SOME KREIN-MILMAN THEOREMS FOR ORDER-CONVEXITY

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1. Introduction

Analogues of the Krein-Milman theorem for order-convexity have been studied by several authors. Franklin [2] has proved a set-theoretic result, while Baker [1] has proved the theorem for posets with the Frink interval topology. We prove two Krein-Milman results on a large class of posets, with the open-interval topology, one for the original order and one for the associated preorder. This class of posets includes all pogroups. Cellular-internity defined in R^n by Miller [3] leads to another notion of convexity, cell-convexity. We generalize the definition of cell-convexity to abelian l-groups and prove a Krein-Milman theorem in terms of it for divisible abelian l-groups.

2. Preliminaries

Let $(X, \leqq)$ be a poset. By taking the family of sets $\{x: x > a\}$, $\{y: y < b\}$ where $a, b \in X$ as a subbase we define the *open-interval topology* $\mathscr{U}$ on X. Denote the set $\{x: a < x < b\}$ for $a < b$ by (a, b).

We say $a \preccurlyeq b$ if $x > b$ implies that $x > a$, and if $y < a$ implies that $y < b$ [4]. Then $(X, \preccurlyeq)$ is a preordered set; $\preccurlyeq$ is called the *associated preorder* and $a \succcurlyeq b$, $b > c$ implies that $a > c$ (also $a > b$, $b \succcurlyeq c$ implies that $a > c$). Also $\mathscr{U}$ is T_0 if and only if $(X, \preccurlyeq)$ is a poset [4]. For example if $X \equiv R^n$ define $(x_1, \cdots, x_n) > 0$ to mean $x_i > 0$ for $i = 1, 2, \cdots, n$, then $\mathscr{U}$ is just the usual euclidean topology and $(x_1, \cdots, x_n) \succcurlyeq 0$ if $x_i \geqq 0$ for $i = 1, 2, \cdots, n$. In fact in a Banach lattice $(B, \leqq')$ with strong unit if we define $x > 0$ to mean x is a strong unit then $\mathscr{U}$ is homeomorphic with the metric topology and $\preccurlyeq \; = \; \leqq'$ [5]. We will denote the closure of $S \subseteq X$ in $\mathscr{U}$ by $S^{-\mathscr{U}}$ or S^- if the latter does not cause confusion.

If $S \subseteq X$ we say S is $\leqq$*-convex* ($\preccurlyeq$*-convex*) if $a, b \in S$ $a \leqq x \leqq b$ $(a \preccurlyeq x \preccurlyeq b)$ implies that $x \in S$. We say $a \in S$ is a $\leqq$*-extreme* ($\preccurlyeq$*-extreme*) *point* of S if $x \leqq a \leqq y$ $(x \preccurlyeq a \preccurlyeq y)$ with $x, y \in S$ implies that $a = x$ or $a = y$ $(a \preccurlyeq x$ or $a \succcurlyeq y)$. We denote the $\leqq$-extreme ($\preccurlyeq$-extreme) points of S by $E(S)$ $(e(S))$. It is clear that $x \in E(S)$ $(x \in e(S))$ if and only if x is a maximal or a minimal element of $(S, \leqq)$ $((S, \preccurlyeq))$. The $\leqq$*-convex hull* of S, denoted by

$C(S)$, is the smallest $\leqq$-convex set containing S. Similarly we define the $\preccurlyeq$-*convex hull*, $c(S)$. It is easily seen that $C(S)$ and $c(S)$ do in fact exist and also $C(S) = \{x: a \leqq x \leqq b \text{ for some } a, b \in S\}$, with a similar formula for $c(S)$. We note that $e(S) \subseteq E(s)$ and $C(S) \subseteq c(S)$.

We shall need one or other of the following conditions:

(α) For all $a, b \in X$ ($y < a$ implies $y < b$) if and only if ($b < x$ implies $a < x$)

(β) $a < b$ implies that $a < x$ for all $x \in \{z: z > b\}^-$
and that $y < b$ for all $y \in \{z: z < a\}^-$.

There exist posets satisfying neither (α) nor (β), however each pogroup satisfies both. We note that our condition (α) is the same as condition (Σ) in [4].

Let $(Y, \preccurlyeq)$ be a preordered set. By taking the sets $\{y: y \succcurlyeq a\}$, $\{y: y \preccurlyeq b\}$ where $a, b \in Y$ as a subbase we define the *Frink interval topology* $\mathscr{F}$ on Y. Baker [1] proved his results using the topology $\mathscr{F}$ on a poset.

In the context of solving a class of functional equations Miller [3] defined cellular internity in R^n; the concept of cell-convexity follows immediately from it. We generalize the definition to abelian l-groups. Let $(G, \preccurlyeq)$ be an abelian l-group. Denote $\{x: x \succcurlyeq 0\}$ by G^+ and for $a, b \in G^+$ write $a \sim b$ if there exist positive integers m and n such that a $\preccurlyeq mb$ and $b \preccurlyeq na$. If $a \in G^+$ write a^0 for the set $\{x: x \sim a\}$. Call the sets x^0 for $x \in G^+$ *archimedean classes* and denote the family of archimedean classes by $\mathscr{A}$ [5]. We define two partial orders, $\preccurlyeq$ and $\lll$, on $\mathscr{A}$. We say $a^0 \preccurlyeq b^0$ if $a \preccurlyeq nb$ for some positive integer n; and $a^0 \ll b^0$ if $na \preccurlyeq b$ for all integers n.We write $[a\,b\,c]$ if $a \preccurlyeq b \preccurlyeq c$ and if $(c-b)^0 = (b-a)^0$. We say $S \subseteq G$ is *cell-convex* if $a, b \in S$ and $[a\,x\,b]$ implies that $x \in S$. We say $a \in S$ is a *cell-extreme point* of S if $[x\,a\,y]$ with $x, y \in S$ implies that $a = x$ or $a = y$. We denote the cell-extreme points of S by $\mathscr{E}(S)$. The intersection of all cell-convex sets containing S is itself cell-convex and is the smallest such set; it is called the *cell-convex hull* of S, denoted by $\mathscr{C}(S)$.

The open-interval topology on $(G, \preccurlyeq)$ is discrete if $(G, \preccurlyeq)$ is not fully ordered. So to obtain a suitable topology on G we consider *compatible tight Riesz orders* (abbreviated CTROs) on $(G, \preccurlyeq)$ [5]. A *tight Riesz group* $(H, \leqq)$ is a directed abelian pogroup which satisfies the following interpolation property: if $a_1, a_2, b_1, b_2 \in H$ such that $a_i < b_j$ for $i, j = 1, 2$ then there exists $c \in H$ such that $a_i < c < b_j$ for $i, j = 1, 2$. A CTRO on $(G, \preccurlyeq)$ is a non-trivial partial order $\leqq$ making $(G, \leqq)$ a tight Riesz group with $\preccurlyeq$ as its associated order. Each CTRO gives rise to an open-intrval topology which is Hausdorff. If $\leqq$ is a CTRO with topology $\mathscr{U}$ and $a < b$ then $(a, b)^{-\mathscr{U}} = \{x: a \preccurlyeq x \preccurlyeq b\}$. Also the family of sets $\{(-a, a): a > 0\}$ forms a base of neighbourhoods at 0 and $\leqq$ is isolated. Also $e(S) \subseteq \mathscr{E}(S)$ and $\mathscr{C}(S) \subseteq c(S)$ for all $S \subseteq G$.

We quote one result on CTROs from [5].

LEMMA 1 [5]. *There is a one-one correspondence between CTROs on* $(G, \preccurlyeq)$ *and sets* $\mathscr{T}$ *with the properties*:

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(i) *$\mathscr{T}$ is a proper dual ideal of* $(\mathscr{A}, \preccurlyeq)$

(ii) *if* $a^0 \in \mathscr{T}$ *then there exist* $b^0, c^0 \in \mathscr{T}$ *such that* $a = b + c$

(iii) *if* $x^0 \preccurlyeq y^0$ *for all* $y^0 \in \mathscr{T}$ *then* $x = 0$.

In fact the set of archimedean classes of the strictly positive elements of a CTRO satisfies (i)–(iii) and vice versa.

It can be shown that every divisible abelian l-group has at least one CTRO [5].

3. The $\leqq$-convex and $\preccurlyeq$-convex cases

THEOREM 1. *Let K be compact in $(X, \leqq, \mathscr{U})$ then*

(i) $CE(K) = C(K)$ *if* $(X, \leqq)$ *satisfies* (α) *or* (β)

(ii) $ce(K) = c(K)$ *if* $(X, \leqq)$ *satisfies* (α) *or if* $(X, \leqq)$ *is dense and satisfies* (β)

PROOF. Franklin [2] has shown that to prove (i) it is sufficient to prove that for each $a \in K$, $(K \cap \{x: x \geqq a\}, \leqq)$ has a maximal element and that $(K \cap \{x: x \leqq a\}, \leqq)$ has a minimal element. Let C be a chain in $(K \cap \{x: x \geqq a\}, \leqq)$, we want to show that C is bounded above in $(K \cap \{x: x \geqq a\}, \leqq)$. C is a net with itself as indexing set so by the definition of $\mathscr{U}$ there exists $b \in K$ such that $\mathscr{U}$-lim $C = b$, we may also assume that C has no largest element. If $x > b$ then $\{y: y < x\}$ is a neighbourhood of b so $c < x$ for all $c \in C$. So if (α) holds then $b \succcurlyeq c$ for all $c \in C$. So in fact $b > c$ for all $c \in C$ and $b \in K \cap \{x: x \geqq a\}$. Now suppose instead that (β) holds. If $c \in C$ there exists $c_1 \in C$ such that $c < c_1$ and so $b \in \{x: x > c_1\}^-$. Hence by (β) $b > c$ for all $c \in C$. Hence (i) follows by applying Zorn's lemma and a dual argument for $K \cap \{x: x \leqq a\}$.

Franklin's result can be shown to be true also for preordered sets. Let $a \in K$ and let D be a chain in $(K \cap \{x: x \succcurlyeq a\}, \preccurlyeq)$, so there exists $f \in K$ such that $\mathscr{U}$-lim $D = f$. If $x > f$ then $x > d$ for all $d \in D$. So if (α) holds then $f \succcurlyeq d$ for all D. Now suppose that $(X, \leqq)$ is dense and (β) holds. Let $d \in D$, if $y < d$ then for some $z \in X$ $y < z < d$. So $f \in \{x: x > z\}^-$ and by (β) $y < f$. So $x > f$ implies that $x > d$, and $y < d$ implies that $y < f$. Hence $f \succcurlyeq d$ for all $d \in D$. Hence (ii) follows.

Baker [1] proved that if K is convex and compact in $(Y, \leqq, \mathscr{F})$ where $(Y, \leqq)$ is a poset then $CE(K) = K$. We restate his result, and for completeness prove it.

LEMMA 2 (Baker). *Let $(Y, \leqq)$ be a preordered set and let K be compact in $(Y, \leqq, \mathscr{F})$ then $CE(K) = C(K)$.*

PROOF. Let $a \in K$ and let C be a chain in $(K \cap \{x: x \geqq a\}, \leqq)$. Then the family of $\mathscr{F}$-closed sets $\{y: y \geqq c\}$, $c \in C$, has the finite intersection property. So by the compactness of K there exists $b \geqq c$ for all $c \in C$, and $b \in K$. The rest of the proof follows the method used above.

We now show that the first part of Theorem 1 (ii) can alternatively be proved using Lemma 2.

LEMMA 3. *If* $(X, \leqq, \mathscr{U})$ *satisfies* (α) *then* $\mathscr{U}$ *is at least as strong as* $\mathscr{F}$ *for* $(X, \preccurlyeq)$; *hence if* K *is* $\mathscr{U}$*-compact then* $ce(K) = c(K)$.

PROOF. We shall show that $S = \{x : x \succcurlyeq a\}$ is $\mathscr{U}$-closed for each $a \in X$. Let $\{y_\alpha\}$ be a net in S with $\mathscr{U}$-limit b. If $c > b$ then $c > y_\alpha$ for some α so $c > a$. Since (α) holds we conclude that $b \succcurlyeq a$ so that S is $\mathscr{U}$-closed. Hence $\mathscr{U}$ is at least as strong as the Frink interval topology on $(X, \preccurlyeq)$. So if K is $\mathscr{U}$-compact then it is $\mathscr{F}$-compact. The rest follows by Lemma 2.

The class of posets satisfying (α) and (β) is quite large as is made clear now.

LEMMA 4. *If* $(X, \leqq)$ *is a pogroup then it satisfies* (α) *and* (β).

PROOF. Now $y < a$ implies $y < b$ means that $a - p < b$ for all $p > 0$; and $b < x$ implies $a < x$ means that $b + p > a$ for all $p > 0$. So (α) is satisfied since these are equivalent.

If $a < b$ and $x \in \{z : z > b\}^-$ then $\{y : y < b - a + x\}$ is a neighbourhood of x. So for some c, $b < c < b - \mathrm{a} + x$, hence $a < x$. The dual follows easily and so (β) is satisfied.

We note that $(X, +)$ need not be abelian.

EXAMPLE. There exists a lattice satisfying neither (α) nor (β). Consider the lattice consisting of the elements $a_1 < a_2 < a_3 < a_4 < a_5$ and b such that $a_1 < b < a_5$, and b not comparable with a_2, a_3 and a_4. Then $x > b$ implies that $x > a_4$ but $y < a_4$ does not imply that $y < b$ (for example $a_3 < a_4$ and $a_3 \nless b$). Also $a_2 < a_3$ and $a_4 \in \{x : x > a_3\}$, so $b \in \{x : x > a_3\}^-$ but $a_2 \nless b$.

4. The cell-convex case

LEMMA 5. *Let* $(G, \preccurlyeq)$ *be a divisible abelian l-group, with a CTRO,* $\leqq$, K *a compact cell-convex subset of* $(G, \leqq, \mathscr{U})$ *and* 0, $a \in K \cap G^+$. *Then* $a^0 \preccurlyeq p^0$ *for all* $p > 0$.

PROOF. Let $L = \{x : x = ra \text{ for } 0 \leqq r \leqq 1, \text{ and } r \text{ rational}\}$.

By cell-convexity of K we have $L \subseteq K$, so L^- is compact. Let $p > 0$ and cover L^- by the family of sets $(x - p, x + p)$ where $x \in L^-$. Then by compactness there exist $0 \leqq r_1, r_2 \leqq 1$, $r_1 \neq r_2$, and some $x \in L^-$ such that $x - p < r_1 a$, $r_2 a < x + p$. Also for some r_3, $-p < x - r_3 a < p$, since $x \in L^-$. So $-2p < (r_1 - r_3)a$, $(r_2 - r_3)a < 2p$. Since at least one of $r_1 - r_3$, $r_2 - r_3$ is non-zero we have $a^0 \preccurlyeq p^0$.

COROLLARY 1. *For the above* a, *the set* $\{x : x^0 \succcurlyeq a^0\}$ *is the strictly positive cone of a CTRO.*

PROOF. Properties (i) and (ii) of Lemma 1 are easily proved. If $y^0 \ll a^0$ then certainly $y^0 \ll p^0$ for all $p > 0$, so $y = 0$.

Denote this CTRO by $\underset{a}{\leqq}$, and the open-interval topology of $(G, \underset{a}{\leqq})$ by $\mathscr{U}_a$.

COROLLARY 2. *The topology $\mathscr{U}_a$ is at least as strong as $\mathscr{U}$.*

PROOF. It will be sufficient to prove that if $p > 0$ then $\{x: -p < x < p\}$ is a neighbourhood of 0 in $\mathscr{U}_a$. There exists a positive integer m such that $p/2 \succcurlyeq a/m$, since $p^0 \succcurlyeq a^0$. So $(a/m) < p$ since $\leqq$ is isolated. If $(-a/m) \underset{a}{<} y \underset{a}{<} (a/m)$ then certainly $(-a/m) \prec y \prec (a/m)$, so $-p < y < p$. Hence $\mathscr{U}_a$ is at least as strong as $\mathscr{U}$.

THEOREM 2. *Let $(G, \preccurlyeq)$ be a divisible abelian l-group, $\leqq$ a CTRO on $(G, \preccurlyeq)$, K a compact cell-convex subset of $(G, \leqq, \mathscr{U})$ then*

$$[\mathscr{C}e(K)]^- = K.$$

PROOF. Let $x \in K$ then by Theorem 1 (ii) and Lemma 4 there exist a, $b \in e(K)$ such that $b \preccurlyeq x \preccurlyeq a$. We may assume without loss of generality that $b = 0$.

Now $\{y: 0 \underset{a}{<} y \underset{a}{<} a\} \subseteq \mathscr{C}e(K)$, since $a^0 \preccurlyeq y^0 \preccurlyeq a^0$ and $a^0 \preccurlyeq (a-y)^0 \preccurlyeq a^0$, so $[0\ y\ a]$. By Lemma 5 Corollary 2

$$\{z: 0 \preccurlyeq z \preccurlyeq a\} = \{y: 0 \underset{a}{<} y \underset{a}{<} a\}^{-\mathscr{U}_a} \subseteq \{y: 0 \underset{a}{<} y \underset{a}{<} a\}^{-\mathscr{U}} \subseteq [\mathscr{C}e(K)]^{-\mathscr{U}},$$

hence $x \in [\mathscr{C}e(K)]^-$; so $K \subseteq [\mathscr{C}e(K)]^-$. The rest is obvious since $e(K) \subseteq K$, K is cell-convex, and $\mathscr{U}$ is Hausdorff.

COROLLARY. *With the hypotheses as above $[\mathscr{C}\mathscr{E}(K)]^- = K$.*

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