Canad. Math. Bull. Vol. **38** (2), 1995 pp. 187–195

ON t- Spec($R[[X]]$)

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ABSTRACT. Let D be an integral domain, and let X be an analytic indeterminate. As usual, if I is an ideal of D, set $I_t = \bigcup\{J_v = (J^{-1})^{-1} \mid J$ is a nonzero finitely generated subideal of $I\}$; this defines the t-operation, a particularly useful star-operation on D. We discuss the t-operation on $R[[X]]$, paying particular attention to the relation between t- dim(R) and t- dim($R[[X]]$). We show that if P is a t-prime of R, then $P[[X]]$ contains a t-prime which contracts to P in R, and we note that this does not quite suffice to show that t- dim($R[[X]]$) $\geq$ t- dim(R) in general. If R is Noetherian, it is easy to see that t- dim($R[[X]]$) $\geq$ t- dim(R), and we show that we have equality in the case of t-dimension 1. We also observe that if V is a valuation domain, then t- dim($V[[X]]$) $\geq$ t- dim(V), and we give examples to show that the inequality can be strict. Finally, we prove that if V is a finite-dimensional valuation domain with maximal ideal M, then $MV[[X]]$ is a maximal t-ideal of $V[[X]]$.

1. **Introduction.** Throughout this paper, R denotes an integral domain with quotient field K. We begin with a brief review of the t-operation. If I is a nonzero fractional ideal of R, the inverse of I is given by $I^{-1} = \{u \in K \mid uI \subseteq R\}$. The v-operation on R is given by $I_v = (I^{-1})^{-1}$ and the t-operation by $I_t = \bigcup\{J_v = (J^{-1})^{-1} \mid J$ is a nonzero finitely generated subideal of $I\}$. An ideal I is called *divisorial* (or a *v-ideal*) if $I = I_v$; I is called a *t-ideal* if $I = I_t$. The v- and t-operations are examples of star-operations, and the reader is referred to [10] and [13] for a discussion of their properties, which we shall use freely (usually without reference). Of particular importance are the standard facts that every t-ideal is contained in a maximal t-ideal, that maximal t-ideals are prime, and that any prime minimal over a t-ideal is a prime t-ideal (t-prime). In particular, height 1 prime ideals are t-primes.

In [11], an attempt was made to relate the t-spectrum of the polynomial ring $R[X]$ to that of R. In this paper, we are interested in studying the t-spectrum of the power series ring $R[[X]]$. One vehicle for this study may be introduced by analogy with the definition of Krull dimension; as in [11] we define t- dim(D), the t-dimension of an integral domain D, to be the supremum of the lengths of chains of prime t-ideals of D. For the purposes of defining t-dimension, we include $\{0\}$ as a t-prime, thus a domain of Krull dimension 1 also has t-dimension 1. We shall be particularly interested in the relation between t- dim(R) and t- dim($R[[X]]$). In Section 2, we show that if P is a t-prime of R, then $P[[X]]$ contains a t-prime which contracts to P in R, but we note that this does not quite suffice to show that t- dim($R[[X]]$) $\geq$ t- dim(R) in general. We are able to show that this inequality does hold in many cases. For example, it is easily seen to hold when R is

Received by the editors September 23, 1993; revised January 12, 1994.
AMS subject classification: Primary: 13C15, 13F25; secondary: 13G05, 13F30.

Noetherian; more generally, it holds for the SFT-rings of Arnold [1]. (Recall that a ring D is an SFT-ring if for every ideal I of D, there is a finitely generated ideal $J \subseteq I$ and a fixed positive integer k with $a^k \in J$ for each $a \in I$. A good reference for this type of material on power series is [6].) We also note that this inequality holds for valuation domains (more generally, for divided domains). We are not able to prove that the inequality holds in general, however. Finally, we give examples showing that the inequality can be strict.

For a Noetherian domain R (which is not a field), it seems reasonable to expect that $t\text{-}\dim(R[[X]]) = t\text{-}\dim(R)$. In Section 3, we prove this equality for the case of t-dimension 1. A key element in this proof is the following result, which is interesting in its own right: if R is a Noetherian domain, then t-primes of $R[[X]]$ which do not contract to zero in R must be extended from R.

The fourth (and final) section is devoted to a study of the t-spectrum of $V[[X]]$ when V is a valuation domain. We have already noted that we have $t\text{-}\dim(V[[X]]) \geq t\text{-}\dim(V)$. In this section we prove that the inequality can be strict. In fact, we show that if V is an n-dimensional valuation domain in which each prime ideal is idempotent, then $t\text{-}\dim(V[[X]]) \geq 2n - 1$. We also show that $t\text{-}\dim(V[[X]]) = t\text{-}\dim(V)$ if V is an n-dimensional discrete valuation domain.

2. General results.

We begin with a result which we shall use frequently.

PROPOSITION 2.1 (D. F. ANDERSON-KANG [5][1]). *Let I be a nonzero fractional ideal of R. Then*

(1) $(IR[[X]])^{-1} = I^{-1}[[X]] = (I[[X]])^{-1}$ *and*

(2) $(IR[[X]])_v = I_v[[X]] = (I[[X]])_v$.

PROOF. We first assume that I is an integral ideal of R. Suppose that $u \in (IR[[X]])^{-1}$. Since $uI \subseteq R[[X]]$, we have $u \in K[[X]]$. Write $u = \sum u_i X^i$. For $a \in I$ we have $ua = u_0 a + u_1 aX + \cdots \in R[[X]]$. It follows that $u_j I \subseteq R$ for each j, that is, that $u \in I^{-1}[[X]]$. Hence $(IR[[X]])^{-1} \subseteq I^{-1}[[X]]$. It is easy to see that $I^{-1}[[X]] \subseteq I[[X]]^{-1}$. Thus $(IR[[X]])^{-1} \subseteq I^{-1}[[X]] \subseteq I[[X]]^{-1} \subseteq (IR[[X]])^{-1}$, proving the result for integral ideals. If I is a fractional ideal of R, then $aI \subseteq R$ for some nonzero element a of R, and we have $\big((aI)R[[X]]\big)^{-1} = (aI)^{-1}[[X]] = \big((aI)[[X]]\big)^{-1}$. Since all a's may be cancelled, this gives (1). Statement (2) follows from (1). ∎

COROLLARY 2.2. *If M is maximal and divisorial in R, then the only divisorial prime of $R[[X]]$ which contracts to M is $M[[X]]$.*

PROOF. By Proposition 2.1, $(MR[[X]])_v = M[[X]]$. Hence any divisorial ideal of $R[[X]]$ which contracts to M must contain $M[[X]]$. On the other hand, the only prime in $R[[X]]$ which properly contains $M[[X]]$ is $M + (X)$, which is not divisorial. ∎

[1] After giving a talk on a preliminary version of this paper which included Proposition 2.1, we were informed that D. F. Anderson and B. G. Kang had also proved this result in an unpublished manuscript. We are grateful to them for allowing us to include it here.

It is natural to ask whether P being a t-prime implies $P[[X]]$ a t-prime. The next four results provide an affirmative answer in the SFT-case and a satisfactory substitute in general, namely that $R \subseteq R[[X]]$ satisfies "lying-over" for t-primes.

PROPOSITION 2.3. *Let P be a prime minimal over (a) in R. Shrink $P[[X]]$ to a prime Q minimal over $PR[[X]]$ in $R[[X]]$. Then Q is minimal over (a) and is therefore a t-prime.*

PROOF. Shrink Q to a prime Q_1 minimal over (a) in $R[[X]]$. Then $Q_1 \cap R \subseteq Q \cap R \subseteq P[[X]] \cap R = P$. Since $a \in Q_1 \cap R$ we have $Q_1 \cap R = P$, whence $Q_1 \supseteq PR[[X]]$ and $Q = Q_1$. ∎

PROPOSITION 2.4. *If I is an ideal of R, then $(IR[[X]])_t \subseteq I_t[[X]]$.*

PROOF. Let $J = (f_1, \ldots, f_k)$ be a finitely generated subideal of $IR[[X]]$. Since each $f_i \in IR[[X]]$, there is a finitely generated subideal J_1 of I with $J \subseteq J_1R[[X]]$. Hence $J_v \subseteq (J_1R[[X]])_v = (J_1)_v[[X]] \subseteq I_t[[X]]$, the equality following from (2) in Proposition 2.1. The conclusion follows from the definition of the t-operation. ∎

PROPOSITION 2.5. *If P is a t-prime of R, then $P[[X]]$ contains a t-prime which contracts to P in R.*

PROOF. We have $(PR[[X]])_t \subseteq P[[X]]$ by Proposition 2.4. Shrink $P[[X]]$ to a prime Q minimal over $(PR[[X]])_t$. Then Q is a t-prime and $Q \cap R = P$. ∎

COROLLARY 2.6. *If R is an* SFT-*ring and P is a t-prime of R, then $P[[X]]$ is a t-prime of $R[[X]]$. Hence, if R is an* SFT-*ring, then t-* $\dim(R[[X]]) \geq t$- $\dim(R)$.

PROOF. By [1, Theorem 1], $P[[X]] = \sqrt{PR[[X]]}$. ∎

We show in Section 4 that the inequality in Corollary 2.6 can be strict.

In contrast to the situation noted for the v-operation in Proposition 2.1, the inclusion in Proposition 2.4 can be proper. Recall that Arnold [1, Theorem 1] demonstrated the existence of domains R containing prime ideals P for which there is an infinite chain of prime ideals between $PR[[X]]$ and $P[[X]]$, and it is easy to arrange that P be a t-prime. This appears to leave a lot of room for the placement of $(PR[[X]])_t$, and this is in fact the case. In Section 4 we show that $(PR[[X]])_t = PR[[X]]$ ($\neq P_t[[X]]$) when P is the maximal (t-)-ideal of any rank 1 nondiscrete (and hence non-SFT) valuation domain. At the other extreme, one can have $(PR[[X]])_t = P[[X]] \neq PR[[X]]$ for a t-prime P, as the following example shows; in this example, P is actually divisorial.

EXAMPLE 2.7. Let F be a subfield of the field k, such that k is not a finite algebraic extension of F. Let $V = k + P$ be a discrete rank 1 valuation domain, where P is the maximal ideal of V, and set $R = F + P$. Write $P = aV$, and let $b = ua$ for some element $u \in V \setminus R$. It is not hard to show that $P = (a,b)_v$, where the v-operation is taken with respect to R. Thus, by Proposition 2.1, $(PR[[X]])_t \subseteq (PR[[X]])_v = P[[X]] = (a,b)_v[[X]] = \big((a,b)R[[X]]\big)_v \subseteq (PR[[X]])_t$; hence $(PR[[X]])_t = P[[X]]$. On the other hand, the fact that k is not a finite algebraic extension of F implies that we may choose elements $v_0, v_1, \ldots$ in k such that the R-module $\sum Rv_i$ is not contained in any finitely generated R-submodule of V; it follows easily that for $f = av_0 + av_1X + \cdots$, we have $f \in P[[X]] \setminus PR[[X]]$. ■

REMARK 2.8. The SFT-hypothesis is not necessary for the final assertion of Corollary 2.6. Indeed, if V is any divided domain (in the sense of [8]), *e.g.*, any valuation domain, we have $t\text{-}\dim(V[[X]]) \geq (t\text{-})\text{-}\dim(V)$. To see this, note that if $P \underset{\neq}{\subset} Q$ are primes in V, then $P[[X]] \underset{\neq}{\subset} QV[[X]]$ since $PV_P = P$; now apply Proposition 2.5. ■

We close this section by illustrating that one may have $t\text{-}\dim(R[[X]]) > t\text{-}\dim(R)$, when R is a one-dimensional domain. Recall that Arnold [1] showed that if V is a rank 1 nondiscrete valuation domain with maximal ideal M, then there exists an infinite chain of primes between $MV[[X]]$ and $M[[X]]$.

EXAMPLE 2.9. Let $V = k+M$ be a nondiscrete rank 1 valuation domain, where k is a field and M is the maximal ideal of V. Let F be a proper subfield of k, and let $R = F+M$. Then R is one dimensional. By [4, Proposition 1] (and its proof), $MV[[X]]$ is a height 1 prime ideal of $V[[X]]$. We claim that $MR[[X]] = MV[[X]]$ and that this is also a height 1 prime ideal of $R[[X]]$.

That $MR[[X]] = MV[[X]]$ follows easily from the fact that $M = M^2$. Suppose that Q is a prime of $R[[X]]$ which is properly contained in $MR[[X]]$. Note that $QV[[X]] \subseteq M[[X]] \subseteq R[[X]]$. Clearly, $QV[[X]] \cdot MV[[X]] \subseteq Q$ and, since $MV[[X]] \not\subseteq Q$, we have $QV[[X]] = Q$. Let $a \in M \setminus Q$. Since $MV[[X]]$ has height 1 (in $V[[X]]$) [4], we have $fa^n \in QV[[X]] = Q$ for some $f \notin MV[[X]]$. Choose $g \in M[[X]] \setminus MV[[X]]$. Then $gf \in R[[X]]$, $gf \cdot a^n \in Q$, and $gf \notin Q$, whence $a \in Q$, a contradiction. Thus, $MR[[X]]$ has height 1 and is therefore a t-prime of $R[[X]]$. On the other hand, since M is divisorial [7, Theorem 4.1], Proposition 2.1 assures that $M[[X]]$ is divisorial and, hence, a t-prime properly containing $MR[[X]]$. Hence $t\text{-}\dim(R[[X]]) \geq 2$. ■

REMARK 2.10. Examples 2.7 and 2.9 can be generalized to pseudo-valuation domains (PVD's). Let R be a PVD whose canonical valuation overring V is one dimensional, and let P denote the common maximal ideal of R and V. With reference to Example 2.7, if V is discrete and R is noncoherent (*i.e.*, V is not finitely generated over R [12, Theorem 1.6]), then $(PR[[X]])_t = P[[X]] \neq PR[[X]]$. For Example 2.9, assume that V is nondiscrete; then $(t\text{-})\dim(R) = 1$, but $t\text{-}\dim(R[[X]]) \geq 2$. ■

3. **The Noetherian case.** In this section, we will study the t-primes of $R[[X]]$ when R is a Noetherian domain. Recall that a prime ideal P in a Noetherian domain is a t-prime $\Leftrightarrow$ P is divisorial $\Leftrightarrow$ P is an associated prime of a principal ideal [15, Theorem 36]. To help put matters in perspective, let us first review what is known about t-primes in the polynomial ring $R[X]$. Let Q be a nonzero prime of $R[X]$ with $Q \cap P = P$. The following facts are well known. (a) If $P \neq 0$, then Q is a t-prime of $R[X]$ $\Leftrightarrow$ P is a t-prime of R and $Q = P[X]$. (b) If $P = 0$, then Q is a t-prime since $\mathrm{ht}(Q) = 1$.

We shall show that the strict analogue of (a) holds in $R[[X]]$. Thus, we will show that if Q is a prime in $R[[X]]$ with $Q \cap R = P \neq 0$, then Q is a t-prime $\Leftrightarrow$ P is a t-prime of R and

$Q = P[[X]]$. The analogue of (b) is much more challenging. The problem is that, unlike the situation in the polynomial ring $R[X]$, in $R[[X]]$ there may well be primes Q of height greater than 1 for which $Q \cap R = 0$. Specifically, [16, Remark 2, p. 118] shows that it is possible to have an n-dimensional Noetherian domain R in which there are prime ideals Q of height n with $Q \cap R = 0$. (It seems to be unknown whether such a prime must exist in every n-dimensional Noetherian domain.) Thus we pose the following question.

QUESTION 3.1. If R is a Noetherian domain and Q is a t-prime of $R[[X]]$ with $Q \cap R = 0$, then must we have $\mathrm{ht}(Q) = 1$?

Although we cannot answer this question, we will show that $\mathrm{ht}(Q) \leq \max\{\mathrm{ht}(P) \mid P$ is a t-prime of $R\}$.

LEMMA 3.2. *If I is a t-ideal of $R[[X]]$, then $I \cap R$ is either 0 or a t-ideal of R.*

PROOF. Let J be a finitely generated nonzero subideal of $I \cap R$. Then $J_v \subseteq J_v R[[X]] \subseteq J_v[[X]] = (JR[[X]])_v \subseteq I$, the equality following from Proposition 2.1. Hence $J_v \subseteq I \cap R$, as desired. ∎

For an element $k \in K[[X]]$ we denote by $c(k)$ the R-submodule of K generated by the coefficients of k.

PROPOSITION 3.3. *If R is a Noetherian domain and Q is a t-prime of $R[[X]]$ with $P = Q \cap R \neq 0$, then (P is a t-prime of R and) $Q = P[[X]]$.*

PROOF. That P is a t-prime of R follows from Lemma 3.2. Choose $a \neq 0$ in P. Since $aQ^{-1} \subseteq R[[X]]$, we have $Q^{-1} \subseteq K[[X]]$. Let $f = \sum b_i X^i \in Q$ and $k = \sum u_i X^i \in Q^{-1}$. Then $b_0 u_0 \in R$. Also, $b_0 u_1 + b_1 u_0 \in R$, from which it follows that $b_0^2 u_1 \in R$. In general, we have $b_0^n u_{n-1} \in R$ for each n. However, since $ak \in R[[X]]$, $c(k)$ is a (necessarily finitely generated) fractional ideal of R. Hence $\exists$ a positive integer m with $b_0^m k \in R[[X]]$. Therefore, since Q^{-1} is a finitely generated fractional ideal of $R[[X]]$, we have $b_0^r Q^{-1} \subseteq R[[X]]$ for some r. Thus $b_0^r \in Q_v \cap R = Q \cap R = P$, and we have that $b_0 \in P$. It follows that $b_1 X + b_2 X^2 + \cdots \in Q$. We claim that $X \notin Q$. This follows from the easily demonstrated fact that $(a, X)_v = R[[X]]$ and the fact that Q is a t-prime. Hence $b_1 + b_2 X + \cdots \in Q$. As above we can show that $b_1 \in P$. Continuing in this manner, we eventually get $f \in P[[X]]$. Hence $Q \subseteq P[[X]]$. Since R is Noetherian, $P[[X]] = PR[[X]] \subseteq Q$, and we have $Q = P[[X]]$. ∎

As Proposition 3.3. shows, the t-primes of $R[[X]]$ which have nonzero intersection with R are easily described. We now turn to what (little) we know about the much harder problem of t-primes which intersect R in 0.

For an ideal I of $R[[X]]$, we shall use I' to denote the ideal of R generated by the constant terms of the elements of I.

LEMMA 3.4. *Let R be a (not necessarily Noetherian) domain, and let $Q \neq (X)$ be a prime of $R[[X]]$ with $Q \cap R = 0$. If $Q_t \neq R[[X]]$, then $(Q')_t \neq R$.*

PROOF. The hypothesis implies that $X \notin Q$. (If $X \in Q$, choose $g = \sum b_i X^i \in Q \setminus (X)$. Then $b_0 \neq 0$. However, $b_0 = g - \sum_{i>0} b_i X^i \in Q$, and since $Q \cap R = 0$, we have $b_0 = 0$,

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a contradiction.) Suppose that $(Q')_t = R$. Then $I^{-1} = R$ for some (nonzero) finitely generated subideal I of Q'. For each (nonzero) element a in a given finite generating set for I, we may choose $f \in Q$ with $f(0) = a$; let J be the ideal generated by these elements f. We shall show that $J^{-1} = R[[X]]$. Let $u \in J^{-1}$. We claim that $u \in K[[X]]$. To see this, choose $g \in J$ with $g(0) \neq 0$. Then g is a unit in $K[[X]]$. Since $ug \in R[[X]]$, $u \in g^{-1}R[[X]] \subseteq K[[X]]$, as claimed. Write $u = u_nX^n + \cdots$, and let f be an element in the finite generating set chosen above for J. Since $uf \in R[[X]]$, we have $u_nf(0) \in R$. It follows that $u_n \in I^{-1} = R$. Write $uf = u_nX^nf + (u - u_nX^n)f$. Then $(u - u_nX^n)f \in R[[X]]$, and the argument just given shows that $u_{n+1} \in R$. Continuing this process, we see that $u \in R[[X]]$. Thus $J^{-1} = R[[X]]$, and so the definition of the t-operation leads to $Q_t = R[[X]]$, the desired contradiction. ∎

PROPOSITION 3.5. *If R is a Noetherian domain which is not a field, and if Q is a t-prime of $R[[X]]$ with $Q \cap R = 0$, then* $\mathrm{ht}(Q) \leq k = \max\{\mathrm{ht}(P) \mid P$ *is a t-prime of* $R\}$.

PROOF. If $Q = (X)$, then $\mathrm{ht}(Q) = 1 \leq k$. Suppose $Q \neq (X)$. By Lemma 3.4, $(Q')_t \neq R$, whence Q' lies in some t-prime P of R. Now $Q \subseteq Q+(X) = Q'+(X) \subseteq P+(X)$. Since $\mathrm{ht}\, P \leq k$, we have $\mathrm{ht}\big(P + (X)\big) \leq k + 1$ [14, Lemma 2.4(i)], from which it follows (since $X \notin Q$—see the proof of Lemma 3.4) that $\mathrm{ht}\, Q \leq k$, as desired. ∎

QUESTION 3.6. Let R be a Noetherian domain. Recall that Corollary 2.6 shows that $t\text{-}\dim(R[[X]]) \geq t\text{-}\dim(R)$. Must equality hold?

REMARK 3.7. (a) If the answer to Question 3.1 is yes, then it is not difficult to see (using Proposition 3.3) that the answer to Question 3.6 is also yes.

(b) Since the t-height of a prime can be less than its height, Proposition 3.5 is not strong enough to answer Question 3.6 affirmatively except in the special case we mention next. ∎

COROLLARY 3.8. *If R is Noetherian domain of t-dimension* 1, *then so is* $R[[X]]$.

PROOF. Of course, $R[[X]]$ inherits the Noetherian property from R [17, Theorem 4, p. 138]. Let Q be a (nonzero) t-prime of $R[[X]]$. If $Q \cap R \neq 0$, then by Lemma 3.2, $Q \cap R$ is a t-prime of R, whence $\mathrm{ht}(Q \cap R) = 1$. By Proposition 3.3, $Q = (Q \cap R)[[X]]$; hence $\mathrm{ht}\, Q = 1$ [14, Lemma 2.4(i)]. If $Q \cap R = 0$, then since $t\text{-}\dim(R) = 1$ implies that every t-prime of R has height 1, we have $\mathrm{ht}\, Q = 1$ by Proposition 3.5. ∎

REMARK 3.9. Although we have been able to weaken the "Noetherian" hypothesis in Proposition 3.3 to the hypothesis "Mori," we do not know whether Proposition 3.5 and Corollary 3.8 can also be generalized to Mori domains. ∎

4. **Valuation domains.**

Throughout this section, V denotes a valuation domain with quotient field K. We have already observed (Remark 2.8) that $t\text{-}\dim(V[[X]]) \geq t\text{-}\dim(V)$. Although we show in Proposition 4.5 that this inequality can be strict when $(t\text{-})\dim(V) > 1$, we have not been able to settle the question when V is one dimensional. Lemma 4.3 shows, however, that *if* the inequality is strict in the one-dimensional case, then the fault must lie with t-primes of $V[[X]]$ which contract to zero in V. We shall use the following idea of Arnold and Brewer [4]. Let V be a rank 1 valuation domain, and let v be a valuation on K associated with V, where (we may assume that) the value group is a subgroup of the additive group of real numbers. For $f = \sum a_i X^i \in V[[X]]$, put $v^*(f) = \inf\{v(a_i)\}$; then v^* is a valuation on $V[[X]]$.

We begin our analysis with the discrete case.

PROPOSITION 4.1. *Let n be a positive integer, and let V be an n-dimensional discrete valuation domain. Then* t- $\dim(V[[X]]) = (t\text{-})\dim(V) = n$.

PROOF. By [9, Theorem 2.7], $\dim(V[[X]]) = n + 1$. Since the maximal ideal $M + (X)$ of $V[[X]]$ is not a t-ideal, we have $\mathrm{ht}(Q) \leq n$ for each t-prime Q of $V[[X]]$; hence t- $\dim(V[[X]]) \leq n$. On the other hand, by [2, Proposition 3.1], V is an SFT-ring, whence t- $\dim(V[[X]]) \geq n$ by Corollary 2.6. ∎

Proposition 4.1 can be generalized to PVD's: if R is an n-dimensional PVD which is an SFT-ring (*i.e.*, its canonical valuation overring is discrete), then t- $\dim(R[[X]]) = (t\text{-})\dim(R) = n$.

We now show that it is possible to have t- $\dim(R[[X]]) > t$- $\dim(R)$ even when R is an SFT-ring.

EXAMPLE 4.2. Let V be a two-dimensional discrete valuation domain, and let $R = V[[Y]]$ be the power series ring over V. By Proposition 4.1, t- $\dim(R) = 2$. In fact, if $M \supsetneq P \supsetneq 0$ are the (t-)primes of V, then $M[[Y]]$ and $P[[Y]]$ are t-primes of R. By [3], $\dim(R[[X]]) = 5$, whence by [1, Theorem 1], R is an SFT-ring. Hence by Corollary 2.6, $M[[Y,X]]$ and $P[[Y,X]]$ are t-primes of $R[[X]]$. However, by the remark at the end of [3], $\mathrm{ht}(P[[Y,X]]) = 2$. Thus $P[[Y,X]]$ contains a height 1 (t-)prime Q. The chain $M[[Y,X]] \supsetneq P[[Y,X]] \supsetneq Q \supsetneq 0$ then shows that t- $\dim(R[[X]]) \geq 3$. ∎

LEMMA 4.3. *Let V be a rank* 1 *nondiscrete valuation domain with maximal ideal M. Then $MV[[X]]$ is a maximal t-ideal of $V[[X]]$. (In particular, $M[[X]]$ is not a t-ideal.)*

PROOF. As noted in Example 2.9, $MV[[X]]$ is a height 1 (and therefore a t-) prime ideal of $V[[X]]$. Pick a nonzero element $c \in M$ and an element $f \in V[[X]] \setminus MV[[X]]$. We claim that $(c,f)_v = V[[X]]$. Since any ideal properly containing $MV[[X]]$ necessarily contains such elements, the proof will be complete as soon as we establish the claim. To verify the claim, it suffices to show that $(c,f)^{-1} = V[[X]]$. Let $u \in (c,f)^{-1}$. Then we may write $u = g/c$ with $g \in V[[X]]$. Since $uf \in V[[X]]$, we obtain an equation $gf = ch$, with $h \in V[[X]]$. Let the valuations v and v^* be as above. Applying v^* to the equation $gf = ch$ and using the fact that $v^*(f) = 0$ yields $v^*(g) \geq v(c)$. Thus $v(b) \geq v(c)$ for each coefficient b of g, whence $u = g/c \in V[[X]]$. Thus $(c,f)^{-1} = V[[X]]$, as we wished to show. ∎

Recall that a nonzero prime ideal M of a valuation domain is branched if and only if M is not the union of the prime ideals properly contained in M. We can next give a generalization of Lemma 4.3.

https://doi.org/10.4153/CMB-1995-027-1 Published online by Cambridge University Press

THEOREM 4.4. *If V is a valuation domain with branched maximal ideal M, then $MV[[X]]$ is a maximal t-ideal of $V[[X]]$.*

PROOF. If M is principal, then the result is trivial. We may therefore assume that $M = M^2$. Since M is branched, [10, Theorem 17.3(d)] supplies a prime ideal P of V for which $\dim(V/P) = 1$. It is easy to see that $W = V/P$ is a rank 1 nondiscrete valuation domain, and it is easy to use this to show that $MV[[X]]$ is a prime ideal of $V[[X]]$ with $MV[[X]] \neq M[[X]]$. Pick $c \in M \setminus P$ and $f \in V[[X]] \setminus MV[[X]]$. It is enough to show that $(c,f)^{-1} = V[[X]]$. For an element $k \in V[[X]]$, denote its canonical image in $W[[X]]$ by $\bar{k}$. As in the proof of Lemma 4.3, if $u = g/c \in (c,f)^{-1}$, then we have an equation $gf = ch$ for some $h \in V[[X]]$. This leads to the equation $\bar{g}\bar{f} = \bar{c}\bar{h}$ in $W[[X]]$. By Lemma 4.3, $\bar{g} \in \bar{c}W[[X]]$, and we may write $g = ck + g_1$ for some $k \in V[[X]]$ and $g_1 \in P[[X]]$. However, since $P = cP$, we have $P[[X]] \subseteq cV[[X]]$, whence $g \in cV[[X]]$; hence $u = g/c \in V[[X]]$. It follows that $(c,f)^{-1} = V[[X]]$. ■

PROPOSITION 4.5. *Let V be an n-dimensional domain, such that each prime ideal of V is idempotent. Then t-* $\dim(V[[X]]) \geq 2n - 1$.

PROOF. Let $0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ be the primes of V. To see that $P_iV[[X]]$ is prime, pass to the ring $(V/P_{i-1})[[X]] \simeq V[[X]]/P_{i-1}[[X]]$ and note that $P_iV[[X]]/P_{i-1}[[X]]$ is prime by the one-dimensional case [4]. This produces the chain of primes $0 \subseteq P_1V[[X]] \subseteq P_1[[X]] \subseteq P_2V[[X]] \subseteq P_2[[X]] \subseteq \cdots \subseteq P_{n-1}V[[X]] \subseteq P_{n-1}[[X]] \subseteq P_nV[[X]]$ in $V[[X]]$. Since each P_i is idempotent, the containments $P_iV[[X]] \subseteq P_{i+1}[[X]]$ are proper; hence the above chain in $V[[X]]$ has length $2n - 1$. It remains to show that each member of the chain is a t-prime. It is well known that each nonmaximal prime of a valuation domain is divisorial. Hence, by assertion (2) in Proposition 2.1, $P_i[[X]]$ is divisorial (and therefore a t-prime) for each $i < n$. Finally, for each i, $P_iV[[X]]$ is minimal over any element of $P_i \setminus P_{i-1}$ and is therefore a t-prime. ■

REMARK 4.6. In the notation of Proposition 4.5, $P_nV[[X]]$ is a maximal t-ideal of $V[[X]]$. This fact might lead one to suspect that the inequality in Proposition 4.5 is an equality. However, as in the Noetherian case, we have not been able to rule out the existence of uppers to zero which simultaneously are t-primes and have large (*i.e.*, greater than $2n - 1$) height. Indeed, it is conceivable that the t-dimensional of $V[[X]]$ may be infinite, even if $\dim(V) = 1$.

ACKNOWLEDGEMENT. The authors would like to extend their gratitude to the referee for many helpful comments and especially for suggesting the present form of Section 3.

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