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THE SHARP BOUND FOR THE HANKEL DETERMINANT OF THE THIRD KIND FOR CONVEX FUNCTIONS

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Abstract

We prove the sharp inequality $|H_{3,1}(f)| \le 4/135$ for convex functions, that is, for analytic functions f with $a_n := f^{(n)}(0)/n!$, $n \in \mathbb{N}$, such that

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{for } z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

where $H_{3,1}(f)$ is the third Hankel determinant

$$H_{3,1}(f) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

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1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be the subclass normalised by $f(0) := 0$, $f'(0) := 1$, that is, functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{for } a_1 := 1, z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ of univalent functions and $\mathcal{S}^c$ the subclass of $\mathcal{S}$ of convex functions, that is, univalent functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is a convex domain in $\mathbb{C}$. By the well-known result of Study [21] (see also [6, page 42]), a function f is in $\mathcal{S}^c$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.2}$$

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Given $q, n \in \mathbb{N}$, the Hankel determinants $H_{q,n}(f)$ of Taylor coefficients of functions $f \in \mathcal{A}$ of the form (1.1) are defined by

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

In particular, the third Hankel determinant $H_{3,1}(f)$ is given by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{1.3}$$

Finding the rate of growth of the Hankel determinant $H_{q,n}(f)$ in terms of q and n for the whole class $\mathcal{S} \subset \mathcal{A}$ of univalent functions as well as for its subclasses is a significant problem. Pommerenke [16] proved a basic result for the class $\mathcal{S}$. Recently many authors have examined the Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ of order 2 (see [4, 5, 8–10, 15]). Also, $H_{2,1}(f) = a_3 - a_2^2$, so the Hankel determinant $H_{2,1}(f)$ reduces to the well-known coefficient functional which for $\mathcal{S}$ was estimated in 1916 by Bieberbach (see [7, Vol. I, page 35]).

The problem of finding the upper bound for the Hankel determinant $H_{3,1}(f)$ of order 3 is more sophisticated if we expect to get a sharp result. By the triangle inequality, the formula (1.3) yields

$$|H_{3,1}(f)| \le |a_3|\,|H_{2,2}(f)| + |a_4|\,|a_4 - a_2a_3| + |a_5|\,|H_{2,1}(f)|. \tag{1.4}$$

This simple observation allows one to estimate $|H_{3,1}(f)|$ for compact subclasses $\mathcal{F}$ of $\mathcal{A}$ (see [1, 2, 18–20, 22]). However, these results are far from sharp. If the given subclass $\mathcal{F}$ of $\mathcal{A}$ has a representation using the Carathéodory class $\mathcal{P}$, that is, the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \text{for } z \in \mathbb{D}, \tag{1.5}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{F}$ can be usefully represented by coefficients of functions in $\mathcal{P}$. Upper bounds for each term in (1.4) then follow from the well-known formulas for the coefficient c_2 (see [17, page 166]) and the formula for c_3 due to Libera and Zlotkiewicz [12].

In order to improve the bound for $|H_{3,1}(f)|$, we have to use (1.3) directly, so we need a formula for c_4 similar to the formulas (2.1) and (2.2). In a recent paper [11] the authors found such a formula for c_4. As far as we know, formulas for the coefficients c_n for $n \ge 5$ analogous to the formulas (2.1) and (2.2) are not known.

Using the formulas for c_2, c_3 and c_4, we prove that $|H_{3,1}(f)| \le 4/135 = 0.0296\ldots$ for $f \in \mathcal{S}^c$ and that the result is sharp. This solves the problem of estimating $H_{3,1}(f)$ in the class of convex functions. Babalola [1] showed $|H_{3,1}(f)| \le 0.714\ldots$. This result was improved by Zaprawa [23], using a suitable grouping and a result of Livingston [14, Lemma 1], to show $|H_{3,1}(f)| \le 49/540 = 0.090\ldots$.

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2. Main result

The key to the proof of the main result is the following lemma. It contains the well-known formula for c_2 (see [17, page 166]), the formula for c_3 due to Libera and Zlotkiewicz [12, 13] and the formula for c_4 found by the authors [11].

Lemma 2.1. *If $p \in \mathcal{P}$ is of the form (1.5) with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta, \tag{2.1}$$

$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2 - \zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{2.2}$$

and

$$\begin{aligned} 8c_4 = c_1^4 &+ (4 - c_1^2)\zeta[c_1^2(\zeta^2 - 3\zeta + 3) + 4\zeta] \\ &- 4(4 - c_1^2)(1 - |\zeta|^2)[c_1(\zeta - 1)\eta + \overline{\zeta}\eta^2 - (1 - |\eta|^2)\xi] \end{aligned} \tag{2.3}$$

for some $\zeta, \eta, \xi \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

We will now estimate the third-order Hankel determinant $H_{3,1}(f)$ for $f \in \mathcal{S}^c$.

Theorem 2.2. *If $f \in \mathcal{S}^c$ is the form (1.1), then*

$$|H_{3,1}(f)| \leq \tfrac{4}{135} = 0.0296\ldots. \tag{2.4}$$

The result is sharp with equality attained by

$$f(z) = \arctan z \quad \textit{for } z \in \mathbb{D}.$$

Proof. Let $f \in \mathcal{S}^c$ be of the form (1.1). Then by (1.2),

$$f'(z) + zf''(z) = p(z)f'(z) \quad \text{for } z \in \mathbb{D}, \tag{2.5}$$

for some function $p \in \mathcal{P}$ of the form (1.5). The class $\mathcal{P}$ is invariant under rotations, so by the Carathéodory Theorem we may assume that $c := c_1 \in [0, 2]$ ([3], see also [7, Vol. I, page 80, Theorem 3]). Substituting the series (1.1) and (1.5) into (2.5) and equating the coefficients,

$$\begin{aligned} &a_2 = \tfrac{1}{2}c, \quad a_3 = \tfrac{1}{6}(c_2 + c^2), \quad a_4 = \tfrac{1}{12}(c_3 + \tfrac{3}{2}cc_2 + \tfrac{1}{2}c^3), \\ &a_5 = \tfrac{1}{20}(c_4 + \tfrac{4}{3}cc_3 + c_1^2c_2 + \tfrac{1}{2}c_2^2 + \tfrac{1}{6}c^4). \end{aligned}$$

Hence,

$$\begin{aligned} H_{3,1}(f) &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \\ &= \tfrac{1}{8640}(72c_2c_4 - 36c^2c_4 + 36cc_2c_3 - 21c^2c_2^2 + 6c^4c_2 - 4c_2^3 \\ &\quad - 60c_3^2 + 12c^3c_3 - c^6). \end{aligned} \tag{2.6}$$

To simplify the computation, let $t := 4 - c^2$. By using (2.1)–(2.3),

$$\begin{aligned} c_2 &= \tfrac{1}{2}(c^2 + t\zeta), \quad c_3 = \tfrac{1}{4}(c^3 + 2ct\zeta - ct\zeta^2 + 2t(1 - |\zeta|^2)\eta), \\ c_4 &= \tfrac{1}{8}[c^4 + 3c^2t\zeta + (4 - 3c^2)t\zeta^2 + c^2t\zeta^3 + 4t(1 - |\zeta|^2)(c\eta - c\zeta\eta - \overline{\zeta}\eta^2) \\ &\quad + 4t(1 - |\zeta|^2)(1 - |\eta|^2)\xi]. \end{aligned}$$

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Hence by straightforward algebraic computation,

$$
\begin{aligned}
72c_2c_4 - 36c^2c_4 &= \tfrac{1}{4}[18c^4t\zeta + 54c^2t^2\zeta^2 + 18(4-3c^2)t^2\zeta^3 + 18c^2t^2\zeta^4 \\
&\quad + 72t^2(1-|\zeta|^2)(c\zeta\eta - c\zeta^2\eta - |\zeta|^2\eta^2) + 72t^2(1-|\zeta|^2)(1-|\eta|^2)\zeta\xi], \\
36cc_2c_3 &= \tfrac{1}{4}[18c^6 + 54c^4t\zeta + 18c^2(8-3c^2)t\zeta^2 - 18c^2t^2\zeta^3 + 36c^3t(1-|\zeta|^2)\eta \\
&\quad + 36ct^2(1-|\zeta|^2)\zeta\eta], \\
21c^2c_2^2 &= \tfrac{1}{4}[21c^6 + 42c^4t\zeta + 21c^2t^2\zeta^2], \\
6c^4c_2 &= \tfrac{1}{4}[12c^6 + 12c^4t\zeta], \\
4c_2^3 &= \tfrac{1}{4}[2c^6 + 6c^4t\zeta + 6c^2t^2\zeta^2 + 2t^3\zeta^3], \\
60c_3^2 &= \tfrac{1}{4}[15c^6 + 60c^4t\zeta + 30c^2(8-3c^2)t\zeta^2 - 60c^2t^2\zeta^3 + 15c^2t^2\zeta^4 \\
&\quad + 60c^3t(1-|\zeta|^2)\eta + 60t^2(1-|\zeta|^2)^2\eta^2 + 120ct^2(1-|\zeta|^2)\zeta\eta \\
&\quad - 60ct^2(1-|\zeta|^2)\zeta^2\eta], \\
12c^3c_3 &= \tfrac{1}{4}[12c^6 + 24c^4t\zeta - 12c^4t\zeta^2 + 24c^3t(1-|\zeta|^2)\eta].
\end{aligned}
$$

Substituting these expressions in (2.6), by simple but tedious computation,

$$
\begin{aligned}
H_{3,1}(f) = \tfrac{1}{34560}\{ & c^2t[27t - 12(8-3c^2) - 12c^2]\zeta^2 + 2t^2[9(4-3c^2) + 21c^2 - t]\zeta^3 \\
& + 3c^2t^2\zeta^4 - 12ct^2\zeta(1+\zeta)(1-|\zeta|^2)\eta - 12t^2(|\zeta|^2+5)(1-|\zeta|^2)\eta^2 \\
& + 72t^2(1-|\zeta|^2)(1-|\eta|^2)\zeta\xi\}.
\end{aligned}
$$

Since $t = 4 - c^2$,

$$
H_{3,1}(f) = \tfrac{1}{34560}[\gamma_1(c,\zeta) + \gamma_2(c,\zeta)\eta + \gamma_3(c,\zeta)\eta^2 + \Gamma(c,\zeta,\eta)\xi], \tag{2.7}
$$

where, for ζ, η, $\xi \in \overline{\mathbb{D}}$,

$$
\begin{aligned}
\gamma_1(c,\zeta) &:= (4-c^2)^2\zeta^2[3c^2 + 2(32-5c^2)\zeta + 3c^2\zeta^2], \\
\gamma_2(c,\zeta) &:= -12c(4-c^2)^2\zeta(1+\zeta)(1-|\zeta|^2), \\
\gamma_3(c,\zeta) &:= -12(4-c^2)^2(5+|\zeta|^2)(1-|\zeta|^2)
\end{aligned}
$$

and

$$
\Gamma(c,\zeta,\eta) := 72(4-c^2)^2(1-|\zeta|^2)(1-|\eta|^2)\zeta.
$$

From (2.7), setting $x := |\zeta| \in [0,1]$, $y := |\eta| \in [0,1]$, and taking into account that $|\xi| \le 1$,

$$
\begin{aligned}
|H_{3,1}(f)| &\le \tfrac{1}{34560}(|\gamma_1(c,\zeta)| + |\gamma_2(c,\zeta)||\eta| + |\gamma_3(c,\zeta)||\eta|^2 + |\Gamma(c,\zeta,\eta)|) \\
&\le \tfrac{1}{34560}F(c,x,y),
\end{aligned} \tag{2.8}
$$

where

$$
F(c,x,y) := f_1(c,x) + f_4(c,x) + f_2(c,x)y + [f_3(c,x) - f_4(c,x)]y^2,
$$

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with

$$\begin{aligned} f_1(c,x) &:= (4-c^2)^2x^2[3c^2+2(32-5c^2)x+3c^2x^2],\\ f_2(c,x) &:= 12c(4-c^2)^2x(1+x)(1-x^2),\\ f_3(c,x) &:= 12(4-c^2)^2(x^2+5)(1-x^2),\\ f_4(c,x) &:= 72(4-c^2)^2(1-x^2)x. \end{aligned}$$

Now, for $c \in [0,2]$, $x \in [0,1]$ and $y \in [0,1]$, we will show that

$$F(c,x,y) \le 1024. \tag{2.9}$$

I. On the face $x = 0$,

$$F(c,0,y) = 60(4-c^2)^2y^2 \le 960 \quad \text{for } c \in [0,2], y \in [0,1]. \tag{2.10}$$

II. On the face $x = 1$,

$$F(c,1,y) = 4(4-c^2)^2(16-c^2) \le 1024 = F(0,1,1) \quad \text{for } c \in [0,2], y \in [0,1]. \tag{2.11}$$

III. On the face $c = 2$,

$$F(2,x,y) = 0 \quad \text{for } x,y \in [0,1]. \tag{2.12}$$

IV. Now let $c \in [0,2)$ and $x \in (0,1)$. Since

$$f_3(c,x) - f_4(c,x) = 12(4-c^2)^2(1-x^2)(1-x)(5-x) > 0$$

and $f_2(c,x) \ge 0$, it follows that for each $c \in [0,2)$ and $x \in (0,1)$,

$$\frac{\partial F}{\partial y} = f_2(c,x) + 2[f_3(c,x) - f_4(c,x)]y \ge 0 \quad \text{for } y \in [0,1].$$

Thus for each $c \in [0,2)$ and $x \in (0,1)$, the function $y \mapsto F(c,x,y)$ is increasing for $y \in [0,1]$ and therefore

$$F(c,x,y) \le F(c,x,1) = f_1(c,x) + f_2(c,x) + f_3(c,x) \quad \text{for } c \in [0,2), x \in (0,1). \tag{2.13}$$

We now consider three cases with respect to c.

(1) Assume that $c = 0$. From (2.13) and easy checking,

$$F(0,x,y) \le F(0,x,1) = 64(-3x^4+16x^3-12x^2+5) \le 1024 \quad \text{for } x \in (0,1). \tag{2.14}$$

(2) Assume that $c \in (0,1)$. Then by (2.13), for $x \in (0,1)$,

$$\begin{aligned} F(c,x,y) \le F_1(c,x) &:= f_1(c,x) + \frac{1}{c}f_2(c,x) + f_3(c,x)\\ &= (4-c^2)^2[3c^2x^4 - 10c^2x^3 + 3c^2x^2 - 24x^4 + 52x^3 - 36x^2 + 12x + 60]. \end{aligned}$$

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Set $t := 4 - c^2$. Clearly, $t \in (3,4)$. For $t \in (3,4)$ and $x \in (0,1)$, define

$$\widetilde{F}_1(t,x) := F_1(\sqrt{4-t}, x) = (-3x^4 + 10x^3 - 3x^2)t^3 - 12(x^4 - x^3 + 2x^2 - x - 5)t^2.$$

Then

$$\frac{\partial}{\partial t}\widetilde{F}_1(t,x) = 3\phi_2(x)t^2 - 24\phi_1(x)t \quad \text{for } t \in (3,4), x \in (0,1), \tag{2.15}$$

where

$$\phi_1(x) := (x+1)(x^3 - 2x^2 + 4x - 5),$$

and

$$\phi_2(x) := -x^2(3x-1)(x-3).$$

Note that

$$\phi_1(x) = (x+1)(x^3 - 2x^2 + 4x - 5) < (x+1)(-2x^2 + 4x - 4) < 0 \quad \text{for } x \in (0,1). \tag{2.16}$$

For $x = 1/3$,

$$\frac{\partial}{\partial t}\widetilde{F}_1(t, 1/3) = \frac{3328}{27}t > 0 \quad \text{for } t \in (3,4).$$

For $x \neq 1/3$,

$$\frac{\partial}{\partial t}\widetilde{F}_1(t,x) = 0$$

if and only if $t = t_1(x) := 8\phi_1(x)/\phi_2(x)$.

Let $x \in (0, 1/3)$. Observe that $t_1(x) > 4$ and consequently

$$\frac{8\phi_1(x) - 4\phi_2(x)}{\phi_2(x)} > 0.$$

Indeed, the above inequality holds since $\phi_2(x) < 0$ in $(0, 1/3)$ and

$$8\phi_1(x) - 4\phi_2(x) = 4(5x^4 - 12x^3 + 7x^2 - 2x - 10) < 4(7x^2 - 2x - 5) < 0$$

in $(0,1)$. Consequently, for each $x \in (0, 1/3)$,

$$\frac{\partial}{\partial t}\widetilde{F}_1(t,x) > 0 \quad \text{for } t \in (3,4). \tag{2.17}$$

Let $x \in (1/3, 1)$. Then $\phi_2(x) > 0$. Hence, by (2.16), we see that $t_1(x) < 0$, so by (2.15) it follows for each $x \in (1/3, 1)$ that the inequality (2.17) holds.

Summarising, we have shown that the inequality (2.17) is true for each $x \in (0,1)$, so for each $x \in (0,1)$ the function $t \mapsto \widetilde{F}_1(t,x)$ is increasing for $t \in (3,4)$. Thus,

$$\widetilde{F}_1(t,x) \le \widetilde{F}_1(4,x) = -384x^4 + 832x^3 - 576x^2 + 192x + 960 \quad \text{for } x \in (0,1).$$

Observe now that

$$\widetilde{F}_1(4,x) < 1024 \quad \text{for } x \in (0,1). \tag{2.18}$$

Indeed, the above inequality holds since

$$\widetilde{F}_1(4,x) - 1024 = 64(1-x)^2(-6x^2 + x - 1) < 0 \quad \text{for } x \in (0,1).$$

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(3) Assume that $c \in [1, 2)$. Then by (2.13), for $x \in (0, 1)$,

$$\begin{aligned} F(c, x, y) \le F_2(c, x) &:= f_1(c, x) + c f_2(c, x) + f_3(c, x) \\ &= (4 - c^2)^2[-9c^2x^4 - 22c^2x^3 + 15c^2x^2 + 12c^2x - 12x^4 + 64x^3 - 48x^2 + 60]. \end{aligned} \tag{2.19}$$

Set $t := 4 - c^2$. Clearly, $t \in (0, 3]$. For $t \in (0, 3]$ and $x \in (0, 1)$, define

$$\begin{aligned} \widetilde{F}_2(t, x) &:= F_2(\sqrt{4 - t}, x) \\ &= (9x^4 + 22x^3 - 15x^2 - 12x)t^3 - 12(4x^4 + 2x^3 - x^2 - 4x - 5)t^2. \end{aligned}$$

Then

$$\frac{\partial}{\partial t}\widetilde{F}_2(t, x) = 3\theta_2(x)t^2 - 24\theta_1(x)t \quad \text{for } t \in (0, 3], x \in (0, 1), \tag{2.20}$$

where

$$\theta_1(x) := 4x^4 + 2x^3 - x^2 - 4x - 5$$

and

$$\theta_2(x) := 9x^4 + 22x^3 - 15x^2 - 12x.$$

Note that

$$\theta_1(x) = 4x^4 + 2x^3 - x^2 - 4x - 5 < 2x^3 - x^2 - 4x - 1 < 0 \quad \text{for } x \in (0, 1). \tag{2.21}$$

One can check that θ_2 has a unique root in $(0, 1)$, namely, $x = x_1 \approx 0.92$. By (2.20),

$$\frac{\partial}{\partial t}\widetilde{F}_2(t, x_1) = -24\theta_1(x_1)t > 0 \quad \text{for } t \in (0, 3].$$

For $x \in (0, 1) \setminus \{x_1\}$,

$$\frac{\partial}{\partial t}\widetilde{F}_2(t, x) = 0$$

if and only if $t = t_2(x) := 8\theta_1(x)/\theta_2(x)$.

Let $x \in (0, x_1)$. Observe that $t_2(x) > 3$, that is,

$$\frac{8\theta_1(x) - 3\theta_2(x)}{\theta_2(x)} > 0 \quad \text{for } x \in (0, x_1).$$

Indeed, the above inequality holds since $\theta_2(x) < 0$ and

$$8\theta_1(x) - 3\theta_2(x) = 5x^4 - 50x^3 + 37x^2 + 4x - 40 < -50x^3 + 8x^2 \le 0$$

in $(0, 1)$. Consequently, for each $x \in (0, x_1)$,

$$\frac{\partial}{\partial t}\widetilde{F}_2(t, x) > 0 \quad \text{for } t \in (0, 3]. \tag{2.22}$$

Let $x \in (x_1, 1)$. Then $\theta_2(x) > 0$. Hence, by (2.21), we see that $t_2(x) < 0$, so by (2.20) it follows that the inequality (2.22) holds for each $x \in (x_1, 1)$.

Summarising, we have shown that the inequality (2.22) is true for each $x \in (0,1)$, so for each $x \in (0,1)$ the function $t \mapsto \widetilde{F}_2(t,x)$ is increasing for $t \in (0,3]$. Thus,

$$\widetilde{F}_2(t,x) \le \widetilde{F}_2(3,x) = -189x^4 + 378x^3 - 297x^2 + 108x + 540 \quad \text{for } x \in (0,1).$$

Observe now that

$$\widetilde{F}_2(3,x) \le 1024 \quad \text{for } x \in (0,1). \tag{2.23}$$

Indeed, the above inequality holds since

$$\begin{aligned}\widetilde{F}_2(3,x) - 1024 &= -189x^4 + 378x^3 - 297x^2 + 108x - 484 \\ &< -297x^2 + 108x - 106 < 0 \quad \text{for } x \in (0,1).\end{aligned}$$

Thus, from (2.10)–(2.12), (2.14), (2.18), (2.19) and (2.23), it follows that the inequality (2.9) holds. Together with (2.8), this proves the inequality (2.4).

Consider the function $f \in \mathcal{S}^c$ given by

$$1 + \frac{zf''(z)}{f'(z)} = p(z) \quad \text{for } z \in \mathbb{D},$$

where

$$p(z) := \frac{1-z^2}{1+z^2} = 1 - 2z^2 + 2z^4 \mp \cdots \quad \text{for } z \in \mathbb{D},$$

that is, the function

$$f(z) = \arctan z = \frac{1}{2\mathrm{i}} \log \frac{1+\mathrm{i}z}{1-\mathrm{i}z} \quad \text{for } z \in \mathbb{D},\ \log 1 := 0.$$

Since $c_1 = c_3 = 0$, $c_2 = -2$ and $c_4 = 2$, by (2.6) we see that the equality in (2.4) holds, which makes the result sharp. □

Remark 2.3. Let us remark that the proof of the basic inequality (2.9) can be made shorter with some numerical computation. Rewrite the inequality (2.12) as

$$\begin{aligned}F(c,x,y) &\le F(c,x,1) \\ &= (4-c^2)^2[3(c^2-4c-4)x^4 - 2(5c^2+6c-32)x^3 \\ &\quad + 3(c^2+4c-16)x^2 + 12cx + 60] =: g(c,x) \quad \text{for } (c,x) \in [0,2] \times [0,1].\end{aligned}$$

Using the inequalities $0 \le c \le 2$ and $0 \le x \le 1$,

$$g(c,x) \le h(c,x),$$

where

$$h(c,x) := 4(4-c^2)\{24cx(1-x^2) + 12(1-x^2)(5+x^2) + x^2(64x + 2c(3-7x))\}.$$

Now, we will show that

$$h(c,x) \le 1024 \tag{2.24}$$

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for $c \in [0,2]$ and $x \in [0,1]$. To see this, we start by finding the critical points of h in $(0,2)\times(0,1)$. Differentiating the function h with respect to x and c, respectively,

$$\frac{\partial h}{\partial x}(c,x) = -24(4-c^2)(8x(2-4x+x^2)+c(-4-2x+19x^2)) \tag{2.25}$$

and

$$\begin{aligned}\frac{\partial h}{\partial c}(c,x) &= 8[4x(12+3x-19x^2)\\ &\quad + 3c^2x(-12-3x+19x^2)+4x(-15+12x^2-16x^3+3x^4)].\end{aligned}$$

From (2.25),

$$\frac{\partial h}{\partial x}(c,x) = 0$$

if and only if

$$c = \tilde{c}(x) := \frac{8x(2-4x+x^2)}{4+2x-19x^2} \quad \text{for } x \in (0,1).$$

Moreover,

$$\frac{\partial h}{\partial x}(\tilde{c}(x),x) = \frac{32x}{(4+2x-19x^2)^2}k(x),$$

where, for $x \in (0,1)$,

$$\begin{aligned}k(x) := &-768+1680x+1472x^2-4556x^3+1324x^4+2239x^5-859x^6\\ &-3136x^7+456x^8.\end{aligned}$$

Solving the equation $k(x)=0$, we find that in $(0,1)$ there are exactly two zeros of k given by

$$x = x_1 := 0.523884\ldots \quad \text{and} \quad x = x_2 := 0.630513\ldots.$$

Furthermore,

$$\tilde{c}(x_1) \approx -4.4937 \quad \text{and} \quad \tilde{c}(x_2) \approx 0.27396.$$

Therefore the function h has a unique critical point in $(0,2)\times(0,1)$ at $(\tilde{c}(x_2),x_2)$ and

$$h(\tilde{c}(x_2),x_2) \approx 898.86 < 1024,$$

which shows (2.24). Finally, we can observe the following inequalities:

(1) $h(2,x) = 0,\ x \in [0,1]$;
(2) $h(0,x) = 64(15-12x^2+16x^3-3x^4) \le 1024,\ x \in [0,1]$;
(3) $h(c,0) = 240(4-c^2) \le 960,\ c \in [0,2]$;
(4) $h(c,1) = 32(8-c)(4-c^2) \le 1024,\ c \in [0,2]$.

References

[1] K. O. Babalola, 'On $H_3(1)$ Hankel determinants for some classes of univalent functions', in: *Inequality Theory and Applications*, Vol. 6 (ed. Y. J. Cho) (Nova Science Publishers, New York, 2010), 1–7.

[2] D. Bansal, S. Maharana and J. K. Prajapat, 'Third order Hankel determinant for certain univalent functions', *J. Korean Math. Soc.* **52**(6) (2015), 1139–1148.

[3] C. Carathéodory, 'Über den Variabilitatsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen', *Math. Ann.* **64** (1907), 95–115.

[4] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko and Y. J. Sim, 'The bounds of some determinants for starlike functions of order alpha', *Bull. Malay. Math. Sci. Soc.*, to appear, doi:10.1007/s40840-017-0476-x.

[5] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko and Y. J. Sim, 'The bound of the Hankel determinant for strongly starlike functions of order alpha', *J. Math. Ineq.* **11**(2) (2017), 429–439.

[6] P. T. Duren, *Univalent Functions* (Springer, New York, 1983).

[7] A. W. Goodman, *Univalent Functions* (Mariner, Tampa, FL, 1983).

[8] A. Janteng, S. A. Halim and M. Darus, 'Coefficient inequality for a function whose derivative has a positive real part', *J. Inequal. Pure Appl. Math.* **7**(2) (2006), 1–5.

[9] A. Janteng, S. A. Halim and M. Darus, 'Hankel determinant for starlike and convex functions', *Int. J. Math. Anal.* **1**(13) (2007), 619–625.

[10] S. K. Lee, V. Ravichandran and S. Supramanian, 'Bound for the second Hankel determinant of certain univalent functions', *J. Ineq. Appl.* **2013**(281) (2013), 1–17.

[11] O. S. Kwon, A. Lecko and Y. J. Sim, 'On the fourth coefficient of functions in the Carathéodory class', *Comp. Methods Funct. Theory*, to appear; doi:10.1007/s40315-017-0229-8.

[12] R. J. Libera and E. J. Zlotkiewicz, 'Early coefficients of the inverse of a regular convex function', *Proc. Amer. Math. Soc.* **85**(2) (1982), 225–230.

[13] R. J. Libera and E. J. Zlotkiewicz, 'Coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$', *Proc. Amer. Math. Soc.* **87**(2) (1983), 251–257.

[14] A. E. Livingston, 'The coefficients of multivalent close-to-convex functions', *Proc. Amer. Math. Soc.* **21**(3) (1969), 545–552.

[15] A. K. Mishra and P. Gochhayat, 'Second Hankel determinant for a class of analytic functions defined by fractional derivative', *Int. J. Math. Math. Sci.* **2008** (2008), Article ID 153280, 1–10.

[16] C. Pommerenke, 'On the coefficients and Hankel determinant of univalent functions', *J. Lond. Math. Soc.* **41** (1966), 111–122.

[17] C. Pommerenke, *Univalent Functions* (Vandenhoeck & Ruprecht, Göttingen, 1975).

[18] J. K. Prajapat, D. Bansal, A. Singh and A. K. Mishra, 'Bounds on third Hankel determinant for close-to-convex functions', *Acta Univ. Sapientae Math.* **7**(2) (2015), 210–219.

[19] M. Raza and S. N. Malik, 'Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli', *J. Inequal. Appl.* **2013**(412) (2013), 8 pages.

[20] G. Shanmugam, B. A. Stephen and K. O. Babalola, 'Third Hankel determinant for α-starlike functions', *Gulf J. Math.* **2**(2) (2014), 107–113.

[21] E. Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie, Zweites Heft; Konforme Abbildung Einfach-Zusammenhängender Bereiche* (Druck und Verlag von BG Teubner, Leipzig und Berlin, 1913).

[22] T. V. Sudharsan, S. P. Vijayalaksmi and B. A Sthephen, 'Third Hankel determinant for a subclass of analytic functions', *Malaya J. Math.* **2**(4) (2014), 438–444.

[23] P. Zaprawa, 'Third Hankel determinants for classes of univalent functions', *Medit. J. Math.* **14**(19) (2017), 1–10.

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