The Cesàro summability of Fourier integrals

By J. Cossar.

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1. The functions $f(t)$ and $h(t)$ that occur in what follows are supposed to be integrable (L) in every finite interval in which they are defined; and the order of summability, which need not be an integer, is not negative.

The following result is proved :

Theorem 1. *Suppose that*

$$g(u) = \frac{2}{\pi}\int_0^\infty f(t) \sin ut\, dt, \tag{1.1}$$

the integral being boundedly summable (C, r) *in every finite range of* u. *Then, for almost all positive* x,

$$\int_0^\infty g(u) \sin ux\, du = f(x), \tag{1.2}$$

the integral being summable $(C,\ r+1)$ *for such* x.

The case $r = 0$ (for a continuous f and uniform, instead of bounded, convergence) is contained in Titchmarsh's *Theory of Fourier integrals* (Oxford, 1937), p. 162; but his proof does not seem to be adaptable to the case of general r. A variant of the corresponding theorem for cosine integrals was stated by Macphail and Titchmarsh, and proved by them in the cases $r = 0$, 1.[1] Their proof made use of the fact, obtained by putting $u = 0$ in the cosine integral corresponding to (1.1), that the integral of $f(t)$ over $(0, \infty)$ was summable (C, r). But no such result is obtainable so simply from (1.1); and we use, instead, Theorem 3 below.

Our argument yields a result obviously more general than Theorem 1, which we state as Theorem 2.

Theorem 2.[2] *Suppose that, in each finite range* $0 \leqq u \leqq \omega$, *the integral* (1.1) *is summable* (C, k), *where* $k = k(\omega)$, *for almost all* u, *and that*

$$\left| \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^k f(t) \sin ut\, dt \right| \leqq M(u) \qquad (\lambda > 0),$$

[1] See Titchmarsh, *op. cit.*, p. 37.

[2] The corresponding theorem for cosine integrals is also true, and may be proved in the same way.

where $M(u)$ is integrable over $(0, \omega)$. Then (1.2) holds for almost all positive x, the integral being summable $(C, r+1)$ for such x, where r is any value of $k(\omega)$.

THEOREM 3. *Suppose that*

$$\left| \int_1^\lambda \left(1 - \frac{t}{\lambda}\right)^r f(t) \sin ut \, dt \right| \leqq M(u) \qquad (\lambda > 1,\ 0 \leqq u \leqq \Omega), \tag{1.3}$$

where $M(u)$ is integrable over $(0, \Omega)$. Then, for each x,

$$\int^\infty \frac{f(t+x)}{t^j} \, dt \tag{1.4}$$

is bounded (C, r) when $j = 1$ and summable (C, r) when $j > 1$; and

$$\int^\infty \frac{f(t+x)}{t^a} e^{iyt} \, dt \qquad (a > 0) \tag{1.5}$$

is summable (C, r) for almost all y in $(0, \Omega)$.

2. Let $\quad d_1 = i, \qquad d_a = \int_0^\infty \theta^{a-1} e^{i\theta} \, d\theta \qquad (0 < a < 1),$

$$d_a = c_a + i s_a = c + is.$$

Clearly $s \neq 0$.

Write also $\quad {}_r\!\int_b^\infty h(t) \, dt = \lim_{\omega \to \infty} \int_b^\omega \left(1 - \frac{t}{\omega}\right)^r h(t) \, dt.$

3. Lemma 1. *If either of the integrals*

$$\int_b^\infty h(t) \, dt, \qquad \int_{b-x}^\infty h(t+x) \, dt \qquad (x \text{ fixed})$$

is bounded (C, r), then so is the other. If either integral is summable (C, r), then so is the other, to the same value.

For $\quad \int_b^\omega \left(1 - \frac{t}{\omega}\right)^r h(t) \, dt = \left(1 - \frac{x}{\omega}\right)^r \int_{b-x}^{\omega - x} \left(1 - \frac{t}{\omega - x}\right)^r h(t+x) \, dt.$

Lemma 2.[1] *Suppose that $a > 0$ and that $\phi(t)$ satisfies the following conditions:*

(i) $\quad \phi(t) \to 0$ *as* $t \to \infty$, $\qquad$ (ii) $\quad p \geqq -1$,

(iii) $\quad \phi^{(p+1)}(t)$ *is absolutely continuous in every finite interval* (a, ω),

(iv) $$\int_a^\infty t^{p+1} \left| \phi^{(p+2)}(t) \right| dt < \infty.$$

[1] This theorem on integration by parts is due to G. H. Hardy in the case of integral r. See J. Cossar, *Journal London Math. Soc.*, 16 (1941), 56.

Then if
$$\int_a^\infty h(t)\,dt$$
is bounded (C, r), *where* $r \leqq p+1$,
$$\int_a^\infty h(t)\,\phi(t)\,dt = (-1)^p \int_a^\infty h_{p+2}(t)\,\phi^{(p+2)}(t)\,dt,$$
where
$$h_{p+2}(t) = \frac{1}{(p+1)!}\int_a^t (t-u)^{p+1}\,h(u)\,du.$$

Corollary.
$$\left|\int_a^\infty h(t)\,\phi(t)\,dt\right| \leqq K \int_a^\infty t^{p+1}\left|\phi^{(p+2)}(t)\right| dt,$$
where K is independent of ϕ.

Lemma 3. *If the first of the integrals*
$$\int^\infty h(t)\,dt, \qquad \int^\infty \frac{h(t)}{t^\gamma}\,dt \qquad (\gamma > 0)$$
is bounded (C, r), then the second is summable (C, r).

This follows from Lemma 2 with $\phi(t) = t^{-\gamma}$ and p any integer not less than $r-1$.

Lemma 4. *If the first of the integrals*
$$\int^\infty \frac{h(t)}{t^\delta}\,dt, \qquad \int^\infty \frac{h(t+x)}{t^\delta}\,dt$$
is bounded (C, r), then so is the second.

By Lemma 1,
$$\int^\infty \frac{h(t+x)}{(t+x)^\delta}\,dt$$
is bounded (C, r). Now apply Lemma 2 with $p \geqq r-1$, $h(t+x)/(t+x)^\delta$ instead of $h(t)$, and $\phi(t) = (1+x/t)^\delta - 1$. This is permissible since $t^{p+1}\phi^{(p+2)}(t) = O(t^{-2})$, as may be seen by expanding $\phi(t)$ in powers of t^{-1}.

Lemma 5. *If $t > 0$, $\lambda > 0$, $0 < a < 1$, then*
$$\int_0^t \theta^{-a} e^{i\lambda\theta}\,d\theta = d\lambda^{a-1} + O(\lambda^{-1}t^{-a}),$$
where $d = d_{1-a}$ (see § 2), and the constant implied by the order symbol does not exceed 2.

Lemma 6. *If $0 < a \leqq 1$, $\lambda > 0$, then*
$$\int_0^1 (1-u)^{a-1}\cos(\lambda u - \kappa)\,du = \lambda^{-a}\left\{c\cos(\lambda-\kappa) + s\sin(\lambda-\kappa) - \psi(\lambda, \kappa, a)\right\},$$

where c, s are the constants of § 2, *and*

$$\psi(\lambda, \kappa, a) = \begin{cases} \int_0^\infty (u+\lambda)^{a-1} \cos(u+\kappa)\, du & (0 < a < 1), \\ -\sin \kappa & (a = 1). \end{cases}$$

LEMMA 7. *If* $0 < a < 1$, $y > 0$, $t > 0$, *then*

$$\int_0^y (y-u)^{a-1} \sin ut\, du = c\frac{\sin yt}{t^a} - s\frac{\cos yt}{t^a} + \frac{y^{a-1}}{t} - \frac{(a-1)(a-2)}{t^{1+a}} \chi(t),$$

where c, s are the constants of § 2, *and*

$$\chi(t) = t\int_0^\infty (\theta + yt)^{a-3} \sin\theta\, d\theta.$$

We now prove Lemmas 5, 6, 7. If

$$y > 0, \qquad \lambda > 0, \qquad 0 < a < 1,$$

then
$$\lambda^a \int_0^y v^{a-1} e^{i\lambda v}\, dv = \left(\int_0^\infty - \int_{\lambda y}^\infty\right) v^{a-1} e^{iv}\, dv$$
$$= c + is - \int_0^\infty (v+\lambda y)^{a-1} e^{i(v+\lambda y)}\, dv. \qquad (3)$$

By the second mean-value theorem,[1] the last integral does not exceed $2(\lambda y)^{a-1}$ in absolute value. This gives Lemma 5 with $1-a$ and y instead of a and t.

We notice that Lemma 6 is obvious when $a = 1$. When $0 < a < 1$, put $y = 1$ in (3), multiply each side by $e^{i(\kappa-\lambda)}$, and equate real parts: then

$$\lambda^a \int_0^1 v^{a-1} \cos(\lambda v - \lambda + \kappa)\, dv = c\cos(\kappa - \lambda) - s \sin(\kappa - \lambda)$$
$$- \int_0^\infty (v+\lambda)^{a-1} \cos(v+\kappa)\, dv.$$

This gives Lemma 6 for $0 < a < 1$ if we put $v = 1 - u$ on the left-hand side.

Again, put $v = y - u$ on the left-hand side of (3), multiply each side by $e^{-i\lambda y}$, and equate imaginary parts: then

$$\lambda^a \int_0^y (y-u)^{a-1} \sin\lambda u\, du = c\sin\lambda y - s\cos\lambda y + \int_0^\infty (v+\lambda y)^{a-1} \sin v\, dv.$$

If the last term is modified by integrating twice by parts, Lemma 7 is obtained with λ instead of t.

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[1] If, in (a, b), F is a bounded positive decreasing function and G (possibly complex) is integrable, then there is a ξ $(a \leqq \xi \leqq b)$ such that

$$\left|\int_a^b FG\, dx\right| = F(a)\left|\int_a^\xi G\, dx\right|.$$

Lemma 8. *If $0 < a \leqq 1$ and $p \geqq -1$, then*

$$\int_0^1 (1-u)^{a+p+1} \cos \rho u \, du = \sum_{j=2}^{p+2} \frac{\mu_j}{\rho^j} + \frac{\mu}{\rho^{p+2}} \int_0^1 (1-u)^{a-1} \cos\left(\rho u - \frac{p\pi}{2}\right) du.$$

In the case $p = -1$, the sum on the right-hand side is to be omitted. The constants μ_j, μ depend on a, p, but not on ρ.

This may be proved by repeated integration by parts.

Lemma 9. *For the function $\chi(t)$ defined in Lemma 7, $\chi(\infty -) = 0$. All derivatives of the function exist when $t \geqq 1$ and, if $p \geqq -1$,*

$$\int_1^\infty t^{p+1} \left| \chi^{(p+2)}(t) \right| dt < \infty .$$

For each integer q ($\geqq 0$),

$$\chi^{(q)}(t) = \int_0^\infty \left\{ Ct\,(\theta + yt)^{a-q-3} + C'\,(\theta + yt)^{a-q-2} \right\} \sin \theta \, d\theta,$$

where C, C' depend only on y, a, q; and so, by the second mean-value theorem, $\chi^{(q)}(t) = O(t^{a-q-2})$. Hence the result.

Proof of Theorem 3.

4. Let w (> 0) be such that $M(w) < \infty$. Then from (1.3) and Lemma 3 we infer that

$$\int_1^\infty f(t) \frac{\sin wt}{t^2} dt \quad \text{exists}, \tag{4.1}$$

By operating on the integral in (1.3) with $\int_0^w dv \int_0^v du$, and using (1.3), we see that

$$\int_1^\infty f(t) \left(\frac{w}{t} - \frac{\sin wt}{t^2} \right) dt$$

is bounded (C, r). This, with (4.1) and the fact that $w \neq 0$, shows that

$$\int_1^\infty \frac{f(t)}{t} dt \quad \text{is bounded } (C, r); \tag{4.2}$$

and so, by Lemma 4, the integral (1.4) is bounded (C, r) in the case $j = 1$. The cases $j > 1$ then follow by Lemma 3.

5. To complete the proof of Theorem 3 it will be enough, in view of Lemma 3, to show that if $0 < a < 1$ the integral (1.5) is *bounded* (C, r) for almost every y in $(0, \Omega)$.

Let y $(0 < y < \Omega)$ be such that

$$M(y) < \infty, \qquad \int_0^y (y-u)^{a-1} M(u)\, du < \infty.$$

The values of y that do not satisfy these requirements form a set of measure zero.

Operating on the integral in (1.3) with $\int_0^y du\,(y-u)^{a-1}$, and using (1.3) and Lemma 7, we see that

$$J = \int_1^\infty f(t) \left\{ c\,\frac{\sin yt}{t^a} - s\,\frac{\cos yt}{t^a} + \frac{y^{a-1}}{t} - \frac{(a-1)(a-2)}{t^{1+a}}\,\chi(t) \right\} dt$$

is bounded (C, r). Write the last equation formally

$$J = J_1 - J_2 + J_3 - J_4,$$

where $J_1, \ldots, J_4$, are integrals corresponding to the four terms in { }.

The integral J_1 is summable (C, r) by (1.3) and Lemma 3, since $M(y) < \infty$; and J_3 is bounded (C, r) by (4.2). The integral J_4 is summable (C, r) by Lemma 2, with $a = 1$, $p \geqq r - 1$, $\phi(t) = \chi(t)$, $h(t) = f(t)/t^{1+a}$; the requirements of Lemma 2 being fulfilled by ϕ in virtue of Lemma 9, and by h in virtue of (4.2) and Lemma 3.

Since J, J_1, J_3, J_4 are bounded (C, r), so also is J_2, and thus, since $s \neq 0$ (§ 2),

$$\int^\infty f(t)\,\frac{\cos yt}{t^a}\, dt$$

is bounded (C, r). So also is the similar integral with $\sin yt$ instead of $\cos yt$ (as we have seen in proving that J_1 is summable (C, r)). Hence the same is true of

$$\int^\infty f(t)\,\frac{e^{iyt}}{t^a}\, dt;$$

and therefore, by Lemma 4, the integral (1.5) is bounded (C, r). This completes the proof of Theorem 3, in view of the opening remark of this section.

Proof of Theorem 2.

6. The theorem is known to be true for a function f that is absolutely integrable over $(0, \infty)$, and so for an f that vanishes outside a finite interval. It will therefore be sufficient to prove that, if b is any positive number, (1.2) is true in $(0, b)$ when f vanishes in $(0, b)$.

We may write $\quad r = k(\Omega) = a + p$

where $\quad 0 < a \leqq 1$

and p is an integer not less than -1.

Then, when $0 \leqq x < b$,

$$\pi \int_0^{\omega} \left(1 - \frac{u}{\omega}\right)^{r+1} g(u) \sin ux \, du = 2 \int_{b}^{\infty}{}_{k(\omega)} f(t)\, dt \int_0^{\omega} \left(1 - \frac{u}{\omega}\right)^{r+1} \sin ux \sin ut \, du$$
$$= I_1 - I_2,$$

where[1]
$$I_1 = {}_r\!\int_b^{\infty} f(t)\, dt\, \omega \int_0^1 (1-u)^{r+1} \cos \omega u(t-x)\, du$$
$$= {}_r\!\int_a^{\infty} F(t)\, dt\, \omega \int_0^1 (1-u)^{a+p+1} \cos \omega t u \, du,$$

by Lemma 1, $F(t)$ being $f(t+x)$, $a = b - x$, so that $a > 0$; and I_2 is a similar expression with $f(t-x)$ instead of $f(t+x)$.

It will be shown that $I_1 \to 0$ as $\omega \to \infty$; and it may be shown similarly that $I_2 \to 0$.

7. By Theorem 3,

$$ {}_r\!\int_a^{\infty} \frac{F(t)}{t^j}\, dt \quad \text{exists} \quad (j > 1), \tag{7.1}$$

and ξ may be chosen so that

$$H_q(t) = \frac{1}{\Gamma(q)} \int_a^t (t-u)^{q-1} \frac{F(u)\, e^{i\xi u}}{u^{a/2}}\, du = O(t^{q-1}) \quad (q \geqq r+1, t \geqq a). \tag{7.2}$$

8. By Lemma 8, I_1 will tend to zero if I_3 and I_4 tend to zero, where[2]

$$I_3 = \sum_{j=2}^{p+2} \frac{\mu_j}{\omega^{j-1}}\, {}_r\!\int_a^{\infty} \frac{F(t)}{t^j}\, dt,$$
$$I_4 = {}_r\!\int_a^{\infty} \frac{F(t)\, dt}{\omega^{p+1}\, t^{p+2}}\, \mu \int_0^1 (1-u)^{a-1} \cos\left(\omega t u - \frac{p\pi}{2}\right) du.$$

It is plain from (7.1) that $I_3 \to 0$. By Lemma 6, $I_4 \to 0$ if

$$I_5 = {}_r\!\int_a^{\infty} \frac{F(t)}{t^{a+p+2}}\, \psi\left(\omega t, \frac{p\pi}{2}, a\right) dt = o(\omega^{a+p+1}) \tag{8}$$

and $\quad I_6(\omega) = {}_r\!\int_a^{\infty} \frac{F(t)\, e^{i\omega t}}{t^{a+p+2}}\, dt = o(\omega^{a+p+1}).$

9. If $a = 1$, it is clear by Lemma 6 that I_5 is independent of ω, and so, by (7.1), (8) is true.

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[1] The convergence of the expressions denoted by I_1 and I_2 will become apparent later. The same remark applies to several of the expressions I^n that follow.

[2] If $p = -1$, $I_3 = 0$.

If $0 < a < 1$, the hypotheses of Lemma 2 are satisfied by $h(t) = F(t)/t^{a+p+2}$, $\phi(t) = \psi\left(\omega t, \frac{p\pi}{2}, a\right)$. As regards h, this follows from (7.1). As regards ϕ we have (see Lemma 6) for $q \geqq 0$

$$\begin{aligned}\phi^{(q)}(t) &= \left(\frac{d}{dt}\right)^q \int_0^\infty (u + \omega t)^{a-1} \cos\left(u + \frac{p\pi}{2}\right) du \\ &= C\omega^q \int_0^\infty (u + \omega t)^{a-q-1} \cos\left(u + \frac{p\pi}{2}\right) du \\ &\leqq 2C\omega^q (\omega t)^{a-q-1} = 2C\omega^{a-1} t^{a-q-1}\end{aligned}$$

by the second mean-value theorem, where C depends only on a and q; so that ϕ fulfils the requirements of Lemma 2. By the corollary to that lemma,

$$I_5 = O(\omega^{a-1}) \int_a^\infty t^{a-2}\, dt,$$

and so (8) is established.

10. Write $\beta = \frac{a}{2} + p + 2$; then, with the notation of (7.2),

$$I_6(\omega + \xi) = \int_a^\infty \frac{F(t)\, e^{i\xi t}}{t^{a/2}} \frac{e^{i\omega t}}{t^\beta}\, dt = \int_a^\infty H_{p+2}(t) \left(\frac{d}{dt}\right)^{p+2} \frac{e^{i\omega t}}{t^\beta}\, dt$$

by Lemma 2, with h, ϕ chosen in the obvious way. Performing the $(p + 2)$-fold differentiation, we see that

$$I_6(\omega + \xi) = o(\omega^{p+1}) + (i\omega)^{p+2} I_7,$$

where $$I_7 = \int_a^\infty H_{p+2}(t)\, t^{-\beta} e^{i\omega t}\, dt.$$

To complete the proof of Theorem 2 we have to show that

$$I_7 = o(\omega^{a-1}).$$

By (7.2) and the Riemann-Lebesgue theorem, $I_7 = o(1)$, and this completes the proof when $a = 1$.

When $0 < a < 1$,

$$\begin{aligned}\Gamma(1 - a) I_7 &= \int_a^\infty t^{-\beta} e^{i\omega t}\, dt \int_a^t (t - u)^{-a} H_{a+p+1}(u)\, du \\ &= \int_a^\infty H_{a+p+1}(u)\, du \int_u^\infty t^{-\beta}(t - u)^{-a} e^{i\omega t}\, dt. \qquad (10)\end{aligned}$$

The first of these repeated integrals is obtained from the theory of fractional integration, and the inversion is justified by absolute convergence. In fact, if the integrand of the first repeated integral is

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replaced by its modulus, we obtain by (7.2) an integral that does not exceed

$$\int_a^\infty t^{-\beta}\,dt \int_a^t (t-u)^{-\alpha}\, C u^{\alpha+p}\,du \leqq C \int_a^\infty t^{-\alpha/2-1}\,dt \int_0^1 (1-v)^{-\alpha}\, v^{\alpha+p}\,dv < \infty .$$

On writing $t+u$ for t in the inner integral in the last member of (10) and then integrating by parts, we obtain

$$\beta \int_0^\infty (t+u)^{-\beta-1}\,dt \int_0^t \theta^{-\alpha}\, e^{i\omega(\theta+u)}\,d\theta,$$

which by Lemma 5 is equal to

$$\beta \int_0^\infty (t+u)^{-\beta-1} \left\{ d\omega^{\alpha-1}\, e^{i\omega u} + O(\omega^{-1}\, t^{-\alpha}) \right\} dt.$$

Hence

$$\Gamma(1-\alpha)\, I_7 = d\omega^{\alpha-1} \int_a^\infty H_{\alpha+p+1}(u)\, \frac{e^{i\omega u}}{u^\beta} du + \frac{\beta}{\omega} \int_a^\infty H_{\alpha+p+1}(u)\,du \int_0^\infty \frac{O(1)\,dt}{t^\alpha\,(t+u)^{\beta+1}}.$$

By (7.2) and the Riemann-Lebesgue theorem, the first term on the right-hand side is $o(\omega^{\alpha-1})$; and the second term does not exceed a constant multiple of

$$\omega^{-1} \int_a^\infty u^{\alpha+p}\,du \int_0^\infty \frac{dt}{t^\alpha\,(t+u)^{\alpha/2+p+3}} \leqq \omega^{-1} \int_a^\infty \frac{du}{u^{2-\alpha/2}} \int_0^\infty \frac{dt}{t^\alpha\,(t+a)} = O(\omega^{-1}).$$

It then follows that $I_7 = o(\omega^{\alpha-1})$, and the proof of Theorem 2 is complete.

[*Added in proof.*]

Theorem 2 is similar to a result published[1], while this paper was in the press, by J. L. B. Cooper (who considers only integral orders of summability). The two theorems overlap, but neither contains the other.

[1] See *Proc. London Math. Soc.* (2), **48** (1944), 292-309.

MATHEMATICAL INSTITUTE,
UNIVERSITY OF EDINBURGH.