

Proof: The implication \Leftarrow is evident (Figures 2 and 4). In order to show the implication \Rightarrow we can argue as follows: Since the P -Lemoine axis is always orthogonal to the straight line PP' (Figure 4) passing through the circumcentre O , to satisfy the property on the left-hand side of the Lemma P must lie on the straight line OX (15). Since P and P' are *inverse* points with respect to the circumcircle it is easy to see that the midpoint of PP' (i.e. Q' in Figure 4) is the nearer to the circumcircle the nearer P lies to it (in other words: on the line OX (15) the distance of Q' to the circumcircle increases *monotonically* with the distance of P from the circumcircle, in both directions). Therefore, there are exactly two possible positions for P , namely X (15) and X (16).

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108.18 A one-line proof of the Finsler-Hadwiger inequality

Every proof is a one-line proof if you start sufficiently far to the left, [1].

The *Finsler-Hadwiger inequality* asserts that, in the triangle ABC with side-lengths a , b , c and area Δ ,

$$\sum a^2 \geq \sum (b - c)^2 + 4\sqrt{3}\Delta, \quad (1)$$

with equality if, and only if, triangle ABC is equilateral.

In this short Note, we adapt an idea from [2] to give a very quick proof.

Let r and R denote the inradius and circumradius of triangle ABC , let r_A be the radius of the excircle opposite A which touches BC (with r_B, r_C similarly defined) and, as usual, let $s = \frac{1}{2}(a + b + c)$.

Then $\sum r_A = r + 4R$, $\sum r_A r_B = s^2$, $r_A r_B r_C = r s^2$: these are reasonably standard triangle identities but, for the convenience of readers, we outline their proofs.

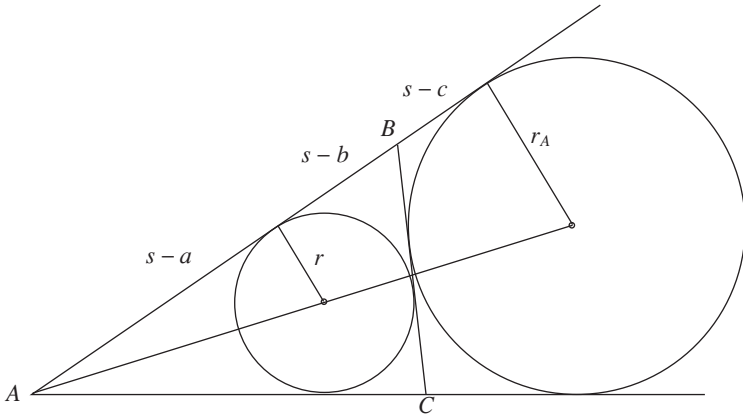


FIGURE: Similar triangles give $\frac{r}{s-a} = \frac{r_A}{s}$

From the Figure, $r_A = \frac{rs}{s-a}$. This gives

- $r_A r_B r_C = \frac{r^3 s^3}{(s-a)(s-b)(s-c)} = r s^2$, since $rs = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$;
- $\sum r_A r_B = \sum \frac{r^2 s^2}{(s-a)(s-b)} = \frac{r^2 s^2}{(s-a)(s-b)(s-c)} \cdot \sum (s-a) = s \cdot s = s^2$;
- $\sum r_A = \frac{rs}{(s-a)(s-b)(s-c)} \cdot \sum (s-a)(s-b) = \frac{1}{r} \cdot \sum (s-a)(s-b)$.

Now, on the one hand,

$$\sum (s-a)(s-b) = \sum [s^2 - s(a+b) + ab] = \sum ab - s^2, \quad (2)$$

but we also have

$$\begin{aligned} r^2 s &= \frac{\Delta^2}{s} = (s-a)(s-b)(s-c) = s^3 - (\Sigma a)s^2 + (\Sigma ab)s - abc \\ &= -s^3 + (\Sigma ab)s - 4rRs, \end{aligned}$$

and hence

$$\sum ab - s^2 = r^2 + 4rR. \quad (3)$$

From (2) and (3), $\sum r_A = \frac{1}{r} \cdot (r^2 + 4rR) = r + 4R$.

It follows that r_A, r_B, r_C are the roots of the cubic

$$f(X) = X^3 - (r + 4R)X^2 + s^2X - rs^2 = 0.$$

In [2, p. 206], the discriminant of this cubic is used to prove the triangle inequalities of Rouché and Blundon. Here we simply observe that, because the graph $y = f(X)$ always has stationary points, the discriminant of the quadratic $f'(X) = 3X^2 - 2(r + 4R)X + s^2$ is non-negative. It follows that $(r + 4R)^2 \geq 3s^2$ or $r + 4R \geq \sqrt{3}s$. This completes the proof because the latter is a known equivalent of (1). Indeed, $\Delta = rs$ and, using (2) and (3),

$$\begin{aligned} \sum a^2 - \sum (b - c)^2 &= \sum (a - b + c)(a + b - c) \\ &= 4 \sum (s - b)(s - c) = 4(r^2 + 4rR). \end{aligned}$$

An alternative to using the discriminant is to employ the standard inequality for three variables, $\sum r_A^2 \geq \sum r_A r_B$. This gives

$$\left(\sum r_A \right)^2 = \sum r_A^2 + 2 \sum r_A r_B \geq 3 \sum r_A r_B,$$

from which $(r + 4R)^2 \geq 3s^2$ again follows.

We also note that the inequality of arithmetic and harmonic means gives $\frac{1}{3} \sum r_A \geq \frac{3r_A r_B r_C}{\sum r_A r_B}$ or $\frac{1}{3}(r + 4R) \geq \frac{3rs^2}{s^2}$ which rearranges to Euler's inequality, $R \geq 2r$.

Finally, it is a remarkable fact that, although the Finsler-Hadwiger inequality seems clearly stronger than the famous Weitzenböck inequality $\sum a^2 \geq 4\sqrt{3}\Delta$, it is actually equivalent to it, [2, 3].

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