THE STONE–ČECH COMPACTIFICATION OF THE RATIONAL WORLD

by M. P. STANNETT†

(Received 12 September, 1986)

1. Introduction. In his paper [**11**], Peter Neumann considered in detail the cycle structures of elements of Aut($\mathbb{Q}$), the group of all homeomorphisms of the "rational world" $\mathbb{Q}$ onto itself, and further analyses of Aut($\mathbb{Q}$) and its subgroups have been given by Mekler [**9**], Bruyns [**1**], and Truss [**13**]. My interest in Aut($\mathbb{Q}$) stems from its utility in proving an at first sight rather startling (to a general topologist) result concerning $\beta\mathbb{Q}$, the so-called Stone–Čech compactification of $\mathbb{Q}$, namely that $\beta\mathbb{Q}\backslash\mathbb{Q}$ is separable, and in fact contains a *homogeneous* countable dense subspace. (A space X is "homogeneous" provided whenever $x, y \in X$, there is some $g \in \mathrm{Aut}(X)$ with $g(x) = y$.) This is in sharp contrast to the spaces $\beta\mathbb{N}\backslash\mathbb{N}$ and $\beta\mathbb{R}\backslash\mathbb{R}$, which are both inseparable.

From the point of view of this paper, the easiest way to think of $\beta\mathbb{Q}$ is simply as a compact Hausdorff space which just happens to contain a dense copy of $\mathbb{Q}$. There are many other such spaces, of course, including such apparently "nice" spaces as the interval $[0, 1]$, but the reason we've picked upon $\beta\mathbb{Q}$ is that every element of Aut($\mathbb{Q}$) actually extends to a homeomorphism of the whole of $\beta\mathbb{Q}$ onto itself. This extension property will be easily derived from the following well-known definition of the Stone–Čech compactification of an arbitrary Tychonov space X. (A space is "Tychonov" provided it exists as a subspace of a compact Hausdorff space, so that for example $\mathbb{Q}$ is Tychonov.)

Extension Principle. *Let X be a Tychonov space. Then βX is that compact Hausdorff space such that*

(a) *βX contains a dense copy of X;*

and

(b) *if Y is any compact Hausdorff space and $f: X \to Y$ any continuous mapping, then there exists a unique continuous extension $f^\beta: \beta X \to Y$ such that the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \\ \beta X \arrow[ur, "f^\beta"'] & \end{tikzcd}$$

For a proof that βX exists, and is essentially unique, as well as an informative review of its structure, consult e.g. Walker's classic text [**14**].

It follows, of course, that Aut($\mathbb{Q}$) has a natural action on $\beta\mathbb{Q}$, and it is this action with which we shall be mainly concerned. For example, we shall see that, given any point

† The author would like to acknowledge the support of the SERC during the preparation of this paper.

x of $\beta\mathbb{Q}\backslash\mathbb{Q}$ the stabiliser of x in $\mathrm{Aut}(\mathbb{Q})$ is highly transitive on $\mathbb{Q}$. It will then follow that $\mathrm{Aut}(\mathbb{Q})$ has no countable orbit in $\beta\mathbb{Q}$, except $\mathbb{Q}$ itself.

I should like to take this opportunity to express my sincerest thanks to Peter Neumann for his helpful and stimulating correspondence concerning $\mathrm{Aut}(\mathbb{Q})$.

2. The action of Aut($\mathbb{Q}$). We begin by proving our claim that $\mathrm{Aut}(\mathbb{Q})$ acts both on $\beta\mathbb{Q}$ and $\beta\mathbb{Q}\backslash\mathbb{Q}$. In fact, we'll see that $\mathrm{Aut}(\mathbb{Q})$ and $\mathrm{Aut}(\beta\mathbb{Q})$ are isomorphic in a very natural way.

LEMMA 1. *Every homeomorphism $g \in \mathrm{Aut}(\mathbb{Q})$ has a unique extension $g^\beta \in \mathrm{Aut}(\beta\mathbb{Q})$. Moreover, the map $\theta: g \to g^\beta$ is a group isomorphism $\mathrm{Aut}(\mathbb{Q}) \to \mathrm{Aut}(\beta\mathbb{Q})$.*

Proof. Each g in $\mathrm{Aut}(\mathbb{Q})$ can be thought of as a continuous map $g: \mathbb{Q} \to \beta\mathbb{Q}$, so the extension principle implies the existence of a continuous extension $g^\beta: \beta\mathbb{Q} \to \beta\mathbb{Q}$. Our task is to show that g^β is actually a homeomorphism.

This follows almost immediately, since g^{-1} also has a continuous extension $(g^{-1})^\beta: \beta\mathbb{Q} \to \beta\mathbb{Q}$. Now $g^\beta \circ (g^{-1})^\beta$, $(g^{-1})^\beta \circ g^\beta$ and $\mathrm{id}_{\beta\mathbb{Q}}$ are all continuous extensions to $\beta\mathbb{Q}$ of the embedding $\mathrm{id}: \mathbb{Q} \to \beta\mathbb{Q}$, whence they are equal, by the uniqueness clause of the extension principle. Thus g^β and $(g^{-1})^\beta$ are both bijections, and are mutual inverses, whence our claim follows.

The uniqueness of extensions also implies that $\theta: g \to g^\beta$ is well-defined, while the equality $g = \theta(g)|_{\mathbb{Q}}$ shows that θ is injective. Since θ is clearly a homomorphism, we need only show that it is also surjective.

To see this, we appeal to a result of Čech [2], that no point of $\beta\mathbb{Q}\backslash\mathbb{Q}$ can have a countable neighbourhood base. Since *every* point of $\mathbb{Q}$ has such a base (because $\mathbb{Q}$ is a metric space), $\mathbb{Q}$ must be a union of $\mathrm{Aut}(\beta\mathbb{Q})$-orbits in $\beta\mathbb{Q}$. Thus whenever h lies in $\mathrm{Aut}(\beta\mathbb{Q})$, we have $h|_{\mathbb{Q}} \in \mathrm{Aut}(\mathbb{Q})$, whence $h = \theta(h|_{\mathbb{Q}})$, and θ is surjective.

It follows immediately from this lemma that $\mathrm{Aut}(\mathbb{Q})$ acts naturally on $\beta\mathbb{Q}$. To see that $\mathrm{Aut}(\mathbb{Q})$ also acts on $\beta\mathbb{Q}\backslash\mathbb{Q}$ we simply note that $\beta\mathbb{Q}\backslash\mathbb{Q}$ is a union of $\mathrm{Aut}(\beta\mathbb{Q})$-orbits in $\beta\mathbb{Q}$, because $\mathbb{Q}$ is, and so there is a homomorphism

$$\phi: \mathrm{Aut}(\mathbb{Q}) \to \mathrm{Aut}(\beta\mathbb{Q}\backslash\mathbb{Q})$$

given by $\phi(g) := g^\beta|_{\beta\mathbb{Q}\backslash\mathbb{Q}}$. In fact, since $\beta\mathbb{Q}\backslash\mathbb{Q}$ is dense in $\beta\mathbb{Q}$—an immediate corollary of our next result—the map ϕ is actually an injection.

The action of $\mathrm{Aut}(\mathbb{Q})$ on $\beta\mathbb{Q}$ and $\beta\mathbb{Q}\backslash\mathbb{Q}$ cannot be transitive, because $\mathrm{Aut}(\mathbb{Q})$ has c elements, while $\beta\mathbb{Q}$ and $\beta\mathbb{Q}\backslash\mathbb{Q}$ both have 2^c elements [5, 9.3]. However, the action is "very nearly" transitive, in that each orbit of $\mathrm{Aut}(\mathbb{Q})$ is dense in its respective space, as we now show.

Note that our proof of Lemma 1 shows that given *any* Tychonov space X, $\mathrm{Aut}(X)$ has a natural action both on βX and $\beta X\backslash X$.

THEOREM 2. *Let X be a Tychonov space, and suppose that $H \leqq \mathrm{Aut}(X)$. If there exists a base $\mathbb{B}$ for the topology on X satisfying*

(i) *$\forall B_1, B_2 \in \mathbb{B}$, $\exists h \in H$ such that $h(B_1) \subseteq B_2$*

and

(ii) $\beta X = \bigcup \{\bar{B}^{\beta X} : B \in \mathbb{B}\}$,

then

(a) *all H-orbits in* $\beta X \backslash X$ *are dense in* $\beta X \backslash X$,

(b) *all H-orbits in* βX *are dense in* βX.

Proof. If X is compact, then $X = \beta X$, so that (a) is trivial, while (b) is an immediate consequence of (i). Suppose then that X is non-compact, and choose $x \in \beta X \backslash X$ and $B_1 \in \mathbb{B}$ with $x \in \bar{B}_1^{\beta X}$. Let U be any non-empty open subset of $\beta X \backslash X$, and choose V, an open subset of X, with $U \cup V$ open in βX. Since βX is compact Hausdorff, we can choose W, a non-empty open set in βX, with $W \subseteq \bar{W}^{\beta X} \subseteq U \cup V$. Now X is dense in βX, so that $W \cap X$ is nonempty, and we may choose $B_2 \subseteq W \cap X (B_2 \in \mathbb{B})$. Let $h \in H$ satisfy $h(B_1) \subseteq B_2$.

Put $h^* := (h^\beta)|_{\beta X \backslash X}$. Then

$$h^*(x) = h^\beta(x) \in h^\beta(\bar{B}_1^{\beta X}) \subseteq \bar{B}_2^{\beta X} \subseteq \bar{W}^{\beta X} \subseteq U \cup V.$$

Since $h^*(x) \notin V$, we must have $h^*(x) \in U$. This proves (a).

To prove part (b), let W_1 be any non-empty open set in βX, and put $U = W_1 \cap (\beta X \backslash X)$ and $V = W_1 \cap X$. Choose any $x \in \beta X$, and $B_1 \in \mathbb{B}$ with $x \in \bar{B}_1^{\beta X}$. As above, choose $B_2 \in \mathbb{B}$ satisfying $\bar{B}_2^{\beta X} \subseteq U \cup V$, and h satisfying $h(B_1) \subseteq B_2$. Once again, we have $h^\beta(x) \in \bar{B}_2^{\beta X} \subseteq W_1$.

Corollary 3. *There exists a countable subgroup H of* Aut($\mathbb{Q}$) *such that*

(a) *every H-orbit in* $\beta\mathbb{Q}$ *is dense,*

(b) *every H-orbit in* $\beta\mathbb{Q} \backslash \mathbb{Q}$ *is dense.*

In particular, then, if $q \in \beta\mathbb{Q} \backslash \mathbb{Q}$, *there exists a countable dense homogeneous set* $\Delta_q \subseteq \beta\mathbb{Q} \backslash \mathbb{Q}$, *with* $q \in \Delta_q$.

Proof. We choose $\mathbb{B}$ to be the collection of all proper nonempty clopen subsets of $\mathbb{Q}$, of the form

$$\mathbb{Q} \cap (I \cup J)$$

where I and J are intervals (in $\mathbb{R}$) with boundary (in $\mathbb{R}$) contained in $\sqrt{2}\mathbb{Q} \backslash \{0\}$. Note that $\mathbb{B}$ is countable, so that we can put $\mathbb{B} = \{B_n : n \in \mathbb{N}\}$.

Now $\beta\mathbb{Q} = \bigcup \{\bar{B}_n^{\beta\mathbb{Q}} : n \in \mathbb{N}\}$. To see this, let (q_λ) be a net in $\mathbb{Q}$ converging to x in $\beta\mathbb{Q}$, and let $U \subseteq \beta\mathbb{Q}$ be a proper open neighbourhood of x. According to [**5**, 16F, 16.11], $\beta\mathbb{Q}$ has a base consisting of clopen sets, and so there is a proper clopen neighbourhood V of x such that $x \in V = \bar{V}^{\beta\mathbb{Q}} \subseteq U$.

Let (q') be the net $(q_\lambda) \cap V$; i.e. the part of (q_λ) lying in V. Then $q' \to x$ and $(q') \subset V \cap \mathbb{Q}$. The latter is clopen in $\mathbb{Q}$, and nonempty. Moreover it is proper, lest $\beta\mathbb{Q} = \bar{\mathbb{Q}}^{\beta\mathbb{Q}} \subseteq \bar{V}^{\beta\mathbb{Q}} \subseteq U \neq \beta\mathbb{Q}$.

Consequently we can find a superset B of $V \cap \mathbb{Q}$ of the desired form, i.e. with $B \in \mathbb{B}$, and now $x \in \bar{B}^{\beta\mathbb{Q}}$.

So $\mathbb{B}$ satisfies condition (ii) of Theorem 2. We now choose H to satisfy condition (i).

For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, choose $h_{i,j} \in \mathrm{Aut}(\mathbb{Q})$, satisfying

$$h_{i,j}(B_i) \subseteq B_j.$$

To see that such $h_{i,j}$ exist, observe that it follows from [**11**, Sierpinski's Theorem] that any two non-empty clopen subsets of $\mathbb{Q}$ are homeomorphic. Hence there are homemorphisms from B_i to B_j and from $\mathbb{Q}\backslash B_i$ to $\mathbb{Q}\backslash B_j$ which can be combined to give a suitable $h_{i,j}$.

We now set $H = \langle h_{i,j} \mid i, j \in \mathbb{N}\rangle$. Since H is countably generated, it is countable, and by Theorem 2, the H-orbits in $\beta\mathbb{Q}$ and $\beta\mathbb{Q}\backslash\mathbb{Q}$ are dense.

In particular, if $q \in \beta\mathbb{Q}\backslash\mathbb{Q}$, set $\Delta_q = \{h(q): h \in H\}$. Then $\Delta_q \subseteq \beta\mathbb{Q}\backslash\mathbb{Q}$ is countable, dense, homeogeneous, and contains q.

An immediate consequence of Corollary 3 is that $\beta\mathbb{Q}\backslash\mathbb{Q}$ is dense in $\beta\mathbb{Q}$, since any H-orbit in $\beta\mathbb{Q}\backslash\mathbb{Q}$ is simultaneously an H-orbit in $\beta\mathbb{Q}$.

In fact, we could have shown the separability of $\beta\mathbb{Q}\backslash\mathbb{Q}$ quite differently, using the idea of π-weight. The proof we now give of the separability of $\beta\mathbb{Q}\backslash\mathbb{Q}$ and its density in $\beta\mathbb{Q}$ does *not* show that the countable dense subset of $\beta\mathbb{Q}\backslash\mathbb{Q}$ can be chosen to be homogeneous, but has advantages in the scope of its application.

3. The iterated remainders of $\mathbb{Q}$. The space $\beta X\backslash X$, where X is a Tychonov space, is called the *growth* or *remainder* of X and is usually denoted X^*. This space is itself Tychonov, and so we can consider $(X^*)^*$, $((X^*)^*)^*$, and so on. This notation is rather cumbersome, and so we introduce the following alternative.

Put $X^{(0)} := X$, and for each $n \in \mathbb{N}$, define $X^{(n+1)} := (X^{(n)})^*$. Both Jackson [**7**] and Hussak [**6**] have considered those spaces for which $X^{(n)}$ is eventually empty for some n. In fact, $\mathbb{Q}^{(n)}$ is never empty for any n, and is "so non-empty" that $\mathbb{Q}^{(n+1)}$ is actually dense in $\beta\mathbb{Q}^{(n)}$, for all $n \in \mathbb{N}$. We prove this now. We'll need the following definition.

A space X is said to be "nowhere locally compact" provided every compact subset of X has empty interior. In particular, the space $\mathbb{Q}$ is nowhere locally compact.

Lemma 4. *For $n \in \mathbb{N}$, $\mathbb{Q}^{(n)}$ is nowhere locally compact.*

Proof. We proceed by induction on n. It's already known that $\mathbb{Q}^{(0)}$ is nowhere locally compact, so we assume that $\mathbb{Q}^{(n)}$ is nowhere locally compact. Suppose $\mathbb{Q}^{(n+1)}$ fails to be nowhere locally compact, so that there is some non-empty open set U in $\mathbb{Q}^{(n+1)}$, and $\bar{U}^{\mathbb{Q}^{(n+1)}}$ is compact. Choose some open W in $\beta\mathbb{Q}^{(n)}$ with $U = W \cap \mathbb{Q}^{(n+1)}$. Now $\mathbb{Q}^{(n)}$ is dense in $\beta\mathbb{Q}^{(n)}$, so that $W \cap \mathbb{Q}^{(n)}$ is nonempty. Moreover, $\bar{U}^{\mathbb{Q}^{(n+1)}}$ is closed in $\beta\mathbb{Q}^{(n)}$, so that $W\backslash\bar{U}^{\mathbb{Q}^{(n+1)}}$ is open in $\beta\mathbb{Q}^{(n)}$, nonempty, and lies entirely in $\mathbb{Q}^{(n)}$. But now, since $\beta\mathbb{Q}^{(n)}$ is compact Hausdorff, we can find some open nonempty $V \subseteq W\backslash\bar{U}^{\mathbb{Q}^{(n+1)}}$ with $V \subseteq \bar{V}^{\beta\mathbb{Q}^{(n)}} \subseteq W\backslash\bar{U}^{\mathbb{Q}^{(n+1)}} \subseteq \mathbb{Q}^{(n)}$. So $\bar{V}^{\beta\mathbb{Q}^{(n)}}$ is a compact subspace of $\mathbb{Q}^{(n)}$ with nonempty interior. But this contradicts our hypothesis that $\mathbb{Q}^{(n)}$ is nowhere locally compact.

Corollary 5. *For each $n \in \mathbb{N}$, $\mathbb{Q}^{(n+1)}$ is dense in $\beta\mathbb{Q}^{(n)}$.*

Proof. If $\mathbb{Q}^{(n+1)}$ isn't dense in $\beta\mathbb{Q}^{(n)}$ for some n, we can find a non-empty open set U in $\beta\mathbb{Q}^{(n)}$ lying entirely in $\mathbb{Q}^{(n)}$. But now U contains compact sets with nonempty interior, so that $\mathbb{Q}^{(n)}$ fails to be nowhere locally compact.

A *π-base* for the topology on a space X is a collection $\mathbb{P}$ of nonempty open sets in X such that every nonempty open set in X contains a member of $\mathbb{P}$. The *π-weight* of a space X, $\pi(X)$, is the smallest cardinal of any π-base for X.

If X is a Tychonov space, and Y is a dense subspace of X then $\pi(Y)=\pi(X)$. (See e.g. **[8]**.) Any space of countable π-weight is separable: just pick one point from each member of some countable π-base; the resulting countable set is dense.

COROLLARY 6. *For each* $n \in \mathbb{N}$, $\mathbb{Q}^{(n)}$ *is separable.*

Proof. We show, by induction on n, that $\pi(\mathbb{Q}^{(n)})$ is countable for each n.

Clearly $\pi(\mathbb{Q})=\aleph_0$. Suppose, therefore, that $\pi(\mathbb{Q}^{(n)})=\aleph_0$. Then $\pi(\beta\mathbb{Q}^{(n)})=\pi(\mathbb{Q}^{(n)})=\aleph_0$, since $\beta\mathbb{Q}^{(n)}$ is Tychonov and $\mathbb{Q}^{(n)}$ is dense in $\beta\mathbb{Q}^{(n)}$. But $\mathbb{Q}^{(n+1)}$ is also dense in $\beta\mathbb{Q}^{(n)}$, by Corollary 5, whence $\pi(\mathbb{Q}^{(n+1)})=\pi(\beta\mathbb{Q}^{(n)})=\aleph_0$.

4. Aut($\mathbb{Q}$) orbits in $\beta\mathbb{Q}$. Recently, John Truss ["The group of autohomeomorphisms of $\mathbb{Q}$: subgroups of small index"—personal correspondence] has succeeded in proving Neumann's conjecture **[3]** that whenever H is a subgroup of Aut($\mathbb{Q}$) with $|\mathrm{Aut}(\mathbb{Q}):H|<2^{\aleph_0}$, then H necessarily contains the pointwise stabiliser of some finite set in $\mathbb{Q}$. This conjecture also prompted the question whether there exists a countable orbit of Aut($\mathbb{Q}$) in $\beta\mathbb{Q}\backslash\mathbb{Q}$ such that if G is the stabiliser of a point in this orbit then G does *not* contain the stabiliser of finitely many points in $\mathbb{Q}$. Truss's result shows that this is not the case, and in fact we can go further; we show that $\beta\mathbb{Q}\backslash\mathbb{Q}$ contains no countable Aut($\mathbb{Q}$)-orbits, so that the question becomes redundant. Before proving this, we shall need the following results of Peter Bruyns **[1]**.

Let H be a subgroup of index $<2^{\aleph_0}$ in Aut($\mathbb{Q}$). Then

(A) there is a finite subset Y_0 of $\mathbb{Q}$ that is invariant under H, and such that $\mathbb{Q}\backslash Y_0$ is a single H-orbit,

(B) if H is transitive on $\mathbb{Q}$ (i.e. if Y_0 is empty) then $H=\mathrm{Aut}(\mathbb{Q})$.

Consequently, as Neumann points out [personal correspondence], if we could find a point of $\beta\mathbb{Q}$ whose Aut($\mathbb{Q}$)-orbit is countable, then its stabiliser in Aut($\mathbb{Q}$) would have finite orbits in $\mathbb{Q}$.

Let $\tau:\mathbb{Q}\to\mathbb{Q}\cap(0,1)$ be a homeomorphism. According to the extension principle, τ has a continuous extension $\tau^\beta:\beta\mathbb{Q}\to[0,1]$, and in fact $\tau^\beta(\beta\mathbb{Q}\backslash\mathbb{Q})=[0,1]\backslash(\mathbb{Q}\cap(0,1))$ [**14**, 6.12].

THEOREM 7. *Let A be any finite subset of* $[0,1]\backslash(\mathbb{Q}\cap(0,1))$. *Then the pointwise stabiliser of* $(\tau^\beta)^{-1}(A)$ *in* Aut($\mathbb{Q}$) *is highly transitive on* $\mathbb{Q}$.

Proof. Let $A:=\{a_1,\ldots,a_n\}$. Putting $G:=\mathrm{Aut}(\mathbb{Q})$, we have to show that $G_{(\tau^\beta)^{-1}(A)}$ is m-transitive on $\mathbb{Q}$, for each m.

Let $(b_1,\ldots,b_m)$ and $(c_1,\ldots,c_m)$ be two m-tuples of distinct points in $\mathbb{Q}$. Then $\tau^\beta(b_i)$ and $\tau^\beta(c_i)$ do not lie in A, for each i, so that we can find irrationals α_i, β_i, γ_i, δ_i with $b_i\in(\alpha_i,\beta_i)$ and $c_i\in(\gamma_i,\delta_i)$, and such that the (α_i,β_i) are mutually disjoint as are the (γ_i,δ_i), and each of the (α_i,β_i) and (γ_i,δ_i) are disjoint from A. Putting $U_i=(\alpha_i,\beta_i)\cap\mathbb{Q}$

https://doi.org/10.1017/S0017089500007205 Published online by Cambridge University Press

and $V_i = (\gamma_i, \delta_i) \cap \mathbb{Q}$, let $\phi \in \mathrm{Aut}(\mathbb{Q})$ be any map which interchanges each U_i and V_i, which satisfies $\phi(b_i) = c_i$ for each i, and which fixes $\mathbb{Q} \setminus (\bigcup U_i \cup \bigcup V_i)$ pointwise. Any net in $\beta\mathbb{Q}$ converging to a point of $(\tau^\beta)^{-1}(A)$ is eventually outside all of the clopen sets of $\beta\mathbb{Q}$ whose intersection with $\mathbb{Q}$ are U_i, V_i, whence $(\tau^\beta)^{-1}(A)$ is fixed pointwise by ϕ^β, and our claim is proven.

Corollary 8. *There are no countable Aut($\mathbb{Q}$)-orbits in $\beta\mathbb{Q}$ except $\mathbb{Q}$ itself.*

Proof. This is immediate from Theorem 7, together with Neumann's interpretation of Bruyns' results, discussed earlier.

5. Special points and subspaces of $\beta\mathbb{Q}$. We saw in Corollary 5 that $\mathbb{Q}^{(n+1)}$ is dense in $\beta\mathbb{Q}^{(n)}$, for each n. It follows from the extension principle that we can find surjections

$$f_n : \beta\mathbb{Q}^{(n+1)} \to \beta\mathbb{Q}^{(n)}$$

fixing $\mathbb{Q}^{(n+1)}$ pointwise, and in fact $f_n(\mathbb{Q}^{(n+2)}) = \mathbb{Q}^{(n)}$ [**14**, 6.12]. Now the spaces $\mathbb{Q}$ and $\mathbb{Q}^*$ are both real compact [**5**, 8*H*], non-compact spaces, which must therefore fail to be pseudocompact [**5**, 5*H*2]. Since the continuous image of a pseudocompact space is again pseudocompact, we see that none of the spaces $\mathbb{Q}^{(n)}$ can be pseudocompact.

Frolik [**4**] has shown that whenever X is a non-pseudocompact Tychonov space, then X^* fails to be homogeneous. Thus, we have shown

Proposition 9. *None of the spaces $\mathbb{Q}^{(n)}$ ($n \geqq 1$) is homogeneous.*

Since $\mathbb{Q}^*$ isn't homogeneous, it makes sense to consider what "kinds of points" occur in $\beta\mathbb{Q}$ and $\mathbb{Q}^*$.

A point x in $\mathbb{Q}^*$ is said to be *remote* provided it lies in the closure of no discrete subspace of $\mathbb{Q}^*$ not already containing x, and is a *weak P-point* provided it lies in the closure of no countable subset of $\mathbb{Q}^*$ not already containing x. Since $\mathbb{Q}^*$ is separable, by Corollary 3, $\mathbb{Q}^*$ contains no weak P-points. On the other hand Plank [**12**] has shown, using the Continuum Hypothesis, that $\beta\mathbb{Q}$ contains 2^c remote points which form a dense subspace of $\mathbb{Q}^*$. Since the homeomorphic image of a remote point is again remote, this density is obvious from Corollary 3, and indeed a homogeneous countable dense subspace of $\mathbb{Q}^*$ exists comprising only remote points of $\beta\mathbb{Q}$. Using Martin's Axiom [:= MA], van Mill [**10**] has shown the existence of a point x_0 of $\mathbb{Q}^*$ which lies in the closure of no countable nowhere dense set in $\beta\mathbb{Q}$ not already containing x_0.

We can use van Mill's result to construct two interesting subspaces of $\mathbb{Q}^*$.

Example 10.[MA]. There exists a compact subspace F of $\mathbb{Q}^*$ with a point $x_0 \in \overline{F \setminus \{x_0\}}^{\beta\mathbb{Q}}$ not in the closure of any countable subset of $F \setminus \{x_0\}$.

To see this, enumerate $\mathbb{Q}$ as $\{q_n : n \in \mathbb{N}\}$. Since $\beta\mathbb{Q}$ is Hausdorff, we can find, for each $n \in \mathbb{N}$, a neighbourhood U_n of q_n with $x_0 \notin \overline{U_n}^{\beta\mathbb{Q}}$. Let V be any open neighbourhood of x_0 in $\beta\mathbb{Q}$, and define

$$F_n := V \setminus \bigcup_{j=0}^{n} \overline{U_j}^{\beta\mathbb{Q}}$$

Put $F = \bigcap \{\overline{F_n}^{\beta\mathbb{Q}} : n \in \mathbb{N}\}$.

Since each $\overline{F_n}^{\beta\mathbb{Q}}$ is a closed neighbourhood of x_0, F is a nonempty compact subset of $\beta\mathbb{Q}$. By construction F misses U_n for each $n \in \mathbb{N}$, so that $F \subseteq \mathbb{Q}^*$.

And since F misses $\mathbb{Q}$, F contains no non-empty open subset of $\beta\mathbb{Q}$, i.e. F is nowhere dense in $\beta\mathbb{Q}$.

By the choice of x_0, x_0 is not in the closure of any countable subset of $F \backslash \{x_0\}$, since subsets of nowhere dense sets are themselves nowhere dense.

However, $x_0 \in \overline{F\backslash\{x_0\}}^{\beta\mathbb{Q}}$, lest x_0 be a G_δ in $\mathbb{Q}^*$; it is shown in [**2**] that no such points exist.

EXAMPLE 11. There exists a space X which is countable, homogeneous, dense-in-itself (since dense in $\beta\mathbb{Q}\backslash\mathbb{Q}$), of countable π-weight, T_4, paracompact, Lindelof and zero-dimensional, but nowhere first-countable.

Moreover, if [MA] is assumed, then every nowhere dense subspace of X is closed, and hence discrete.

Proof. Let H be the group constructed in Corollary 3, and take X to be any H-orbit in $\mathbb{Q}^*$. Then X is countable and homogeneous. Since X is dense in $\mathbb{Q}^*$, we have $\pi(X) = \pi(\mathbb{Q}^*) = \aleph_0$, and X is zero-dimensional, since $\mathbb{Q}^*$ is; [**5**, 16*F*, 16.11]. Since X is countable, it's paracompact (because Lindelof and Tychonov [**15**, 20.8]) and so T_4 [**15**, 20.10]. Finally, since no point of $\mathbb{Q}^*$ is the limit of a sequence of distinct points of $\mathbb{Q}^*$ [**14**, 2.2], X is nowhere first countable.

Under [MA], we can choose X to contain the point x_0 of Example 10, whence no point $x \in X$ lies in the closure of any nowhere dense $F \subseteq X$ satisfying $x \notin F$. Thus, every nowhere dense subset of X is closed; since subsets of nowhere dense subsets are again nowhere dense, each nowhere dense subset of X is discrete.

REFERENCES

1. P. Bruyns; *Aspects of the group of homeomorphisms of the rational numbers,* Doctoral thesis (Oxford, 1986).

2. E. Čech, On bicompact spaces, *Ann. of Math.* (*2*) **38** (1937), 823–844.

3. J. D. Dixon, P. M. Neumann and S. Thomas, Subgroups of small index in infinite symmetric groups, *Bull. London Math. Soc.*, to appear.

4. Z. Frolik, Non-homogeneity of $\beta P \backslash P$, *Comment. Math. Univ. Carolin.* **8** (1967), 705–709.

5. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, 1960).

6. W. Hussak, *Iteratively defined generalisations of locally compact and discrete topological spaces,* Doctoral thesis (Sheffield, 1983).

7. P. P. Jackson, *Iterated remainders in compactifications,* Doctoral thesis (Sheffield, 1980).

8. I. Juhasz, *Cardinal functions in topology – ten years later,* Math. Centre Tracts 123 (Mathematisch Centrum, Amsterdam, 1980).

9. A. H. Mekler, Groups embeddable in the autohomeomorphisms of $\mathbb{Q}$, *J. London Math. Soc.* (*2*) **33** (1986), 49–58.

10. J. van Mill, Weak P-points in Čech-Stone compactifications, *Trans. Amer. Math. Soc.* **273** (1982), 657–678.

11. P. M. Neumann, Automorphisms of the rational world, *J. London Math. Soc.* (*2*) **32** (1985), 439–448.

12. D. Plank, On a class of subalgebras of $C(X)$ with applications to $\beta X \backslash X$, *Fund. Math.* **64** (1969), 41–54.

13. J. K. Truss, Embeddings of infinite permutation groups, *Proceedings of Groups 1985 at St Andrews* (to appear).
14. R. C. Walker, The Stone-Čech compactification, Ergebnisse der Mathematik 83 (Springer, 1974).
15. S. Willard, *General Topology* (Addison-Wesley, 1970).

DEPARTMENT OF COMPUTER SCIENCE
THE UNIVERSITY
SHEFFIELD
S3 7RH