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CURVATURE AND RADIUS OF CURVATURE FOR FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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1. Introduction. For $k \geqq 2$ denote by V_k the class of functions f regular in $U = \{z : |z| < 1\}$ and having the representation

$$f(z) = \int_0^z \exp\left\{- \int_0^{2\pi} \log(1 - \xi e^{-it})^2 d\mu(t)\right\} d\xi \tag{1.1}$$

where μ is a real-valued function of bounded variation on $[0, 2\pi]$ with

$$\int_0^{2\pi} d\mu(t) = 1, \qquad \int_0^{2\pi} |d\mu(t)| \leqq k/2. \tag{1.2}$$

V_k is the class of functions with boundary rotation at most $k\pi$.

The boundary rotation of a domain may be thought of as the total variation (under a complete circuit) of the argument of the boundary tangent vector, whenever such a tangent exists. (See [**4**] for a more detailed description.) It is clear geometrically that $k \geqq 2$ and that V_2 is the class of functions mapping U onto a convex domain.

Consider now any function f regular and locally schlicht in U, and consider the Jordan curve $\Gamma_r = \{ f(re^{i\psi}) : 0 \leqq \psi \leqq 2\pi\}$. Let $s(r, \psi)$ measure arc length along Γ_r, and let $\varphi(r, \psi)$ measure the angle between the positive real axis and the tangent vector to Γ_r at $f(re^{i\psi})$. Then the curvature of Γ_r at $f(re^{i\psi})$ is $\kappa(r, \psi, f) = d\varphi/ds$, and the radius of curvature is $\rho(r, \psi, f) = \kappa(r, \psi, f)^{-1}$. It is known [**2**, p. 359] that with $z = re^{i\psi}$,

$$\kappa(r, \psi, f) = \frac{\operatorname{Re}(1 + zf''(z)/f'(z))}{|zf'(z)|}. \tag{1.3}$$

The problem of estimating $\kappa(r, \psi, f)$ for various classes of functions has attracted considerable attention. For example, if $\mathscr{S}^*$ denotes the class of functions starlike with respect to the origin, then it is known that the problem

$$\max_{f \in \mathscr{S}^*} \max_{\psi} \kappa(r, \psi, f)$$

is solved for each $0 < r < 1$ by the function $f(z) = z/(1 - z)^2$. (See [**1**, pp. 599-601] for a partial history of this problem.) From the definitions of boundary rotation and curvature, it is evident that the two concepts are closely related. The purpose of this note is to establish sharp upper and lower bounds for $\kappa(r, \psi, f)$ and $\rho(r, \psi, f)$ when $f \in V_k$.

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2. Statement of results. For $0 < r < 1$, set $H(r) = (1 + r^2)/2r - \{\log((1 + r)/(1 - r))\}^{-1}$. Simple calculations show that $H(r)$ increases strictly with r and that $0 < H(r) < 1$.

THEOREM 2.1. *Let $f \in V_k$. Then*

$$\left[\frac{r^2 - kr + 1}{r}\right]\left[\frac{1 + r}{1 - r}\right]^{k/2} \leqq \kappa(r, \psi, f)$$
$$\leqq \frac{1 - r^2}{r}\left[\frac{(1 + r)^{H(r)-1}}{(1 - r)^{H(r)+1}} \frac{2r}{\log((1 + r)/(1 - r))}\right]^{(k+2)/4}.$$

Both bounds are sharp for all $0 < r < 1$.

To the best of the author's knowledge, the problem of determining $\max_{f\in\mathscr{S}} \max_\psi \kappa(r, \psi, f)$, where $\mathscr{S}$ is the class of normalized schlicht functions, has not been solved. In this direction we have the following negative result.

COROLLARY 2.2. *For each $0 < r < 1$,*

$$\max_{f\in\mathscr{S}} \max_{\psi} \kappa(r, \psi, f)$$

is not attained by a Koebe function $f_\theta(z) = z/(1 + e^{i\theta}z)^2$.

It is known [**5**] that for $f \in V_k$, $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ if $|z| < R_k = [k - (k^2 - 4)^{\frac{1}{2}}]/2$. Combining this result with Theorem 2.1, we can estimate $\rho(r, \psi, f)$.

THEOREM 2.3. *Let $f \in V_k$. Then for $0 < r < R_k$ we have*

$$\frac{r}{1 - r^2}\left[\frac{(1 - r)^{H(r)+1}}{(1 + r)^{H(r)-1}} \frac{\log((1 + r)/(1 - r))}{2r}\right]^{(k+2)/4} \leqq \rho(r, \psi, f)$$
$$\leqq \frac{r}{r^2 - kr + 1}\left(\frac{1 - r}{1 + r}\right)^{k/2}.$$

If $r = R_k < 1$, the lower bound given above remains valid, but we may have $\sup_\psi \rho(r, \psi, f) = +\infty$. *If $R_k < r < 1$, we may have* $\inf_\psi \rho(r, \psi, f) = -\infty$ and $\sup_\psi \rho(r, \psi, f) = +\infty$. *All bounds are sharp.*

For $k = 2$, F. R. Keogh [**3**] found the sharp upper bound for $\rho(r, \psi, f)$. This was also found by V. A. Zmorovič [**6**], who in addition determined the sharp lower bound for $\rho(r, \psi, f)$ in the case $k = 2$. By employing different methods of proof, we extend these results to V_k.

3. Proofs. Let $m \geqq 3$ be a fixed integer, and define $V_k(m)$ to be that subclass of V_k such that the integrator μ in (1.1) is a step function with at most m jump discontinuities. We first establish Theorem 2.1 for the class $V_k(m)$. Since Theorem 2.1 is independent of m and since the step functions are dense in the functions of bounded variation, we see by allowing $m \to \infty$ that the theorem

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is also true for V_k. Also, in proving Theorem 2.1 for the class $V_k(m)$, it suffices to consider integrators μ satisfying

$$\int_0^{2\pi} |d\mu(t)| = k/2,$$

since both the upper and lower bounds in the theorem are monotone functions of k.

Let μ be such a step function, with jumps $c_j (1 \leqq j \leqq m)$ at the points $\theta_j (1 \leqq j \leqq m)$ respectively. Then (1.2) becomes

$$\sum_{j=1}^{m} c_j = 1, \qquad \sum_{j=1}^{m} |c_j| = k/2. \tag{3.1}$$

We now fix $z = re^{i\psi}$. Defining the vectors $\Theta = (\theta_1, \dots, \theta_m)$ and $C = (c_1, \dots, c_m)$, we see directly from (1.1) and (1.3) that

$$\frac{r}{1 - r^2} \kappa(r, \psi, f) = G(\Theta, C) \tag{3.2}$$

where

$$G(\Theta, C) = \left[\sum_{j=1}^{m} c_j (1 + r^2 - 2r\cos(\psi - \theta_j))^{-1} \right] \times \prod_{j=1}^{m} (1 + r^2 - 2r\cos(\psi - \theta_j))^{c_j}. \tag{3.3}$$

If

$$\Phi(m) = \max\left\{ \frac{r}{1 - r^2} \kappa(r, \psi, f) : f \in V_k(m) \right\}$$

and

$$\varphi(m) = \min\left\{ \frac{r}{1 - r^2} \kappa(r, \psi, f) : f \in V_k(m) \right\},$$

then

$$\begin{aligned} \Phi(m) &= \max G(\Theta, C) \\ \varphi(m) &= \min G(\Theta, C) \end{aligned} \tag{3.4}$$

where we maximize and minimize over the variable vectors Θ and C, subject to the constraints (3.1).

We first evaluate $G(\Theta, C)$ for special values of Θ.

LEMMA 3.1. *Let $z = re^{i\psi}$ be fixed, and define $G(\Theta, C)$ by* (3.3). *If $\Theta = (\theta_1, \dots, \theta_m)$ and if $\theta_j \equiv \psi \pmod{\pi}$ for all j $(1 \leqq j \leqq m)$, then for any C satisfying* (3.1) *we have*

$$(r^2 - kr + 1) \frac{(1 + r)^{k/2 - 1}}{(1 - r)^{k/2 + 1}} \leqq G(\Theta, C) \leqq \frac{(1 + r)^{H(r) - 1}}{(1 - r)^{H(r) + 1}} \frac{2r}{\log((1 + r)/(1 - r))}$$

where $H(r) = (1 + r^2)/2r - (\log((1 + r)/(1 - r)))^{-1}$. *In particular,*

$$\varphi(m) \leqq 1 < \Phi(m), \tag{3.5}$$

with $\varphi(m) = 1$ *only if* $k = 2$.

Proof. Let $J = \{j : \cos(\psi - \theta_j) = 1\}$, $J' = \{j : \cos(\psi - \theta_j) = -1\}$. Set $C = (c_1, \ldots, c_m)$, $p_j = \max(c_j, 0)$, $n_j = \max(-c_j, 0)$, $A = \sum_{j \in J} p_j$, $B = \sum_{j \in J} n_j$, and $E = A - B$. From (3.1) we have $0 \leqq A \leqq (k + 2)/4$ and $0 \leqq B \leqq (k - 2)/4$, and hence $-(k - 2)/4 \leqq E \leqq (k + 2)/4$. After some computation, in which we use the hypothesis that $J \cup J' = \{1, \ldots, m\}$, we find that

$$G(\Theta, C) = \left[\frac{1 - r}{1 + r}\right]^{2E} \left[\frac{4rE}{(1 - r)^2} + 1\right]. \tag{3.6}$$

We now differentiate with respect to E. If follows that the only critical point occurs at $E_0 = (1 - H(r))/2$, and it is simple to verify that E_0 corresponds to a maximum. As noted previously, $0 < H(r) < 1$, and so $-(k - 2)/4 \leqq E_0 \leqq (k + 2)/4$. Evaluating (3.6) at E_0, we find

$$G(\Theta, C) \leqq \frac{(1 + r)^{H(r)-1}}{(1 - r)^{H(r)+1}} \frac{2r}{\log((1 + r)/(1 - r))}.$$

We now minimize $G(\Theta, C)$. From above it follows that the minimum occurs when $E = -(k - 2)/4$ or $E = (k + 2)/4$. After comparing functional values at these two points, we see that

$$G(\Theta, C) \geqq (r^2 - kr + 1) \frac{(1 + r)^{k/2-1}}{(1 - r)^{k/2+1}}.$$

Also note that if in (3.6) we set $E = 0$, then $G(\Theta, C) = 1$, from which (3.5) follows easily. This proves the lemma.

We now examine problem (3.4). Our technique is conceptually simple, although the details are cumbersome. We shall first study the problem of determining $\Phi(m)$. Clearly $G(\Theta, C)$ attains its maximum, say at $\Theta^* = (\theta_1^*, \ldots, \theta_m^*)$, $C^* = (c_1^*, \ldots, c_m^*)$. From (3.4) it is clear that

$$\Phi(m) = \max G(\Theta, C^*) \tag{3.7}$$

where we now maximize only with respect to the vector Θ. In other words, we now allow μ to have m jumps with fixed heights c_j^* $(1 \leqq j \leqq m)$, but we allow the respective positions θ_j of the jumps to vary freely in $[0, 2\pi]$. A similar technique is used in [**4**]. Recall that C^* is a fixed vector, and that θ_j is the position of the jump of height c_j^*.

LEMMA 3.2. *Let* $z = re^{i\psi}$ *be fixed, and consider the problem defined by* (3.7).

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Let $\Theta^* = (\theta_1^*, \ldots, \theta_m^*)$ *denote any solution vector to the problem* (3.7). *Then one of the following conditions must hold*:

(i) *For each* j $(1 \leqq j \leqq m)$, *we have* $\theta_j^* \equiv \psi \pmod{\pi}$.

(ii) *There exists a constant* A *satisfying* $0 < A \leqq (k+2)/4$ *and nonnegative constants* a_1, a_2 *satisfying* $a_1 + a_2 = A$, *such that*

$$\Phi(m) = \frac{(1 + r^2 + 2r(a_1 - a_2)/A)^A}{(1-r)^{2a_2}(1+r)^{2a_1}}.$$

Proof. Suppose that (i) does not hold, and choose h such that $\cos(\psi - \theta_h^*) \neq \pm 1$. For each j $(1 \leqq j \leqq m)$, set $x_j = \cos(\psi - \theta_j)$, so $x_j \in [-1, +1]$. Also set $\Delta(x_j) = 1 + r^2 - 2rx_j$. After changing variables in (3.7), we see that

$$\Phi(m) = \max_X G(X) \tag{3.8}$$

where $X = (x_1, \ldots, x_m)$ and

$$G(X) = \left[\sum_{j=1}^{m} c_j^* \Delta(x_j)^{-1}\right] \prod_{j=1}^{m} \Delta(x_j)^{c_j^*}. \tag{3.9}$$

If we now set $x_j^* = \cos(\psi - \theta_j^*)$ for $1 \leqq j \leqq m$, and $X^* = (x_1^*, \ldots, x_m^*)$, it is clear that $\Phi(m) = G(X^*)$.

We now claim that if j satisfies $|x_j^*| \neq 1$, then

$$x_j^* = x_h^*, \quad \Delta(x_j^*) = \Delta(x_h^*). \tag{3.10}$$

To see this, we note that $|x_j^*| \neq 1$ implies $dG/dx_j = 0$ when evaluated at X^*, which is equivalent to

$$\Delta(x_j^*) \sum_{i=1}^{m} c_i^* \Delta(x_i^*)^{-1} = 1. \tag{3.11}$$

In particular, we know $|x_h^*| \neq 1$, and so

$$\Delta(x_h^*) \sum_{i=1}^{m} c_i^* \Delta(x_i^*)^{-1} = 1. \tag{3.12}$$

From (3.11) and (3.12) it follows immediately that (3.10) holds.

Next, since X^* maximizes $G(X)$, we have for each j $(1 \leqq j \leqq m)$ that $x_j(dG/dx_j) \geqq 0$ when evaluated at X^*. Straightforward computation shows this to be equivalent to

$$x_j^* c_j^* (x_j^* - x_h^*) \geqq 0 \quad (1 \leqq j \leqq m). \tag{3.13}$$

We now claim that $c_j^* < 0$ implies $|x_j^*| \neq 1$. To prove this, suppose $x_j^* = 1$. From (3.13), $c_j^*(1 - x_h^*) \geqq 0$, and since $x_h^* < 1$, we have $c_j^* \geqq 0$. Similarly, if $x_j^* = -1$, then $c_j^*(1 + x_h^*) \geqq 0$, and so $c_j^* \geqq 0$. This establishes the claim. Combining this fact with (3.10), we see that $c_j^* < 0$ implies $\Delta(x_j^*) = \Delta(x_h^*)$.

We now combine this latter fact with (3.8), (3.12), and (3.1). We find that

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with $d = \Delta(x_h^*)$ and $p_i^* = \max\{c_i^*, 0\}$ $(1 \leqq i \leqq m)$ we have

$$\Phi(m) = \max_X G(X) = d^{-(k+2)/4} \prod_{i=1}^{m} \Delta(x_i^*)^{p_i^*}, \tag{3.14}$$

and

$$d \sum_{i=1}^{m} p_i^* \Delta(x_i^*)^{-1} = (k+2)/4, \qquad \sum_{i=1}^{m} p_i^* = (k+2)/4. \tag{3.15}$$

Next put $T_0 = \{j : |x_j^*| \neq 1\}$, $T_1 = \{j : x_j^* = 1\}$, and $T_2 = \{j : x_j^* = -1\}$. Clearly $j \in T_1$ implies $\Delta(x_j^*) = (1-r)^2$, $j \in T_2$ implies $\Delta(x_j^*) = (1+r)^2$, and (by (3.10)) $j \in T_0$ implies $\Delta(x_j^*) = \Delta(x_h^*) = d$. Also put

$$a_0 = \sum_{j \in T_0} p_j^*, \; a_1 = \sum_{j \in T_1} p_j^*, \; a_2 = \sum_{j \in T_2} p_j^*,$$

so $a_0 + a_1 + a_2 = (k+2)/4$. (If T_i is empty, we use the convention $a_i = 0$.) With $A = (k+2)/4 - a_0$, we have from (3.14) and (3.15) that

$$\Phi(m) = \frac{(1-r)^{2a_1}(1+r)^{2a_2}}{d^A}, \qquad d\left(\frac{a_1}{(1-r)^2} + \frac{a_2}{(1+r)^2}\right) = A. \tag{3.16}$$

If $A = 0$, then $a_0 = (k+2)/4$, and since the nonnegative numbers a_i satisfy $a_0 + a_1 + a_2 = (k+2)/4$, we must have $a_1 = a_2 = 0$, and hence from (3.16) $\Phi(m) = 1$. However, from (3.5), we know $\Phi(m) > 1$, and so $0 < A \leqq (k+2)/4$. After some simplification, we then see from (3.16) that

$$d = \frac{(1-r^2)^2}{1 + r^2 + 2r(a_1 - a_2)/A},$$

and the lemma follows upon substituting this into (3.16).

We now begin the proof of Theorem 2.1. As noted previously, it suffices to prove the theorem for $V_k(m)$, where the integrator μ in (1.1) satisfies (3.1). Let $z = re^{i\psi}$ be fixed, and as before put

$$\Phi(m) = \max\left\{\frac{r}{1-r^2}\,\kappa(r, \psi, f) : f \in V_k(m)\right\}.$$

Using the notation of Lemma 3.2, we have two cases to consider. Suppose first that $\theta_j^* \equiv \psi \pmod{\pi}$ for all $j(1 \leqq j \leqq m)$. By Lemma 3.1,

$$\Phi(m) \leqq \frac{(1+r)^{H(r)-1}}{(1-r)^{H(r)+1}} \frac{2r}{\log((1+r)/(1-r))} \tag{3.17}$$

where $H(r) = (1+r^2)/2r - (\log((1+r)/(1-r)))^{-1}$.

Suppose now that condition (ii) of Lemma 3.2 holds, so that

$$\Phi(m) = \frac{(1 + r^2 + 2r(a_1 - a_2)/A)^A}{(1 - r)^{2a_2}(1 + r)^{2a_1}} \tag{3.18}$$

where $a_1 \geqq 0$, $a_2 \geqq 0$, $A = a_1 + a_2$, and $0 < A \leqq (k + 2)/4$. Fixing A and allowing a_1 and a_2 to vary subject to the above conditions, we see by the usual methods of the calculus that the maximum in (3.18) occurs when $2a_1 = A(1 - H(r))$, $2a_2 = A(1 + H(r))$, which gives

$$\Phi(m) \leqq \left[\frac{2r(1 - r^2)^{-1}}{\log((1 + r)/(1 - r))}\right]^A \left(\frac{1 + r}{1 - r}\right)^{A \cdot H(r)}.$$

A straightforward but rather tedious calculation shows that the quantity in brackets is at least 1, and hence we have

$$\Phi(m) \leqq \left[\frac{2r}{\log((1 + r)/(1 - r))} \frac{(1 + r)^{H(r)-1}}{(1 - r)^{H(r)+1}}\right]^{(k+2)/4} \tag{3.19}$$

since $A \leqq (k + 2)/4$. In Lemma 3.1 we showed that the quantity in brackets in (3.19) is at least 1, so comparing (3.17) and (3.19) we see that the right-hand inequality of Theorem 2.1 is valid.

We now minimize $(r/(1 - r^2))\kappa(r, \psi, f)$ for $f \in V_k$. Again it suffices to consider the class $V_k(m)$, and we may assume (3.1) holds. We first suppose $k > 2$. With the notation of (3.4), suppose

$$\begin{aligned} \varphi(m) &= \min\left\{\frac{r}{1 - r^2} \kappa(r, \psi, f) : f \in V_k(m)\right\} \\ &= \min G(\Theta, C) \end{aligned}$$

occurs when $\Theta^* = (\theta_1^*, \ldots, \theta_m^*)$, $C^* = (c_1^*, \ldots, c_m^*)$. If for all j $(1 \leqq j \leqq m)$ we have $\theta_j^* \equiv \psi \pmod{\pi}$, then from Lemma 3.1

$$\varphi(m) \geqq (r^2 - kr + 1) \frac{(1 + r)^{k/2-1}}{(1 - r)^{k/2+1}}. \tag{3.20}$$

If there exists h such that $\theta_h^* \not\equiv \psi \pmod{\pi}$, then proceeding as in the proof of Lemma 3.2 we see that

$$\Delta(x_h^*) \sum_{i=1}^{m} c_i^* \Delta(x_i^*)^{-1} = 1,$$

and we also see that $\Delta(x_j^*) = \Delta(x_h^*)$ for any j such that $c_j^* > 0$. Also, defining $G(X)$ as in (3.9). we must have $d^2G/dx_h^2 \geqq 0$ when evaluated at X^* since X^* minimizes G and $|x_h^*| \neq 1$. Using these facts to simplify the expression for d^2G/dx_h^2, we conclude that we must in fact have $k \leqq 2$. This contradicts our assumption that $k > 2$, and hence $\theta_j^* \equiv \psi \pmod{\pi}$ for all j $(1 \leqq j \leqq m)$. Therefore (3.20) holds, and this is equivalent to the inequality in Theorem 2.1.

In order to prove the theorem for $k = 2$, we note that $V_2 \subset V_k$ for all $k > 2$.

Hence

$$\min_{f \in V_2} \kappa(r, \psi, f) \geqq \frac{r^2 - kr + 1}{r} \left[\frac{1+r}{1-r}\right]^{k/2}$$

for all $k > 2$, which implies that the same inequality holds with $k = 2$.

The function

$$f(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right] \tag{3.21}$$

shows that the lower bound in Theorem 2.1 is sharp for all r and k. In order to show that the upper bound is sharp, let r be given. Construct a step function μ with positive jumps $(k+2)(1 - H(r))/8$ and $(k+2)(1 + H(r))/8$ at $t = 0$ and $t = \pi$ respectively, and with a negative jump of magnitude $(k-2)/4$ at any value of t satisfying

$$1 + r^2 - 2r\cos(t) = \frac{(1-r^2)^2}{1 + r^2 - 2rH(r)}.$$

The resulting function $f_r \in V_k$ shows that the upper bound is sharp. Note that the function f_r varies with r.

In order to prove Corollary 2.2, let $re^{i\psi}$ be given, and set $f_\theta(z) = z/(1 - ze^{i\theta})^2$. A straightforward calculation shows

$$\kappa(r, \psi, f_\theta) \leqq \frac{2r}{\log((1+r)/(1-r))} \frac{(1+r)^{H(r)-1}}{(1-r)^{H(r)+1}}. \tag{3.22}$$

Since the quantity on the right-hand side of (3.22) is strictly greater than 1, we see from Theorem 2.1 that for any $k > 2$ there exists $f_r \in V_k$ such that

$$\max_\psi \kappa(r, \psi, f_r) > \max_\psi \kappa(r, \psi, f_\theta).$$

However, it is well-known (See [**5**] for references.) that if $2 \leqq k \leqq 4$, then V_k contains only schlicht functions. This proves the corollary.

We now prove Theorem 2.3. If $0 < r < R_k = [k - (k^2 - 4)^{\frac{1}{2}}]/2$. then $r^2 - kr + 1 > 0$, and Theorem 2.3 follows immediately from Theorem 2.1. For $r = R_k < 1$, the lower bound for $\rho(r, \psi, f)$ also follows directly from Theorem 2.1. In addition, we have $\sup_\psi \rho(r, \psi, f) = +\infty$ for f given by (3.21). Suppose now that $r > R_k$. For f given by (3.21) we have

$$\mathrm{Re}[1 + zf''(z)/f'(z)] = \mathrm{Re}[(z^2 + kz + 1)/(1 - z^2)].$$

Since at $z = r$ this quantity is positive and at $z = -r$ it is negative, a change in sign must occur on $|z| = r$. Hence, for this function,

$$\sup_\psi \rho(r, \psi, f) = +\infty \quad \text{and} \quad \inf_\psi \rho(r, \psi, f) = -\infty.$$

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