This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University Montreal, P. Q.

A DISCRETE ANALOGUE OF OPIAL'S INEQUALITY

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In a number of papers [1] - [7], successively simpler proofs were given for the following inequality of Opial [1], in case $p = 1$.

THEOREM 1. *If* $x(t)$ *is absolutely continuous with* $x(0) = 0$, *then for any* $p \geq 0$,

$$(1) \qquad \int_0^a |x'(t)x^p(t)|\,dt \leq \frac{a^p}{p+1} \int_0^a |x'(t)|^{p+1}\,dt \,;$$

Equality holds only if $x(t) = Kt$ *for some constant* K.

Proof. Let $z(t) = \int_0^t |x'(s)|\,ds$ and note that $z(t) \geq |x(t)|$ for all $t \geq 0$. Observe that

$$\int_0^a |x'(t)x^p(t)|\,dt \leq \int_0^a z'(t)z^p(t)\,dt = \frac{z^{p+1}(a)}{p+1} .$$

By Holder's inequality, we have $z^{p+1}(a) \leq a^p \int_0^a |z'(t)|^{p+1}dt$, from which (1) readily follows.

We remark that a proof of (1) for p a positive integer is given in [7], but the same proof fails for general p. The purpose of the present note is to prove the following discrete analogue of (1).

THEOREM 2. *Let* u_i *be a non-decreasing sequence of*

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non-negative numbers. Then for $p \geq 1$, we have

$$(2) \qquad \sum_{i=1}^{n} (u_i - u_{i-1}) u_i^p \leq \frac{(n+1)^p}{p+1} \sum_{i=1}^{n} (u_i - u_{i-1})^{p+1},$$

where $u_o = 0$.

Proof. Let $x_i = u_i - u_{i-1}$; then $u_i = \sum_{j=1}^{i} x_j$ where $x_i \geq 0$. We may now rewrite (2) as

$$(3) \qquad \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{i} x_j \right)^p \leq \frac{(n+1)^p}{p+1} \sum_{i=1}^{n} x_i^{p+1}.$$

We shall proceed to prove (3) by induction. Clearly (3) holds with $n = 1$. Now we assume (3) holds for n, and observe

$$(4) \quad \sum_{i=1}^{n+1} x_i \left(\sum_{j=1}^{i} x_j \right)^p \leq \frac{(n+1)^p}{p+1} \left\{ \sum_{i=1}^{n} x_i^{p+1} + (p+1) x_{n+1}^{*p} x_{n+1} \right\}$$

where $x_k^* = \frac{1}{k} \sum_{i=1}^{k} x_i$. By Young's inequality, one easily sees that $(p+1) x_{n+1}^{*p} x_{n+1} \leq x_{n+1}^{p+1} + p\, x_{n+1}^{*p+1}$. Using Hőlder's inequality, we may show that

$$(x_{n+1}^*)^{p+1} \leq \frac{1}{n+1} \sum_{j=1}^{n+1} x_j^{p+1}.$$

Substituting these estimates into (4), we find

$$\sum_{i=1}^{n+1} x_i \left(\sum_{j=1}^{i} x_j \right)^p \leq \frac{(n+1)^p}{p+1} \left\{ \sum_{i=1}^{n+1} x_i^{p+1} + \frac{p}{n+1} \sum_{i=1}^{n+1} x_i^{p+1} \right\}$$

$$\leq \frac{(n+2)^p}{p+1} \sum_{i=1}^{n+1} x_i^{p+1},$$

which is what we wish to prove.

REMARK 1. Inequality (3) fails to hold for $p < 1$.

Consider $p = \frac{1}{2}$, $n = 2$, $u_1 = 1$, and $u_2 = 2$.

REMARK 2. From the above proof, one readily sees that for all $p > 1$, strict inequality in fact holds in (2). In case $p = 1$, equality occurs only when $u_i = Ki$ for some constant $K \geq 0$ and for all $i = 1, 2, 3, \ldots$.

REMARK 3. To see that (2) is indeed a useful inequality, set $x_i = 1$ for all $i = 1, 2, 3, \ldots$ in (3) and obtain for all $p > 1$

$$\sum_{k=1}^{n} k^p < \frac{(n+1)^p n}{p+1} < \frac{(n+1)^{p+1} - 1}{p+1} = \int_1^{n+1} x^p dx ,$$

which shows that (2) yields a better estimate than that obtained by simply comparing areas. In case $p = 1$, (3) reduces to the familiar identity

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} .$$

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