \overline{F(C) \cap F(O)} \subseteq \overline{\overline{\mathrm{conv}}(H([0,1] \times \overline{C}) \cup V) \cap F(O)}.$$

This is also the second inclusion of (4) holds in view of (7). Since $C \subseteq U_0 \cap O$, the condition 5 of the theorem now implies by (7) and Lemma 1.2 that the set $\overline{\mathrm{conv}}(H([0,1] \times \overline{C}) \cup V)$ is compact. This contradicts the fact that this set contains the sequence $\{y_n\}$ which has no convergent subsequence. This shows that $\overline{\mathrm{conv}}(H([0,1] \times (\overline{U_0 \cap O})) \cup V)$ is indeed compact, as claimed.

Replacing the set $U$ from the assumption by the subset $U_0 \in \mathfrak{U}$ if necessary, we may thus assume without loss of generality that $U \in \mathfrak{U}$ has the property that $W = \overline{\mathrm{conv}}(H([0,1] \times (\overline{U \cap O})) \cup V)$ is compact. (Actually, the condition 5 means that we may pass to such a set $U \in \mathfrak{U}$; note that in general the originally given set $U$ does not have this property: consider for example $U = D$.)

By condition 2, the set $\overline{S}$ does not intersect $B = \overline{B}$, and so Urysohn's lemma provides a continuous function $\lambda : D \to [\lambda_0, 1]$ with $\lambda|_B \equiv \lambda_0$ and $\lambda|_S \equiv 1$. Then the function $\psi(x) = H(\lambda(x), x)$ is continuous and satisfies $\psi(x) = H(\lambda_0, x)$ for $x \in B$.

Put $M = \overline{O \cap U}$. Since $U \supseteq A$ and $O \cap A \neq \emptyset$, the sets $M$ and $W$ are not empty. In particular, Lemma 1.1 implies the existence of a retraction $\rho$ from $Y$ onto $W$. Now we define continuous functions $\varphi_0, f : D \to W$ by putting $\varphi_0(x) = \rho(\psi(x))$ and $f(x) = \rho(H(\lambda_0, x))$. Note that for $x \in M$ we have $\psi(x) \in W$, and so $\varphi_0(x) = \psi(x)$. Moreover, for $x \in B$, we have in view of (1) and $A \subseteq U$ that $H(\lambda_0, x) \in \overline{\mathrm{conv}}(H([0,1] \times (\overline{A \cap O})) \cup V) \subseteq W$, and so $f(x) = H(\lambda_0, x)$. Now we define $\varphi : D \to W$ in the following way: if $B = \emptyset$, put $\varphi = \varphi_0$, and otherwise

$$\varphi(x) = \begin{cases} \dfrac{\mathrm{dist}(x,B)}{\mathrm{dist}(x,M) + \mathrm{dist}(x,B)}\varphi_0(x) + \dfrac{\mathrm{dist}(x,M)}{\mathrm{dist}(x,M) + \mathrm{dist}(x,B)} f(x) & \text{if } x \in D \setminus (M \cap B), \\ f(x) & \text{if } x \in M \cap B. \end{cases}$$

Then $\varphi$ is continuous, and its range is contained in $\mathrm{conv}(\varphi_0(D) \cup f(D)) \subseteq \mathrm{conv}(W \cup W) = W$. Since $W$ is compact and $\varphi(x) = f(x) = H(\lambda_0, x)$ for $x \in B$, the assumption implies that the equation $F(x) = \varphi(x)$ has a solution $x \in O$. Since $F(x) = \varphi(x) \in W$, we have $x \in F^{-1}(W) = F^{-1}(\overline{\mathrm{conv}}(H([0,1] \times (\overline{U \cap O})) \cup V)) \subseteq U$, because $U \in \mathfrak{U}$. Hence, $x \in U \cap O \subseteq M$ which implies $F(x) = \varphi(x) = \varphi_0(x) = H(\lambda(x), x)$. Since $\lambda(x) \in [\lambda_0, 1]$, we thus must have either $x \in S$ or $\lambda(x) = 1$. In view of $\lambda|_S \equiv 1$, we have $\lambda(x) = 1$ in both cases. Hence, $x \in U \cap O$ is a desired solution. □

The set $U_0$ in the previous proof plays a similar role to the *V-ultimate fundamental* set in [**23**].

We point out that for the special case $\lambda_0 = 1$, we could have replaced the homotopy $H$ by a single function $G = H(1, \cdot)$ throughout (but the proof is not easier, except for the construction of the function $\lambda$). We shall see that this special case already contains the fixed point theorem of Darbo and many of its generalizations. This special case also contains the earlier mentioned coincidence result [**10**, Theorem 4.2.1].

We prove now a theorem which is "dual" to Theorem 1.1 where, roughly speaking, the compactness assumptions are not imposed on the "map $F^{-1} \circ H$" but instead on the "map $H \circ F^{-1}$". Note that $U$ is now a subset of $Y$ (not of $D$).

Theorem 1.2. *Let $D$ be a metric space, $Y$ a metrizable locally convex space, and $B, O \subseteq D$ with $B$ closed ($B = \emptyset$ is not excluded). Let $H : [0,1] \times D \to Y$ be continuous* *and $F: D \to Y$ be arbitrary. Assume that there is some $V \subseteq Y$ (possibly empty), some $\lambda_0 \in [0, 1]$ and some $U \subseteq Y$ such that the properties hold.*

1. *For any continuous function $\varphi : D \to \overline{\text{conv}}(H([0, 1] \times (F^{-1}(U) \cap O)) \cup V)$ that satisfies $\varphi(x) = H(\lambda_0, x)$ for each $x \in B$ and for which $\overline{\text{conv}}(\varphi(D))$ is compact, the equation $F(x) = \varphi(x)$ has a solution $x$ in $O$.*
2. *The union $S$ of all coincidence sets $\{x \in O : F(x) = H(\lambda, x) \in U\}$ with $\lambda_0 \leq \lambda < 1$ satisfies $\overline{S} \cap B = \emptyset$.*
*This is satisfied if either $\lambda_0 = 1$, or if for each $x \in B \cap (\overline{O \cap F^{-1}(U)})$ the function $F$ is continuous at $x$ and satisfies $F(x) \neq H(\lambda, x)$ for all $\lambda \in [\lambda_0, 1]$.*
3. *There is some $A \subseteq U$ with $F^{-1}(A) \cap O \neq \emptyset$ and*

$$A \cup H(\{\lambda_0\} \times B) \subseteq \overline{\text{conv}}(H([0, 1] \times (\overline{F^{-1}(A) \cap O})) \cup V). \tag{14}$$

4. *The set $U$ satisfies*

$$\overline{\text{conv}}(H([0, 1] \times (F^{-1}(U) \cap O)) \cup V) \subseteq U. \tag{15}$$

5. *Any subset $U_0 \subseteq U$ satisfying*

$$U_0 = \overline{\text{conv}}(H([0, 1] \times (F^{-1}(U_0) \cap O)) \cup V) \tag{16}$$

*is compact.*
*Then there is some $x \in O \cap F^{-1}(U)$ with $F(x) = H(1, x)$.*

We remark that, using a result from [**23**], it is possible also in Theorem 1.2 to formulate the compactness condition 5 in terms of countable sets. However, the latter is more technical than in Theorem 1.1 and appears not so useful for applications.

*Proof*. The proof is analogous to that of Theorem 1.1, and so we confine ourselves to sketching the main steps. In view of (7), the inclusion (15) is equivalent to

$$\overline{\text{conv}}(H([0, 1] \times (\overline{F^{-1}(U) \cap O})) \cup V) \subseteq U. \tag{17}$$

Let $U_0$ be the intersection of all sets $U \subseteq Y$ that contain $A$ and satisfy (17). As in the proof of Theorem 1.1, we find that

$$\overline{\text{conv}}(H([0, 1] \times (\overline{F^{-1}(U_0) \cap O})) \cup V) = U_0.$$

Hence, (7) implies that (16) holds. Replacing $U$ by $U_0$, we may thus assume that $U = U_0$ is compact and convex. Choose a continuous map $\lambda : D \to [\lambda_0, 1]$ with $\lambda|_B \equiv \lambda_0$ and $\lambda|_S \equiv 1$, and put $\psi(x) = H(\lambda(x), x)$. Note that $\psi(x) = H(\lambda_0, x)$ for $x \in B$.

Since $U \supseteq A$, the sets $M = \overline{F^{-1}(U) \cap O}$ and $U$ are not empty. Let $\rho$ be a retraction of $Y$ onto $U$, and define continuous functions $\varphi_0, f : D \to U$ by $\varphi_0(x) = \rho(\psi(x))$ and $f(x) = \rho(H(\lambda_0, x))$. For $x \in M$, we have $\psi(x) \in H([0, 1] \times (\overline{F^{-1}(U) \cap O})) \subseteq U$, and so $\varphi_0(x) = \psi(x)$. Moreover, for $x \in B$, we have in view of (14) and $A \subseteq U$ that $H(\lambda_0, x) \in \overline{\text{conv}}(H([0, 1] \times (\overline{F^{-1}(A) \cap O})) \cup V) \subseteq U$, and so $f(x) = H(\lambda_0, x)$. Now define $\varphi : D \to U$ by the same formula as in the proof of Theorem 1.1. Then $\varphi$ is continuous, and its range is contained in $\text{conv}(\varphi_0(D) \cup f(D)) \subseteq \text{conv}(U \cup U) = U = U_0 = \overline{\text{conv}}(H([0, 1] \times (F^{-1}(U_0) \cap O)) \cup V)$. Since $U$ is compact and $\varphi(x) = f(x) =$

$H(\lambda_0, x)$ for $x \in B$, the assumption 1 implies that the equation $F(x) = \varphi(x)$ has a solution $x \in O$. Since $F(x) = \varphi(x) \in U$, we have $x \in F^{-1}(U)$; so $x \in F^{-1}(U) \cap O \subseteq M$ which implies that $F(x) = \varphi(x) = \varphi_0(x) = H(\lambda(x), x)$. Since $\lambda(x) \in [\lambda_0, 1]$, we thus must have either $x \in S$ or $\lambda(x) = 1$. In view of $\lambda|_S \equiv 1$, we have $\lambda(x) = 1$ in both cases. Hence, $x \in F^{-1}(U) \cap O$ is a desired solution. □

We can also prove multivalued variants of the previous results. To this end, we need an extension theorem for upper semicontinuous maps. Such a result is Ma's generalization of Dugundji's extension theorem. See [**16**, (2.1)]. We give a constructive proof for the situation that we need.

By $\mathcal{K}(Y)$, we denote the set of all nonempty convex and compact subsets of $Y$. For multivalued maps $F : D \to 2^Y$, we put as usual $F(A) = \bigcup\{F(x) : x \in A\}$.

If we speak of completeness in a metrizable locally convex space $Y$, we have to take care how the metric is chosen. In the following Proposition 1.2, we assume to this end that the metric in $Y$ generates the same uniform structure as the countable family of seminorms generating the topology. (See, for example, [**19**].) These considerations will not be important later, since in our applications of Proposition 1.2, the set $M$ is compact and thus complete with respect to *any* metric generating the topology of $Y$.

Proposition 1.2. *Let $X$ be a metric space, and $Y$ be a locally convex metric space. Let $A \subseteq X$ be closed and separable, and $f : A \to \mathcal{K}(Y)$ be upper semicontinuous. If $M = \overline{\operatorname{conv}} f(A)$ is complete (in the sense described above), then $f$ has an upper semicontinuous extension to a function $F : X \to \mathcal{K}(Y)$ with $F(X) \subseteq M$.*

*Proof*. Let the metric in $Y$ be generated by the countable family $\|\cdot\|_k$ of seminorms. By our assumption, a sequence is a Cauchy sequence (resp. converges or is bounded) in $Y$ if and only if it is a Cauchy sequence (resp. converges or is bounded) with respect to each seminorm $\|\cdot\|_k$.

Since $A$ is separable, there exists a dense subset $\{a_1, a_2, \dots\} \subseteq A$. Choose a sequence of numbers $c_n > 0$ such that $\sum c_n$ and $\sum c_n s_n$ converge, where $s_n$ denotes the supremum of all numbers $\|y\|_k$ with $y \in f(a_k)$, where $k = 1, \dots, n$. For $x \in A$, put $F(x) = f(x)$, and for $x \notin A$, put $\lambda_n(x) = \max\{2 - d(x, a_n)/(x, A), 0\}$ and

$$F(x) = \left\{ \frac{\sum_{n=1}^{\infty} c_n \lambda_n(x) y_n}{\sum_{n=1}^{\infty} c_n \lambda_n(x)} \mid y_n \in f(a_n) \right\}. \tag{18}$$

This defines a subset of $M$. Indeed, let $y_n \in f(a_n)$. Since $\{a_1, a_2, \dots\}$ is dense in $A$, the expression

$$h_N = \frac{\sum_{n=1}^{N} c_n \lambda_n(x) y_n}{\sum_{n=1}^{N} c_n \lambda_n(x)}$$

is defined for sufficiently large $N$, and $h_N \in M$. Moreover, $h_N$ forms a Cauchy sequence with respect to each seminorm $\|\cdot\|_k$, since $\lambda_n$ is bounded by 2 and $\sum 2c_n\lambda_n\|y_{nk}\|$ and $\sum 2c_n$ converge. Hence, $h_N$ is a Cauchy sequence in $M$ and thus convergent. $F(x)$ consists precisely of all limits obtained in this way.

The set $F(x)$ is compact: if $z_j = \sum_n c_n\lambda_n(x)y_{n,j}$ with $y_{n,j} \in f(a_n)$, then we may by a diagonal argument pass to a subsequence of $j$ such that $y_{n,j} \to y_n \in f(a_n)$ $(j \to \infty)$ for

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each $n$. Using the majorant $\sum c_n\lambda_n(x)s_n$, we thus find that $z_j \to \sum_n c_n\lambda_n(x)y_n =: z$ with respect to each seminorm $||\cdot||_k$, and so $z_j \to z$ in the space $Y$.

Let us prove now that $F$ is upper semicontinuous at each $x_0 \in X$. We have to prove that for each $\varepsilon$ and each $k$ we can find some $\delta > 0$ such that $d(x, x_0) \le \delta$ and $y \in F(x)$ implies that there is some $y_0 \in F(x_0)$ with $||y - y_0||_k \le \varepsilon$.

In case $x_0 \notin A$, this is easily seen. The denominator in (18) depends continuously upon $x$, since $\lambda_n$ is continuous, and we have the majorant $\sum 2c_n$. An analogous argument for the numerator, using the majorant $\sum 2c_n s_n$, thus implies that $F$ is upper semicontinuous: we just associate to each $y \in F(x)$ with $x \notin A$ the value $y_0 \in F(x_0)$ that is generated corresponding to (18) with the same coefficients $y_n$ (since $A$ is closed, it suffices to consider points $x \notin A$).

It remains to consider the case $x_0 \in A$ and (since $F|_A$ is upper semicontinuous by assumption) we only have to consider points $x \notin A$. Thus, let $\varepsilon > 0$ and some $k$ be given. Since $f$ is upper semicontinuous, we can find some $\delta > 0$ such that $a_0 \in A$ and $d(x_0, a_0) \le 3\delta$ implies that for each $\tilde{z} \in f(a_0)$ there is some $\tilde{y} \in f(x_0)$ with $||\tilde{z} - \tilde{y}||_k \le \varepsilon$.

If $x \in X \setminus A$ and $n$ are such that $\lambda_n(x) \neq 0$, then $d(x, a_n) \le 2(x, A) \le 2d(x, x_0)$, and so $d(x_0, a_n) \le d(x_0, x) + d(x, a_n) \le 3d(x, x_0)$. Hence, if $x \in X \setminus A$ satisfies $d(x, x_0) \le \delta$ and $y \in F(x)$, say $y = \sum_{n=1}^{\infty} c_n\lambda_n(x)z_n / \sum_{n=1}^{\infty} c_n\lambda_n(x)$ with $z_n \in f(a_n)$, we find for each $n$ some $y_n \in f(x_0)$ with $\lambda_n(x)||z_n - y_n||_k \le \lambda_n(x)\varepsilon$. Putting $y_0 = \sum_{n=1}^{\infty} c_n\lambda_n(x)y_n / \sum_{n=1}^{\infty} c_n\lambda_n(x)$, we thus have

$$\left\|y - y_0\right\|_k = \left\|\frac{\sum_{n=1}^{\infty} c_n\lambda_n(x)(z_n - y_n)}{\sum_{n=1}^{\infty} c_n\lambda_n(x)}\right\|_k \le \varepsilon.$$

Hence, $F$ is upper semicontinuous at $x_0$. □

The construction of $F$ in Proposition 1.2 is a variation of a construction which is well known for single-valued maps with bounded images in finite-dimensional spaces (see [**6**, Proposition 1.1]).

We call a subset $M$ of a (not necessarily complete) metric space *precompact*, if its completion is compact; i.e. if each sequence in $M$ has a Cauchy subsequence.

Let $\mathcal{C}(Y)$ denote the system of all nonempty closed and convex subsets of $Y$.

THEOREM 1.3. *Theorem 1.1 holds if one considers in place of continuous functions $H$ and $\varphi$ upper semicontinuous multivalued functions $H : [0, 1] \times D \to \mathcal{C}(Y)$ and $\varphi : D \to \mathcal{K}(Y)$, respectively (replace "$F(x) =$" by "$F(x) \in$"), under the following modifications. The conditions 4 and 5 have to be replaced by 4′ and 5′, respectively.*

*4′. The set $U$ satisfies (8).*

*5′. For any subset $U_0 \subseteq U$ given by (10) we have that each value $H(\lambda, x)$ is separable for $0 \le \lambda \le 1$ and $x \in \overline{O \cap U_0}$, and that for any countable $C \subseteq O \cap U_0$ the relation*

$$\overline{\operatorname{conv}(H([0, 1] \times C) \cup V) \cap F(O)} \subseteq \overline{F(C)} \subseteq \overline{\overline{\operatorname{conv}}(H([0, 1] \times \overline{C}) \cup V) \cap F(O)} \qquad (19)$$

*implies that $\overline{\operatorname{conv}} H([0, 1] \times \overline{C})$ is compact.*

*Moreover, assume either the axiom of choice, or that the following two conditions hold.*

*6. $O \cap U_0$ is separable for any subset $U_0 \subseteq U$ satisfying (10), or the relation (19) for some countable $C \subseteq O \cap U_0$ implies that $C$ is precompact.*

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7. *B is separable, or the restriction* $f = H(\lambda_0, \cdot)|_B$ *has an extension to an upper semicontinuous function* $f: D \to \mathcal{C}(Y)$ *with* $f(D) \subseteq \overline{\text{conv}}(H([0,1] \times (\overline{A \cap O})) \cup V)$.

*If* $\overline{\text{conv}}(H([0,1] \times D) \cup V)$ *is compact and* $\overline{U \cap O} = D$*, then the assumptions 3–7 may be dropped.*

*Proof*. The proof is similar to the proof of Theorem 1.1. Define $\mathfrak{U}$ and $U_0$ as in the proof of Theorem 1.1. The same argument as in that proof shows that $U_0$ satisfies (10) and that $\overline{\text{conv}}(H([0,1] \times (\overline{U_0 \cap O})) \cup V)$ is compact. (For the construction of $C_n$ recall that upper semicontinuous maps in metric spaces with separable values map separable sets into separable sets [**25**, Lemma 1.1]). Moreover, if the last alternative of assumption 6 holds, an analogous argument shows that $U_0 \cap O$ is precompact. Indeed, otherwise $U_0 \cap O$ contains a sequence $x_n$ without a Cauchy subsequence. Put $C_1 = \{x_1, x_2, \ldots\}$ and proceed as in the proof of Theorem 1.1 to define a countable set $C \subseteq U_0 \cap O \subseteq U \cap O$ that contains $C_1$ such that (19) holds. The assumption implies that $C$ is precompact, contradicting the fact that $x_n \in C$.

Now replacing $U$ by $U_0$ if necessary, we may assume without loss of generality that $W = \overline{\text{conv}}(H([0,1] \times (\overline{U \cap O})) \cup V)$ is compact (hence $H: [0,1] \times (\overline{U \cap O}) \to \mathcal{K}(Y)$) and (if we do not want to assume the axiom of choice) that $M = \overline{O \cap U}$ is separable or even precompact or, alternatively, $M = D$. Observe that $A \subseteq U$ implies $\overline{\text{conv}}(H([0,1] \times (\overline{A \cap O})) \cup V) \subseteq W$, and so (1) implies that $f = H(\lambda_0, \cdot)|_B$ satisfies $f(B) \subseteq W$ (hence $f: B \to \mathcal{K}(Y)$). In view of our assumption 7, we may assume that $f$ has an upper semicontinuous extension to a function $f: D \to \mathcal{K}(Y)$ with $f(D) \subseteq W$.

Indeed, for the case $M = D$ this is trivial, since we may just put $f = H(\lambda_0, \cdot)$. In general, we have $f(B) \subseteq W$. If $B$ is separable, we may apply Proposition 1.2 to extend $f$, and if we assume the axiom of choice, we may apply Ma's generalization of Dugundji's extension theorem [**16**, (2.1)].

As in the proof of Theorem 1.1, we define continuous functions $\lambda: D \to [\lambda_0, 1]$ and $\psi(x) = H(\lambda(x), x)$ such that $\psi|_B = f|_B$ and $\lambda|_S \equiv 1$.

We define $\varphi_0$ differently from the proof of Theorem 1.1: we first put $\varphi_0(x) = \psi(x)$ for $x \in M$. Observing that $\varphi_0(M) \subseteq W$, we may use either Proposition 1.2 (if $M$ is separable) or Ma's theorem to extend $\varphi_0$ to an upper semicontinuous function $\varphi_0: D \to \mathcal{K}(Y)$ with $\varphi_0(D) \subseteq W$; in case $M = D$, no extension theorem is required, of course.

The rest of the proof is analogous to that of Theorem 1.1. □

Note that if we assume the axiom of choice, Theorem 1.3 contains Theorem 1.1 as a special case in view of the equality (7). However, the proof is somewhat different: the retraction argument used in the proof of Theorem 1.1 fails in the multivalued case, since it is not clear whether the composition $\rho \circ f$ of a multivalued function $f: \overline{\Omega} \to \mathcal{K}(Y)$ and of a retraction $\rho: Y \to W$ must attain convex values.

If the restriction of $H(\lambda, \cdot)$ to $\overline{U \cap O}$ is lower semicontinuous for each $\lambda \in [0,1]$, we have the equality (7), and thus may replace (19) by (4). In this case an analogue of Proposition 1.1 holds. However, without the lower semicontinuity of $H(\lambda, \cdot)$, we cannot prove an analogue to that proposition, since we do not know whether it is possible to replace (19) in Theorem 1.3 by the stronger relation

$$\overline{\text{conv}(H([0,1] \times \overline{C}) \cup V) \cap F(O)} \subseteq \overline{F(C)} \subseteq \overline{\overline{\text{conv}}(H([0,1] \times \overline{C}) \cup V) \cap F(O)}. \tag{17}$$

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The problem in the proof is that for $x \in (H([0,1] \times \overline{C}) \cup V)$ the relations $C_n \subseteq C_{n+1}$ and $C = \bigcup C_n$ do not imply that $x \in \overline{\text{conv}}(H([0,1] \times \overline{C}_n) \cup V)$ for some $n$.

For applications, the interesting part in the relations (4) resp. (19) is usually the second inclusion. If we confine ourselves to this part, we can even prove a result where also the map $F$ is multivalued. Here, we use for multivalued maps $F : D \to 2^Y$ the notation $F^{-1}(M) = \{x \in D : F(x) \subseteq M\}$ which we define for any $M \subseteq Y$ (even if $F(D) \not\subseteq M$).

Theorem 1.4. *Theorem 1.3 holds if we replace $F : D \to Y$ by an arbitrary function $F : D \to 2^Y$ (and "$F(x) \in$" by "$F(x) \subseteq$") provided that we replace (19) by*

$$\overline{F(C)} \subseteq \overline{\text{conv}}(H([0,1] \times \overline{C}) \cup V). \tag{20}$$

*The condition 2 is satisfied in this case if either $\lambda_0 = 1$, or if for each $x \in B \cap (\overline{O \cap U})$ one of the following two properties holds.*

(a) *$F$ is upper semicontinuous at $x$, $\overline{F(x)}$ does not intersect $H(\lambda, x)$ for each $\lambda \in [\lambda_0, 1]$, and $F$ attains nonempty values in a neighbourhood of $x$.*

(b) *$F$ is lower semicontinuous at $x$ and satisfies $F(x) \not\subseteq H(\lambda, x)$ for each $\lambda \in [\lambda_0, 1]$.*

*Proof.* For the statement concerning condition 2, assume there is some $x \in \overline{S} \cap B$. Then $x \in B \cap (\overline{O \cap U})$, and we find sequences $x_n \in O \cap U$ and $\lambda_n \in [\lambda_0, 1]$ with $x_n \to x$ and $F(x_n) \subseteq H(\lambda_n, x_n)$. Passing to a subsequence, we may assume that $\lambda_n \to \lambda \in [\lambda_0, 1]$. If the property (a) holds at $x$, then $\overline{F(x)}$ is disjoint from the compact set $H(x, \lambda)$, and so we find disjoint neighbourhoods $U_1$ and $U_2$ of $F(x)$ and $H(x, \lambda)$, respectively. The upper semicontinuity of $F$ and $H$ then implies that $F(x_n) \subseteq U_1$ and $H(\lambda_n, x_n) \subseteq U_2$ for sufficiently large $n$, so that $F(x_n)$ and $H(\lambda_n, x_n)$ are disjoint. This contradicts $F(x_n) \subseteq H(\lambda_n, x_n)$, since $F(x_n) \neq \emptyset$ for sufficiently large $n$, by assumption. If the property (b) holds, we find some $y \in F(x)$ that is not contained in the compact set $H(\lambda, x)$. Hence, we find disjoint neighbourhoods $U_1$ and $U_2$ of $y$ and $H(\lambda, x)$, respectively. For sufficiently large $n$, the lower semicontinuity of $F$ implies $F(x_n) \cap U_1 \neq \emptyset$ while the upper semicontinuity of $H$ gives $H(\lambda_n, x_n) \subseteq U_2$. This contradicts $F(x_n) \subseteq H(\lambda_n, x_n)$.

The proof of the theorem is analogous to that of Theorem 1.3. The only problems arise in the construction of the set $C$ in that proof, if $F(x)$ is separable for each $x \in O \cap U_0$. It suffices to define inductively, starting from any countable set $C_1 \subseteq O \cap U_0$, a sequence of countable sets $C_n \subseteq O \cap U$ satisfying (11) and (12).

If $C_n$ is already defined, we have $C_n \subseteq U_0 \subseteq F^{-1}(\overline{\text{conv}}(H([0,1] \times (\overline{U_0 \cap O})) \cup V))$ by (10). Since $F(C_n)$ is the countable union of separable sets and thus separable, it contains a countable dense subset $F_n$. Since $F_n \subseteq F(C_n) \subseteq \overline{\text{conv}}(H([0,1] \times (\overline{U_0 \cap O})) \cup V)$, each of the countably many elements from $F_n$ may be approximated by a sequence of (finite) convex combinations of elements from $H([0,1] \times (\overline{U_0 \cap O})) \cup V$, i.e. we can find some countable $E_n \subseteq \overline{U_0 \cap O}$ with $F_n \subseteq \text{conv}(H([0,1] \times E_n) \cup V)$. Choose some countable $A_n \subseteq U_0 \cap O$ with $\overline{A}_n \supseteq E_n$. Then $F(C_n) \subseteq \overline{F}_n \subseteq \overline{\text{conv}}(H([0,1] \times \overline{A}_n) \cup V)$. Hence, we may choose $C_{n+1} = A_n \cup C_n$.

Note that it is not clear whether we can construct the sequence $C_n$ such that additionally (13) holds. The argument from the proof of Theorem 1.1 fails, because the relation $F^{-1}(H_n) \subseteq U_0$ does not imply $H_n \cap F(O) \subseteq F(U_0 \cap O)$. However, we do not need (13), since we are not interested in the first inclusion of (19). $\square$

Also Theorem 1.4 has a "dual" version (for the proof one just has to repeat the proof of Theorem 1.2 with the changes sketched in the proof of Theorems 1.3 and 1.4).

THEOREM 1.5. *Theorem 1.2 holds if one replaces $F: D \to Y$ by an arbitrary function $F: D \to 2^Y$, and $H$, $\varphi$ by upper semicontinuous functions $H: [0,1] \times D \to \mathcal{C}(Y)$ respectively $\varphi: D \to \mathcal{K}(Y)$ (replace "$F(x) =$" by "$F(x) \subseteq$") under the following modifications.*

*The relations* (15) *and* (16) *have to be replaced by* (17) *and*

$$U_0 = \overline{\operatorname{conv}}(H([0,1] \times (\overline{F^{-1}(U_0) \cap O})) \cup V), \tag{21}$$

*respectively. Moreover, assume either the axiom of choice, or the following two conditions.*

6. *$F^{-1}(U_0) \cap O$ is separable for any subset $U_0 \subseteq U$ that satisfies* (21).

7. *$B$ is separable, or the restriction $f = H(\lambda_0, \cdot)|_B$ has an extension to an upper semicontinuous function $f: D \to \mathcal{C}(Y)$ with $f(D) \subseteq \overline{\operatorname{conv}}(H([0,1] \times (\overline{F^{-1}(A) \cap O})) \cup V)$.*

*The condition 2 is satisfied if either $\lambda_0 = 1$, or if for each $x \in B \cap (\overline{F^{-1}(U) \cap O})$ one of the alternatives (a) or (b) from Theorem 1.4 holds.*

Let us close this section by showing that the earlier mentioned result [**10**, Theorem 4.2.1] is indeed a special case of our coincidence theorems. We have a slightly more general result.

COROLLARY 1.1. *Let $X$ be a metric space, $Y$ a locally convex Fréchet space, $K \subseteq Y$ closed and convex, and $F: X \to Y$ be such that the equation $F(x) = \varphi(x)$ has a solution $x$ in $X$ for any continuous $\varphi: X \to K$ with precompact range. Let $D \subseteq X$ be closed, and $G: D \to K$ be continuous. Assume that*

1. *$F^{-1}(\overline{\operatorname{conv}} G(D)) \subseteq D$,*

2. *there is some nonempty precompact $V \subseteq G(D)$ such that for any countable $C \subseteq D$ the relation*

$$\overline{F(C)} = \overline{\operatorname{conv}}(G(C) \cup V) \tag{22}$$

*implies that $\overline{C}$ is compact.*

*Then the equation $F(x) = G(x)$ has a solution $x$ in $D$.*

*Proof*. We apply Theorem 1.1 with $O = U = D$, $B = \emptyset$, $H(\lambda, \cdot) = G$, $\lambda_0 = 1$, and $A = F^{-1}(V)$. Putting $M = \overline{\operatorname{conv}}(H([0,1] \times (\overline{O \cap U})) \cup V)$, we have $M = \overline{\operatorname{conv}} G(D) \subseteq K$, and so $F^{-1}(M) \subseteq U$. Since the range of $F$ must contain $K$, $A$ is a nonempty subset of $O = U$, and we have $F(U) \supseteq M$. Then assumptions 2, 3, and 4 of Theorem 1.1 are readily verified. Concerning 5, note that for any $C \subseteq D$ the relation (4) is in our case equivalent to (22), since $F(O) = F(U) \supseteq M$. Moreover, if $\overline{C}$ is compact, then also $\overline{\operatorname{conv}}(H([0,1] \times \overline{C}) \cup V) = \overline{\operatorname{conv}}(G(\overline{C}) \cup V)$ is compact by Mazur's lemma, since $G$ maps compact sets into compact sets.

We now attack assumption 1 of Theorem 1.1. If we assume the axiom of choice or if $D$ is separable, this assumption is satisfied. Indeed, if $\varphi: D \to M$ is continuous with compact $W = \overline{\operatorname{conv}} \varphi(D)$, we may then extend $\varphi$ to a continuous function $\varphi: X \to W$. Then $F(x) = \varphi(x)$ has a solution $x \in X$ by assumption, and we have $x \in F^{-1}(W) \subseteq F^{-1}(M) \subseteq U = O$.

The general case can be reduced to the case of separable $D$ (and so the axiom of choice is not required for our proof): under the verified assumptions, we have seen in the proof of Theorem 1.3 that we may pass to a subset $U_0$ of $U$ that still contains $A$ and satisfies (8) but which has the additional property that $U_0 = U_0 \cap O$ is pre-compact. The assumptions of our corollary are then even satisfied if we replace $D$ by $\overline{U}_0$. Indeed, (8) (for $U_0$) implies that $F^{-1}(\overline{\operatorname{conv}} G(\overline{U}_0)) \subseteq \overline{U}_0$, and in view of the surjectivity of $F$ and $A \subseteq U_0$, we have $V = F(A) \subseteq F(\overline{U}_0)$. Since $D_0 = \overline{U}_0$ is separable, the statement now follows by what we already have proved. $\square$

**2. $f$-epi maps.** Throughout this section, let $X$ be a metric space, $M \subseteq X$, and $\Omega \subseteq M$. Topological notions are always understood with respect to the relative topology on $M$. In particular, $\overline{\Omega}$ and $\partial\Omega$ denote the closure and boundary of $\Omega$ with respect to the metric space $M$. Let $Y$ be a metrizable locally convex space, and $K$ a closed convex subset of $Y$. By $\mathcal{K}(K)$, we denote the system of all nonempty convex and compact subsets of $K$. Depending on the context, let either $f : \partial\Omega \to K$ be continuous, or $f : \partial\Omega \to \mathcal{K}(K)$ be upper semicontinuous. In both cases, we assume that $\overline{\operatorname{conv}} f(\partial\Omega)$ is compact.

Definition 2.1. We call a map $F : \overline{\Omega} \to Y$

1. *$f$-epi* (on $\Omega$ with respect to $M$ and $K$) if for any continuous function $\varphi : \overline{\Omega} \to K$ with $\varphi|_{\partial\Omega} = f$ for which $\overline{\operatorname{conv}} \varphi(\Omega)$ is compact, the equation $F(x) = \varphi(x)$ has a solution $x$ in $\Omega$.
2. *$f$-multiepi* (on $\Omega$ with respect to $M$ and $K$) if for any upper semicontinuous function $\varphi : \overline{\Omega} \to \mathcal{K}(K)$ with $\varphi|_{\partial\Omega} = f$ for which $\overline{\operatorname{conv}} \varphi(\Omega)$ is compact, the inclusion $F(x) \in \varphi(x)$ has a solution $x$ in $\Omega$.
3. *$f$-admissible* if $F(x) \neq f(x)$ (resp. $F(x) \notin f(x)$) for $x \in \partial\Omega$.

If $f(x) \equiv p \in K$, we call $F$ *$p$-epi*, *$p$-multiepi*, or *$p$-admissible*, respectively.

We point out that in contrast to the usual definition in the literature, we do not require that 0-epi maps be 0-admissible.

The example mentioned in the beginning of the paper is the case $p = 0$: a map $F$ is 0-epi if and only if the equation $F(x) = \varphi(x)$ has a solution $x$ in $\Omega$ for any continuous map $\varphi$ vanishing on the boundary $\partial\Omega$ with compact $\overline{\operatorname{conv}} \varphi(\overline{\Omega})$.

In this connection, we emphasize that if $\varphi$ is as in Definition 2.1, then $\overline{\operatorname{conv}} \varphi(\overline{\Omega})$ is compact (not only $\overline{\operatorname{conv}} \varphi(\Omega)$). Indeed, this follows from Lemma 1.2 with $A = \overline{\operatorname{conv}} \varphi(\Omega)$, $B = \overline{\operatorname{conv}} f(\partial\Omega)$, since $\varphi(\overline{\Omega}) \subseteq A \cup B$.

If $K$ is a subspace of $Y$, the general case can be reduced to the case $f \equiv 0$: a map $F$ is $f$-epi if and only if $F - f$ is 0-epi. This follows easily by replacing $\varphi$ with $\varphi + f$ respectively $\varphi - f$ in Definition 2.1. Note, however, that if $K \pm K \not\subseteq K$ the function $\varphi \pm f$ need not take values in $K$.

The concept of 0-epi mappings has been introduced and developed by M. Furi, M. Martelli, M. P. Pera, A. Vignoli, and others. See, for example, [**11**]. We refer to the recent monograph [**14**] and its references. In the cited references, the definition of 0-epi maps is usually restricted to 0-admissible continuous maps with $M = X$ and $K = Y$ being Banach spaces and bounded $\Omega \subseteq X$. However, it turns out to be useful to consider also the case that $M$ and $K$ are cones defining the order in Banach spaces, since this leads to existence results for positive solutions; see [**14**]. All these cases are of course contained in the above Definition 2.1.

The generalization to 0-multiepi maps may appear somewhat artificial, and it is not clear whether the class of 0-epi maps and of 0-multiepi maps differ. (We prove in

[3] that they do not differ for proper maps.) However, at least for multivalued functions $f$ the class of $f$-multiepi maps has no single-valued analogue, and this case is of some interest in view of the (multivalued) fixed point theorems below.

The class of $p$-epi maps can be considered as a generalization of maps $F$ with nonzero degree $\deg(F, \partial\Omega, p)$, and this class has similar properties like invariance under admissible compact homotopies. However, there exist also 0-epi maps for which the degree is not defined, if $X \neq Y$. In [12] we prove (using Theorem 1.1) that for so-called strictly condensing perturbations of the identity the class of 0-epi maps coincides with the class of maps which have nonzero degree on a component of $\Omega$. (The fact that the connectedness plays a role is seen by the simple example $K = M = X = Y = \mathbb{R}$, $\Omega = (-2, -1) \cup (1, 2)$ and $F(x) = x^2 - 2$ which was given in [11]).

Actually, the coincidence theorems of Section 1 may be considered as a generalization of the homotopy invariance of 0-epi maps. In fact, these theorems immediately imply the following result.

We call the closed convex set $K \subseteq Y$ a *cone*, if $K + K \subseteq K$ and $0 \in K$. Note that we do not require that $K \cap (-K) = \{0\}$ so that in particular $K$ may be a closed subspace of $Y$ (or $K = Y$).

Corollary 2.1 *Let $K$ be a cone, and $F : \overline{\Omega} \to Y$ be continuous and $f$-epi (resp. $f$-multiepi). Let $H : [0, 1] \times \overline{\Omega} \to K$ be continuous with $H(0, \cdot) = 0$ such that each of the functions $F_\lambda(x) = F(x) - H(\lambda, x)$ is $f$-admissible. If $\overline{\operatorname{conv}} H([0, 1] \times \overline{\Omega})$ is compact, then all $F_\lambda$ are $f$-epi (resp. $f$-multiepi).*

*Proof*. Let $\varphi : \overline{\Omega} \to K$ be continuous with $\varphi|_{\partial\Omega} = f$ and compact $\overline{\operatorname{conv}}\, \varphi(\overline{\Omega})$ (Lemma 1.2). Now fix $\lambda \in [0, 1]$ and apply Theorem 1.1 with the homotopy $H_0(t, x) = H(t\lambda, x) + \varphi(x)$ (and $U = D$, $V = \emptyset$). Since Lemma 1.2 implies that $\overline{\operatorname{conv}}\, H_0([0, 1] \times \overline{\Omega})$ is compact, we find that there is some $x \in \Omega$ satisfying $F(x) = H_0(\lambda, x)$ which means $F_\lambda(x) = \varphi(x)$. If $F$ is $f$-multiepi, the statement follows analogously, using Theorem 1.3. □

The standard fixed point theorems imply the following result.

Proposition 2.1 (Normalization property). *Let $X = Y$, $M = K$, and $\Omega \neq \emptyset$. If the set $\overline{\operatorname{conv}} f(\partial\Omega)$ is a compact subset of $\Omega$, then $F = id$ is $f$-epi and $f$-multiepi.*

*In particular, $F = id$ is $p$-epi (and $p$-multiepi) if and only if $p \in \Omega$.*

*Proof*. Let $\varphi : \overline{\Omega} \to K$ be continuous with $\varphi|_{\partial\Omega} = f$ and compact $A = \overline{\operatorname{conv}} \varphi(\overline{\Omega})$ (Lemma 1.2). Since the compact set $A \cap \partial\Omega$ is separable, Proposition 1.2 implies that we may extend the restriction of $\psi = \varphi|_{A \cap \partial\Omega}$ to a continuous function $\psi : A \to \overline{\operatorname{conv}}\, f(\partial\Omega)$. Observe that $f(\partial\Omega) = \varphi(\partial\Omega) \subseteq A$, and so $\psi : A \to A$. Hence, we may define a function $G : A \to A$ by putting $G(x) = \psi(x)$ for $x \in A \setminus \Omega$ and $G(x) = \varphi(x)$ for $x \in A \cap \Omega$. Note that the boundary of $A \cap \Omega$ in the metric space $A$ is contained in $A \cap \partial\Omega$, and so $G$ is continuous. Thus, the fixed point theorem of Tychonoff (respectively of Schauder if $X = Y$ is a Banach space) implies that $G$ has some fixed point $x$ in $A$. We have either $x \in A \cap \Omega$ or $x = G(x) = \psi(x) \in \overline{\operatorname{conv}}\, f(\partial\Omega) \subseteq \Omega$. Hence, we have in both cases $x \in A \cap \Omega$ and thus $F(x) = x = G(x) = \varphi(x)$. An analogous argument shows that $F = id$ is even $f$-multiepi (apply the fixed point theorem of Ky Fan). □

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REMARK 2.1. If $f(\partial\Omega) \subseteq \Omega$ but not necessarily $\overline{\operatorname{conv}} f(\partial\Omega) \subseteq \Omega$, we still find that $F = id$ is $f$-epi, provided that $f$ has an extension to a continuous function $f: (M \setminus \Omega) \cup \partial\Omega \to \Omega$ such that $\overline{\operatorname{conv}} f(M \setminus \Omega)$ is compact.

Indeed, putting $G(x) = \varphi(x)$ for $x \in \Omega$ and $G(x) = f(x)$ for $x \in M \setminus \Omega$, we have a continuous self-map $G$ of the nonempty, closed, and convex set $M$ with compact $\overline{\operatorname{conv}}\, G(M)$. Hence, $G$ has a fixed point in $M$ which must belong to $G(M) \subseteq \Omega$.

An analogous argument can of course be used to prove that $F = id$ is $f$-multiepi. In particular, this argument can be used to simplify the proof of Proposition 2.1 for the case $f(x) \equiv p$ (or even the general proof, if we assume the axiom of choice, so that we may apply Dugundji's extension theorem).

The coincidence Theorem 1.1 from Section 1 implies in view of Proposition 2.1 the following fixed point Theorem 2.1. Note that the only ingredient needed for its proof is the fixed point theorem of Tychonoff (respectively of Schauder if $X = Y$ is a Banach space).

THEOREM 2.1. *Let $X = Y$, and $\Omega$ be nonempty and open in $M = K$. Put $B = \overline{\Omega} \setminus \Omega$. Let the function $H: [0, 1] \times \overline{\Omega} \to K$ be continuous. Assume there is some $V \subseteq K$ with compact $\overline{\operatorname{conv}}\, V \supseteq H(\{0\} \times B)$ and there is some $U \subseteq \overline{\Omega}$ with $\overline{\Omega} \cap \overline{\operatorname{conv}}\, (H([0, 1] \times (U \cap \Omega)) \cup V) \subseteq U$ such that the following conditions hold.*

1. *The set $\overline{\operatorname{conv}}\, H(\{0\} \times D)$ is compact and contained in $\overline{\Omega}$.*
2. *We have $x \neq H(\lambda, x)$ for all $x \in B$ and all $\lambda \in [0, 1)$.*
3. *For any countable $C \subseteq \Omega \cap U$ the relation*

$$C = \overline{\operatorname{conv}(H([0, 1] \times C) \cup V) \cap \Omega} \tag{23}$$

*implies that $\overline{\operatorname{conv}}\, H([0, 1] \times C)$ is compact.*

*Then $x = H(1, x)$ for some $x \in \overline{\Omega}$.*

*Proof*. Put $f = H(0, \cdot)|_B$. By Proposition 2.1, the map $F = id$ is $f$-epi on $\overline{\Omega}$. But $F$ is even $f$-epi on $O = \Omega$. Indeed, our assumption 2 for $\lambda = 0$ implies that $f$ has no fixed point on $B$. In particular, if $F(x) = \varphi(x)$ with $\varphi|_B = f$ and some $x \in \overline{\Omega}$, we have $x \notin B$, and so $x \in \Omega$.

If $H(1, \cdot)$ has a fixed point in $B$, we are done. Otherwise, the assumptions of Theorem 1.1 are satisfied with $D = \overline{\Omega}$. Since $F$ is $f$-epi on $O$, the equation $F(x) = \varphi(x)$ with $\varphi = H(0, \cdot)$ has a solution $x$ in $O$. Hence, (1) holds with $A = \{x\}$. The compactness assumption follows from Proposition 1.1. $\square$

Theorem 2.1 generalizes the main fixed point theorem from [**25**] slightly. Note that the choice $V = \emptyset$ is possible in the most important case $\Omega = K = M$ (since then $B = \emptyset$).

It is easily seen (for details see [**25**]) that Theorem 2.1 contains as special cases the fixed point theorem of Darbo [**5**] and Sadovskiĭ [**20**] as well as the corresponding countable version from [**4**]. Note that for the proof of these special cases, a constant homotopy $H(\lambda, x) = G(x)$ suffices. Theorem 2.1 contains also the two fixed point theorems from [**17**]. (See also [**6**, Theorem 18.1 and Theorem 18.2].) Both are special cases.

COROLLARY 2.2. *Let $X = Y$, and $\Omega$ be nonempty and open in $M = K$. Assume that $\varphi: \overline{\Omega} \to K$ is continuous and there is some $x_0 \in \Omega$ with the following properties.*

1. *The Leray-Schauder boundary condition holds on $\partial\Omega$:*

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$$\varphi(x) - x_0 \neq \lambda(x - x_0) \qquad (x \in \overline{\Omega} \setminus \Omega,\ \lambda > 1).$$

2. *If $C \subseteq \Omega$ is countable and satisfies*

$$\overline{C} = \overline{\Omega \cap (\varphi(C) \cup \{x_0\})}, \tag{24}$$

*then $\overline{C}$ is compact.*

*Then $\varphi$ has a fixed point in $\overline{\Omega}$.*

*Proof.* Put $V = \{x_0\}$ and $H(\lambda, x) = \lambda\varphi(x) + (1-\lambda)x_0$ in Theorem 2.1. The Leray-Schauder boundary condition means that $H(\lambda, \cdot)$ has no fixed points on $\partial\Omega$ for $0 \leq \lambda < 1$. Now observe that $H([0, 1] \times C) = (\varphi(C) \cup \{x_0\})$, and so we have $(H([0, 1]\times C) \cup V) = \mathrm{conv}(\varphi(C) \cup \{x_0\})$, for each $C \subseteq \Omega$. □

For the multivalued case, we apply Theorem 1.3 analogously.

Theorem 2.2. *Theorem 2.1 holds for upper semicontinuous functions $H : [0, 1] \times \overline{\Omega} \to \mathcal{K}(K)$ (with the conclusion $x \in H(1, x)$ for some $x \in \overline{\Omega}$) if one replaces condition 3 by the folowing condition.*

3′. *For any countable $C \subseteq \Omega \cap U$ the relation*

$$\overline{\mathrm{conv}(H([0, 1] \times C) \cup V) \cap \Omega} \subseteq \overline{C} \subseteq \overline{\overline{\mathrm{conv}}\,(H([0, 1] \times \overline{C}) \cup V) \cap \Omega}$$

*implies that $\overline{\mathrm{conv}}\, H([0, 1] \times \overline{C})$ is compact and that $C$ is precompact.*

Note that if $X = Y$ is a Fréchet space (in particular complete), then it suffices to require in the assumption 3′ of Theorem 2.2 only that $C$ is precompact. Indeed, if $C \subseteq \overline{\Omega}$ is precompact, then $\overline{\mathrm{conv}}\, H([0, 1] \times C)$ is compact by Mazur's Theorem [**9**, V.2, Theorem 6] (since then $[0, 1] \times \overline{C}$ is compact, and $H$ maps compact sets into compact sets).

From Theorem 2.2 one may obtain of course a fixed point theorem for multivalued maps analogous to Corollary 2.2. However, the compactness condition is not so nice as (24); it appears not possible to prove a complete analogue to Corollary 2.2 for multivalued maps by our methods. Nevertheless, in [**23**] we shall see that such an analogue is true, but the proof is not so elementary (it uses a fixed point index).

Let $R$ be a partially ordered set. To each set $A \subseteq K$, we associate a value $\gamma_K(A) \in R$ with the following properties.

1. $\gamma_K(A) = \gamma_K(\overline{\mathrm{conv}}A)$.
2. $\gamma_K(A_1) \leq \gamma_K(A_2)$ whenever $A_1 \subseteq A_2$.
3. $\gamma_K(A \cup C) = \gamma_K(A)$ whenever $C \subseteq K$ is precompact.

Using the notation of [**1**,**20**], the first two properties mean that $\gamma_K$ is a monotone measure of noncompactness.

A typical example of $\gamma_K$ is the *Kuratowski measure of noncompactness* $\gamma_K = \alpha$: $\alpha(A)$ is defined as the infimum of all numbers $\varepsilon > 0$ such that $A$ can be covered by finitely many sets with diameter less than $\varepsilon$. It might appear artificial that we do not restrict ourselves to the Kuratowski measure of noncompactness throughout, or at least to $R = [0, \infty]$. But this generalization has an important practical purpose, if $Y$ is not a Banach space.

Example 2.1. If the topology of $Y$ is generated by the seminorms $||\cdot||_k$ $(k = 1, 2, \ldots)$, one may choose $\gamma_K(A) = (\alpha_1(A), \alpha_2(A), \ldots)$, where $\alpha_k(A)$ denotes the Kuratowski measure of noncompactness of $A$ with respect to the seminorm $\|\cdot\|_k$. Here, we have $R = [0, \infty]^{\mathbb{N}}$.

For further details and other examples of measures of noncompactness, we refer to [**1**,**20**].

Recall that an operator $F : K \to K$ is called *L-contracting* with respect to $\alpha$ (for some $L \geq 0$), if $\alpha(F(A)) \leq L\alpha(A)$ holds for each $A \subseteq K$. For example, if $F$ is the sum of a Lipschitz-continuous operator (with Lipschitz constant $L$) and of a compact operator, then $F$ is $L$-contracting with respect to $\alpha$.

We intend to generalize this definition in three aspects: we want to consider operators $F : \Omega \to K$ (also if $\Omega \not\subseteq K$), more general "measures of noncompactness" than $\alpha$, and we intend to replace $[0, \infty]$ by some partially ordered set $\Lambda$. To this end, we have to fix some "measure of noncompactness" on $\Omega$, and we have to introduce some "multiplication" in $R$ by elements from $\Lambda$.

We do this in one, slightly more general, step by fixing some partially ordered set $\Lambda$ and some function $\gamma_\Omega : \Lambda \times 2^\Omega \to R$ such that the following requirements are met.

1. Each nonempty subset of $\Lambda$ has an infimum.
3. $\gamma_\Omega(\lambda_1, A_1) \leq \gamma_\Omega(\lambda_2, A_2)$ whenever $\lambda_1 \leq \lambda_2$ and $A_1 \subseteq A_2$.
4. $\gamma_\Omega(\lambda, A) \geq r$ for any $\lambda > \lambda_0$, then $\gamma_\Omega(\lambda_0, A) \geq r$.

Now, if $\varphi : \Omega \to Y$ or $\varphi : \Omega \to 2^Y$, we denote by $[\varphi]$ the infimum of all $\lambda \in \Lambda$ with

$$\gamma_K(\varphi(A)) \leq \gamma_\Omega(\lambda, A) \qquad (A \subseteq \Omega \text{ and } \varphi(A) \subseteq K) \tag{25}$$

(if no such $\lambda$ exists, we write $[\varphi]_\gamma = \infty$). By $[\varphi]_\gamma^c$, we denote the corresponding infimum when we restrict the estimate (25) to countable subsets $A \subseteq \Omega$. Evidently,

$$[\varphi]_\gamma^c \leq [\varphi]_\gamma. \tag{26}$$

The infimum is actually a minimum.

Proposition 2.2. *If* $[\varphi]_\gamma \neq \infty$, *then*

$$\gamma_K(\varphi(A)) \leq \gamma_\Omega([\varphi]_\gamma, A) \qquad (A \subseteq \Omega \textit{ and } \varphi(A) \subseteq K). \tag{27}$$

*Similarly, if* $[\varphi]_\gamma^c \neq \infty$, *then*

$$\gamma_K(\varphi(A)) \leq \gamma_\Omega([\varphi], A) \qquad (A \subseteq \Omega \textit{ countable and } \varphi(A) \subseteq K). \tag{28}$$

*Proof*. Putting $r = \gamma_K(\varphi(A))$ and $\lambda_0 = [\varphi]_\gamma^c$ (respectively, $\lambda_0 = [\varphi]_\gamma$), we have for any $\lambda > \lambda_0$ that $\gamma_\Omega(\lambda, A) \geq r$. Hence, $\gamma_\Omega(\lambda_0, A) \geq r$. □

Let us illustrate the above definition by some typical examples.

Example 2.2. For the choice $\gamma_K = \alpha$, $\Lambda = [0, \infty)$, and $\gamma_\Omega(\lambda, A) = \lambda\alpha(A)$, the constant $[\varphi]_\gamma$ is the minimum of all $L \geq 0$ such that $\gamma_\Omega(\varphi(A)) \leq L\gamma_K(A)$ holds for all $A \subseteq \Omega$ with $\varphi(A) \subseteq K$. If $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 : \Omega \to Y$ is Lipschitz continuous with constant $L$, and $\varphi_2 : \Omega \to Y$ maps bounded sets into precompact sets, then $[\varphi]_\gamma \leq L$.

Example 2.3. If we choose in Example 2.2 $\gamma_\Omega(\lambda, A) = |\lambda|^\beta\alpha(A)$ with some

$\beta \in (0,1)$, then $\varphi = \varphi_1 + \varphi_2$ satisfies $[\varphi]_\gamma \leq L$, if $\varphi_1 : \Omega \to Y$ is *Hölder* continuous with constant $L$ and exponent $\beta$, and $\varphi_2 : \Omega \to Y$ maps bounded sets into precompact sets.

EXAMPLE 2.4. A natural choice is also $\gamma_K = \chi_Y$ or $\gamma_K = \chi_K$, and $\gamma_\Omega(\lambda, A) = \lambda\chi_\Omega(A)$, $\gamma_\Omega(\lambda, A) = \lambda\chi_M(A)$, or $\gamma_\Omega(\lambda, A) = \lambda\chi_X(A)$. Here, $\chi_S(A)$ denotes the *Hausdorff measure of noncompactness* of $A$ in $S$; that is the infimum of all $\varepsilon > 0$ such that $A$ has a finite $\varepsilon$-net in $S$.

EXAMPLE 2.5. If $X$ is a locally convex space generated by the family of seminorms $\|\cdot\|_{X,k}$ $(k = 1, 2, \ldots)$, and $\gamma_K$ is as in Example 2.1, two choices for $\Lambda$ and $\gamma_\Omega$ are natural.

On the one hand, one may put $\Lambda = [0, \infty)$, and $\gamma_\Omega(\lambda, A) = \lambda(\alpha_{X,1}(A), \alpha_{X,2}(A), \ldots)$, where $\alpha_{X,k}$ denotes the Kuratowski measure of noncompactness with respect to the seminorm $\|\cdot\|_{X,k}$. On the other hand, one may put $\Lambda = [0, \infty)^{\mathbb{N}}$ or $\Lambda = [0, \infty]^{\mathbb{N}}$, and

$$\gamma_\Omega((\lambda_k)_k, A) = (\lambda_1\alpha_{X,1}(A), \lambda_2\alpha_{X,2}(A), \ldots).$$

In the latter case, we have $[\varphi]_\gamma = (\lambda_k)_k$ where $\lambda_k$ is the minimum of all $\lambda \geq 0$ such that $\alpha_k(\varphi(A)) \leq \lambda\alpha_{k,X}(A)$, for all $A \subseteq \Omega$. One has good estimates of this type if $\varphi$ is a Volterra operator of vector functions and $X = Y = C([0, \infty), Z)$. See, for example, [**27**].

DEFINITION 2.2. Let $k \in \Lambda$. We call a map $F : \overline{\Omega} \to Y$

1. $(f, k)$-*epi* (on $\Omega$ with respect to $\gamma_K$, $\gamma_\Omega$ and $K$) if for any continuous function $\varphi : \overline{\Omega} \to K$ with $\varphi|_{\partial\Omega} = f$ and $[\varphi]_\gamma \leq k$, the equation $F(x) = \varphi(x)$ has a solution $x$ in $\Omega$;
2. $(f, k)$-*multiepi* if for any upper semicontinuous function $\varphi : \overline{\Omega} \to \mathcal{K}(K)$ with $\varphi|_{\partial\Omega} = f$ and $[\varphi]_\gamma \leq k$, the inclusion $F(x) \in \varphi(x)$ has a solution $x$ in $\Omega$;
3. $(f, k)^c$-*epi* respectively $(f, k)^c$-*multiepi* if we may replace in the above definition the assumption $[\varphi]_\gamma \leq k$ even by $[\varphi]^c_\gamma \leq k$.

If $f(x) \equiv p \in K$, we call $F$ $(p, k)$-*epi*, $(p, k)$-*multiepi*, etc.

We emphasize that the condition $[\varphi]_\gamma \leq k$ is a requirement on $\varphi$ only on the set $\Omega$ (not on $\overline{\Omega}$).

It follows from (26) that each $(f, k)^c$-epi map is $(f, k)$-epi. The converse need not hold. The class of $(f, k)^c$-epi maps is of particular interest in the context of integral and differential operators of vector functions, since for such operators one has "good" estimates for measures of noncompactness only on countable subsets. See, for example, [**13**,**17**,**18**,**27**].

For the situation of Example 2.2, the class of $(p, k)$-epi maps has been introduced in [**22**]. (See also [**14**].) This class has certain desirable properties like homotopy invariance, normalization, etc. In the special case mentioned, the $(f, 0)$-epi maps are precisely the $f$-epi maps.

In the following, we shall assume that $K$ is complete with respect to a translation invariant metric on $Y$; this is of course satisfied if $Y$ is a Fréchet space.

THEOREM 2.3. *Let $K$ be complete as described above. Let $F : \overline{\Omega} \to Y$ be $f$-epi. If $k \in \Lambda$ satisfies*

$$\gamma_K(F(C)) \not\leq \gamma_\Omega(k, C) \quad (\textit{if } C \subseteq \Omega \textit{ is countable}, F(C) \subseteq K, \textit{and } \overline{C} \textit{ is not compact}), \tag{29}$$

*then $F$ is $(f, k)^c$-epi.*

*Proof.* Let $\varphi : \overline{\Omega} \to K$ be continuous with $[\varphi]_\gamma^c \leq k$ and $\varphi|_{\partial\Omega} = f$. We have to prove that $F(x) = \varphi(x)$ has a solution in $O = \Omega$. To this end, we apply Theorem 1.1 with the constant homotopy $H(\lambda, \cdot) = \varphi$, $\lambda_0 = 1$, $D = U = \overline{\Omega}$, and $B = \partial\Omega$.

By Lemma 1.1, there exists a retraction $\rho$ of $Y$ onto $\overline{\operatorname{conv}} f(\partial\Omega)$. Since $F$ is $f$-epi, the equation $F(x) = \rho(\varphi(x))$ has a solution $x$ in $O$. Then (1) holds for $A = \{x\}$ and $V = f(\partial\Omega) \cup \{F(x)\}$. For assumption 1 of Theorem 1.1, note that $V \subseteq K$ because $F(x) = \rho(\varphi(x)) \in K$ (and recall Lemma 1.2).

To verify the compactness assumption, let $C \subseteq \Omega$ be countable and satisfy (4). We have $F(C) \subseteq \overline{\operatorname{conv}}(\varphi(C) \cup V) \subseteq K$. In view of (28), it follows that

$$\gamma_K(F(C)) \leq \gamma_K(\overline{\operatorname{conv}}(\varphi(C) \cup V)) = \gamma_K(\varphi(C)) \leq \gamma_\Omega([\gamma]_\gamma, C) \leq \gamma_\Omega(k, C).$$

By (29), this is only possible if $\overline{C}$ is compact. Since $\varphi$ maps compact sets into compact sets (and $K$ is complete), the set $\overline{\operatorname{conv}}(\varphi(C) \cup V) \subseteq \overline{\operatorname{conv}}(\varphi(\overline{C}) \cup V)$ is compact. $\square$

Recall that [**10**, Theorem 4.2.1] plays an important role in the definition of the Furi-Martelli-Vignoli spectrum for nonlinear operators that is based on so-called stably solvable maps [**2**,**10**]. The above Theorem 2.3 plays a similar role for the definition of a spectrum based on 0-epi maps. We discuss this in [**21**].

The multivalued version of Theorem 2.3 does not require many more assumptions.

THEOREM 2.4. *Let $K$ be complete as described above. Assume either the axiom of choice or that $\partial\Omega$ is separable or that $f$ has an extension to an upper semicontinuous function $f : \overline{\Omega} \to \mathcal{K}(K)$ with precompact range. Let $F : \overline{\Omega} \to Y$ be $f$-multiepi, $k \in \Lambda$ satisfy* (29) *and*

$$\gamma_\Omega(k, C) = \gamma_\Omega(k, \overline{C} \cap \Omega) \qquad (C \subseteq \Omega \text{ countable}). \tag{30}$$

*Then $F$ is $(f, k)$-multiepi.*

*Proof.* In all cases we may extend $f$ to an upper semicontinuous function $f : \overline{\Omega} \to \mathcal{K}(K)$ with precompact range. (If $\partial\Omega$ is separable, this follows from Proposition 1.2, and if we assume the axiom of choice, this follows from Ma's generalization of Dugundji's extension theorem [**16**, (2.1)].)

Let $\varphi : \overline{\Omega} \to \mathcal{K}(K)$ be upper semicontinuous with $[\varphi]_\gamma \leq \lambda_0$ and $\varphi|_{\partial\Omega} = f$. To prove that $F(x) = \varphi(x)$ has a solution in $O = \Omega$, we apply Theorem 1.3 with the constant homotopy $H(\lambda, \cdot) = \varphi$, $\lambda_0 = 1$, $D = U = \overline{\Omega}$, and $B = \partial\Omega$.

Since $F$ is $f$-multiepi, the inclusion $F(x) \in f(x)$ has a solution $x$ in $O = \Omega$. Then (1) holds with $A = \{x\}$ and $V = f(\overline{\Omega})$, and also the condition 7 of Theorem 1.3 is satisfied. Note that $V \subseteq K$. To verify the compactness assumption, let $C \subseteq \Omega$ be countable with (19). Then $F(C) \subseteq \overline{\operatorname{conv}}(\varphi(\overline{C}) \cup V)$. Since $\overline{C} \subseteq (\overline{C} \cap \Omega) \cup \partial\Omega$ and $\varphi|_{\partial\Omega} = f$, it follows that

$$F(C) \subseteq \overline{\operatorname{conv}}(\varphi(\overline{C} \cap \Omega) \cup f(\partial\Omega) \cup V) = \overline{\operatorname{conv}}(\varphi(\overline{C} \cap \Omega) \cup V) \subseteq K.$$

In view of (27) and (30), this implies that

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$$\gamma_K(F(C)) \le \gamma_K(\overline{\mathrm{conv}}(\varphi(\overline{C} \cap \Omega) \cup V)) = \gamma_K(\varphi(\overline{C} \cap \Omega)) \le \gamma_\Omega(\varphi, \overline{C} \cap \Omega) \le \gamma_\Omega(k, C).$$

By (29), this is only possible if $\overline{C}$ is compact. Since $\varphi$ maps compact sets into compact sets (and $K$ is complete), $\overline{\mathrm{conv}}(\varphi(\overline{C}) \cup V)$ is compact too. □

One may swap the roles of the strict inequalities in (5) and (29).

Definition 2.3. Let $k \in \Lambda$. We call a map $F : \overline{\Omega} \to Y$

1. $(f, k^-)$-*epi* (on $\Omega$ with respect to $\gamma_K, \gamma_\Omega$ and $K$) if for any continuous function $\varphi : \overline{\Omega} \to K$ with $\varphi|_{\partial\Omega} = f$ and

$$\gamma_K(\varphi(C)) \not\ge \gamma_\Omega(k, C) \qquad (C \subseteq \Omega \text{ and } \overline{C} \text{ is not compact}) \tag{31}$$

the equation $F(x) = \varphi(x)$ has a solution $x$ in $\Omega$.

2. $(f, k^-)$-*multiepi* if for any upper semicontinuous function $\varphi : \overline{\Omega} \to \mathcal{K}(K)$ with $\varphi|_{\partial\Omega} = f$ and (31) the inclusion $F(x) \in \varphi(x)$ has a solution $x$ in $\Omega$.

3. $(f, k^-)^c$-*epi* resp. $(f, k^-)^c$-*multiepi* if we require (31) only for countable sets $C$.

If $f(x) \equiv p \in K$, we call $F$ $(p, k^-)$-*epi*, $(p, k^-)$-*multiepi*, etc.

Each $(f, k^-)^c$-epi map is $(f, k^-)$-epi. In the situation of Example 2.2, $(p, k^-)$-epi maps have been introduced in [**22**]. In this situation, the relation (31) for $k = 1$ means that $\varphi$ is condensing with respect to $\alpha$ (if $\Omega$ is bounded). The notation $k^-$ is explained by the following observation.

Proposition 2.3. *If $F$ is $(f, k^-)$-epi, and $k_0 \in \Lambda$ is such that $\gamma_\Omega(k_0, C) < \gamma_\Omega(k, C)$ holds for each set $C \subseteq \Omega$ for which $\overline{C}$ is not compact, then $F$ is $(f, k_0)$-epi.*

*Proof*. The relation $[\varphi]_\gamma \le k_0$ implies (31). □

Similarly, $(f, k^-)^c$-epi maps are $(f, k_0)^c$-epi if $k_0 \le k$ is such that $\gamma_\Omega(k_0, C) \ne \gamma_\Omega(k, C)$ for each countable set $C \subseteq \Omega$ for which $\overline{C}$ is not compact.

In the situation of Examples 2.2–2.4, the conditions of Proposition 2.3 hold if $\Omega$ is bounded, $\overline{\Omega}$ is complete, and $k_0 < k$. The same is true for Example 2.5, provided one understands "bounded" with respect to each seminorm (i.e. in the metric vector space $X$), and if one understands the relation $k_0 < k$ componentwise.

Theorem 2.5. *If the assumptions of Theorem 2.3 hold with* (29) *replaced by*

$$\gamma_K(F(C)) \ge \gamma_\Omega(k, C) \quad (\textit{if } C \subseteq \Omega \textit{ is countable and } F(C) \subseteq K), \tag{32}$$

*then $F$ is $(f, k^-)^c$-epi.*

*Proof*. The proof is analogous to the proof of Theorem 2.3. To verify the compactness assumption of Theorem 1.1, let $C \subseteq \Omega$ be countable and satisfy (4). Since $F(C) \subseteq \overline{\mathrm{conv}}(\varphi(C) \cup V) \subseteq K$, we have

$$\gamma_\Omega(k, C) \le \gamma_K(F(C)) \le \gamma_K(\overline{\mathrm{conv}}(\varphi(C) \cup V)) = \gamma_K(\varphi(C)).$$

By (31), this implies that $\overline{C}$ is compact. Hence, $\overline{\operatorname{conv}}\,(\varphi(C) \cup V) \subseteq \overline{\operatorname{conv}}\,(\varphi(\overline{C}) \cup V)$ is compact. $\square$

Corollary 2.3. (Normalization property for $(f, k)$-epi maps). *Let $X = Y$ and $K = M$. Let $K$ be complete in the sense described earlier, $\Omega \neq \emptyset$, and $k \in \Lambda$ be such that for any countable $C \subseteq \Omega$ the relation $\gamma_\Omega(k, C) \geq \gamma_K(C)$ holds.*

*If the set $\overline{\operatorname{conv}} f(\partial\Omega)$ is a compact subset of $\Omega$, then $F = \mathrm{id}$ is $(f, k^-)^c$-epi.*

*In particular, $F = id$ is $(p, k^-)^c$-epi if and only if $p \in \Omega$.*

*Proof*. Proposition 2.1 shows that $F$ is $f$-epi. Hence, the statement follows by Theorem 2.5. $\square$

For the choice $\gamma_\Omega(\lambda, A) = \lambda\gamma_K(A)$, Corollary 2.3 implies again the fixed point theorems of Sadovskiĭ [**20**] and its countable generalization from [**4**]. The statement that $F = \mathrm{id}$ is $(0, 1^-)$-epi on $\Omega = K$ means in this case precisely that any continuous map $\varphi : K \to K$ has a fixed point if it is condensing with respect to $\gamma_K$ (if $K$ is bounded).

The multivalued variants of the previous results read as follows.

Theorem 2.6. *If the assumptions of Theorem 2.4 hold with (29) replaced by (32), then $F$ is $(f, k^-)$-multiepi.*

*Proof*. The proof is analogous to the proof of Theorem 2.4. To verify the compactness assumption of Theorem 1.3, let $C \subseteq \Omega$ be countable with (19). As in the proof of Theorem 2.4, we then find that $F(C) \subseteq \overline{\operatorname{conv}}(\varphi(\overline{C} \cap \Omega) \cup V) \subseteq K$, and so

$$\gamma_\Omega(k, \overline{C} \cap \Omega) = \gamma_\Omega(k, C) \leq \gamma_K(F(C)) \leq \gamma_K(\overline{\operatorname{conv}}(\varphi(\overline{C} \cap \Omega) \cup V)) = \gamma_K(\varphi(\overline{C} \cap \Omega)).$$

By (31), we find that $\overline{\overline{C} \cap \Omega}$ is compact. Then also $\varphi(\overline{C}) \subseteq \varphi(\overline{\overline{C} \cap \Omega}) \cup f(\partial\Omega)$ is precompact, and so $\overline{\operatorname{conv}}(\varphi(\overline{C}) \cup V)$ is compact. $\square$

Corollary 2.4. (Normalization property for $(f, k)$-multepi maps). *Let $X = Y$ and $K = M$. Let $K$ be complete in the sense described earlier, $\Omega \neq \emptyset$, and $k \in \Lambda$ be such that for any countable $C \subseteq \Omega$ the relation $\gamma_\Omega(k, C) \geq \gamma_K(C)$ holds. Assume that also* (30) *holds.*

*If the set $\overline{\operatorname{conv}}\, f(\partial\Omega)$ is a compact subset of $\Omega$, and if we assume either the axiom of choice or that $\partial\Omega$ is separable or that $f$ has an extension to an upper semicontinuous function $f : \overline{\Omega} \to \mathcal{K}(K)$ with precompact range, then $F = id$ is $(f, k^-)$-multiepi.*

*In particular, $F = id$ is $(p, k^-)$-multiepi if and only if $p \in \Omega$.*

## REFERENCES

**1.** R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskiĭ, *Measures of noncompactness and condensing operators* (Birkhäuser, 1992).

**2.** J. Appell, E. Giorgieri, and M. Vath, On a class of maps related to the Furi-Martelli-Vignoli spectrum, *Ann. Mat. Pura Appl. (4)* **179** (2001), 215–228.

**3.** J. Appell, M. Väth, and A. Vignoli, $\mathcal{F}$-epi Maps, *Topol. Methods Nonlinear Anal.*, to appear.

**4.** S. J. Daher, On a fixed point principle of Sadovskĭ, *Nonlinear Anal.* **2** (1978), 643–645.

**5.** G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Sem. Mat. Univ. Padova* **24** (1955), 84–92.

**6.** K. Deimling, *Nonlinear functional analysis* (Springer-Verlag, 1985).

**7.** J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* **1** (1951), 353–367.

**8.** J. Dugundji, *Topology*, 8th ed., (Allyn and Bacon, 1973).

**9.** N. Dunford and J. T. Schwartz, *Linear operators I*, 3rd ed., (Interscience, 1966).

**10.** M. Furi, M. Martelli, and A. Vignoli, Contributions to the spectral theory for nonlinear operators in Banach spaces, *Ann. Mat. Pura Appl.* **118** (1978), 229–294.

**11.** M. Furi, M. Martelli, and A. Vignoli, On the solvability of nonlinear operator equations in normed spaces, *Ann. Mat. Pura Appl.* **124** (1980), 321–343.

**12.** E. Giorgieri and M. Väth, A Characterization of 0-epi maps with a degree, *J. Funct. Anal* **187** (2001), 183–199.

**13.** H.-P. Heinz, On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions, *Nonlinear Anal.* **7** (1983), 1351–1371.

**14.** D. H. Hyers, G. Isac, and T. M. Rassias, *Topics in nonlinear analysis and applications*, (World Scientific Publ., 1997).

**15.** T. J. Jech, *The axiom of choice* (North-Holland, 1973).

**16.** T.-W. Ma, *Topological degrees of set-valued compact fields in locally convex spaces*, Dissertationes Math. (Rozprawy Mat.), no. 92, (Polish Scientific Publ., 1972).

**17.** H. Mönch, Boundary value problems for nonlinear ordinary differential equations in banach spaces, *Nonlinear Anal.* **4** (1980), 985–999.

**18.** H. Mönch and G.-F. von Harten, On the Cauchy problem for ordinary differential equations in Banach spaces, *Arch. Math. (Basel)* **39** (1982), 153–160.

**19.** W. Rudin, *Functional analysis,* 14th ed. (McGraw-Hill, 1990).

**20.** B. N. Sadovskiĭ, Limit-compact and condensing operators (in Russian), *Uspekhi Mat. Nauk* **27** (1972), 81–146, Engl. transl.: *Russian Math. Surveys* (1) **27** (1972), 85–155.

**21.** P. Santucci and M. Väth, Grasping the phantom – A new approach to nonlinear spectral theory, *Ann. Mat. Pura Appl.*, to appear.

**22.** E. U. Tarafdar and H. B. Thompson, On the solvability of nonlinear, noncompact operator equations, *J. Austral. Math. Soc. Ser.* A **43** (1987), 103–126.

**23.** M. Väth, An axiomatic approach to a coincidence index for noncompact function pairs, *Topol. Methods Nonlinear Anal* (2) **16** (2000), 307–338.

**24.** M. Väth, The Furi-Martelli-Vignoli spectrum vs. the phantom, *Nonlinear Anal.* **47** (2001), 2237–2248.

**25.** M. Väth, Fixed point theorems and fixed point index for countably condensing maps, *Topol. Methods Nonlinear Anal.* (2) **13** (1999), 341–363.

**26.** M. Väth, On the connection of degree theory and 0-epi maps, *J. Math. Anal. Appl.* **257** (2001), 223–237.

**27.** M. Väth, *Volterra and integral equations of vector functions* (Marcel Dekker, 2000).

**28.** P. P. Zabreĭko, Rotation of Vector Fields: Definition, Basic Properties, and Calculation, *Topological Nonlinear Analysis II. Degree, Singularity and Variations* (M. Matzeu and A. Vignoli, eds.), (Birkhäuser), 1997, 445–601.