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ON VALUATIONS OF $K(x)$

by SUDESH K. KHANDUJA

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For a valued field (K,v), let K_v denote the residue field of v and G_v its value group. One way of extending a valuation v defined on a field K to a simple transcendental extension $K(x)$ is to choose any α in K and any μ in a totally ordered Abelian group containing G_v, and define a valuation w on $K[x]$ by $w(\sum_i c_i(x-\alpha)^i)=\min_i (v(c_i)+i\mu)$. Clearly either G_v is a subgroup of finite index in $G_w=G_v+\mathbb{Z}\mu$ or G_w/G_v is not a torsion group. It can be easily shown that $K(x)_w$ is a simple transcendental extension of K_v in the former case. Conversely it is well known that for an algebraically closed field K with a valuation v, if w is an extension of v to $K(x)$ such that either $K(x)_w$ is not algebraic over K_v or G_w/G_v is not a torsion group, then w is of the type described above. The present paper deals with the converse problem for any field K. It determines explicitly all such valuations w together with their residue fields and value groups.

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0. Introduction

Let K be a field with a (Krull) valuation v, with residue field K_v and value group G_v. One way of extending v to a valuation w on the rational function field $K(x)$ is to choose any α in K and any μ in a totally ordered Abelian group G containing G_v and define w on $K[x]$ by

$$w\left(\sum_i c_i(x-\alpha)^i\right)=\inf_i(v(c_i)+i\mu), \tag{1}$$

then extend w to $K(x)$ in the natural way. If $n\mu$ lies in G_v for some positive integer n, then $K(x)_w$ is a simple transcendental extension of K_v and $[G_w\colon G_v]<\infty$ (cf. [**4**, p. 209, Prop. 4.3]). In case μ is free mod G_v then $K(x)_w=K_v$ and $G_w=G_v+\mathbb{Z}\mu$ ([cf. **1**, §10.1, Prop. 1]). Conversely it is well-known that if K is algebraically closed and w is a valuation of $K(x)$ extending v on K, and if either $K(x)_w$ is not algebraic over K_v or G_w/G_v is not a torsion group, then w is of the type described in (1) (see [**4**, p. 205, §2.5]). The present paper deals with the problem when K is not algebraically closed. It is then clear that every extension w of v to $K(x)$ such that $K(x)_w$ is not algebraic over K_v or that G_w/G_v is not a torsion group can be obtained by extending v to a valuation $\bar{v}$ on the algebraic closure $\bar{K}$ of K, extending $\bar{v}$ to a valuation $\bar{w}$ of $\bar{K}(x)$ in the standard way as described in (1) and letting w be the restriction of $\bar{w}$ to $K(x)$. But it is not clear from this approach how the valuation can be described for elements of $K(x)$. This paper gives a direct description of all such valuations w of $K(x)$ as well as of their residue fields and value groups. From this we quickly deduce the analogue of the well-known Ruled Residue Theorem [5] for value groups. It is also shown that Theorems 1 and 2 of [2] are immediate consequences of the results of this paper.

1. Notations, Definitions and Statements of Results

Throughout $K(x)$ is a simple transcendental (abbreviated simple tr.) extension of a field K, v is a valuation of K with value group G_0 and residue field k_0. Let $\bar{K}$ be an algebraic closure of K and $\bar{v}$ an extension of v to $\bar{K}$. We fix any element α of $\bar{K}$ and any element μ of a totally ordered Abelian group G which contains the value group of $\bar{v}$ as an ordered subgroup. Let D denote the subset of $\bar{K}$ defined by

$$D=\{\gamma\in\bar{K}:\bar{v}(\gamma-\alpha)\geqq\mu\};$$

this set depends on α and μ.

An element β of D is chosen so that $[K(\beta):K]\leqq[K(\gamma):K]$ for all γ in D. We shall denote by $P(x)$ the minimal polynomial of β over K of degree n(say); its roots $\beta=\beta_1,\dots,\beta_n$ are arranged so that $\bar{v}(\alpha-\beta_i)\geqq\mu$ for $1\leqq i\leqq m$ and $\bar{v}(\alpha-\beta_i)<\mu$ for $m+1\leqq i\leqq n$. Let θ be the element of G defined by

$$\theta=m\mu+\sum_{i=m+1}^{n}\bar{v}(\alpha-\beta_i). \tag{2}$$

We now define a valuation w of $K(x)$ which extends v. Any non-zero polynomial $f(x)$ of $K[x]$ can be uniquely represented as

$$f(x)=\sum_{i=0}^{r}f_i(x)P(x)^i$$

where the polynomial $f_i(x)$ is either zero or has degree less than that of $P(x)$. The above representation of $f(x)$ will be referred to as the canonical representation of $f(x)$. We define w on $K[x]$ by

$$w(f(x))=\inf_i(\bar{v}(f_i(\alpha))+i\theta). \tag{3}$$

In the second section, we prove:

Theorem 1.1. *w is a valuation of $K[x]$.*

It is not clear at the moment that the valuation w does not depend on the choice of β or $P(x)$ however this turns out to be an immediate consequence of Theorem 1.4.

The unique extension of w to $K(x)$ will again be denoted by w or by $w_{\alpha\mu}$. Also k_1 and

G_1 will respectively denote the residue field and the value group of the valuation $\bar{v}$ restricted to $K(\beta)$. With the above notations we shall prove:

Corollary 1.2. *The value group of the valuation $w_{\alpha\mu}$ is $G_1+\mathbb{Z}\theta$.*

Theorem 1.3. (i) *If μ is torsion modulo G_0 with s as the smallest positive integer such that $s\theta$ is in G_1, say $s\theta=\bar{v}(q(\beta))$, $q(x)$ in $K[x]$, then the residue field of $w_{\alpha\mu}$ is $k_1(t)$ where $t=$ the residue class of $P(x)^s/q(x)$ is transcendental over k_0.*

(ii) If *μ is free modulo G_0, then the residue field of $w_{\alpha\mu}$ is isomorphic to k_1.*

For a fixed α in $\bar{K}$ and μ in a totally ordered Abelian group containing the value group of $\bar{v}$, $\bar{w}_{\alpha\mu}$ will denote the valuation of $\bar{K}(x)$ which is defined for any polynomial $g(x)=\sum_i c_i(x-\alpha)^i$ over $\bar{K}$ by

$$\bar{w}_{\alpha\mu}(g(x))=\inf_i(\bar{v}(c_i)+i\mu).$$

It will be referred to as the valuation defined by inf. $\bar{v}$, α and μ. Theorem 1.1 is proved as soon as we prove:

Theorem 1.4. *The restriction of the valuation $\bar{w}_{\alpha\mu}$ defined by* inf. *α, $\bar{v}$ and μ to $K(x)$ is $w_{\alpha\mu}$.*

2. Proof of Theorem 1.4.

Lemma 2.1. *Let $\bar{w}=\bar{w}_{\alpha\mu}$ be the valuation of $\bar{K}(x)$ defined by* inf. *$\bar{v}$, α and μ, and let $f(x)$ be any non-zero element of $\bar{K}[x]$. Then the following hold for all γ in D.*

(i) $\bar{w}(f(x))\leqq\bar{w}(f(\gamma))$.

(ii) *If $f(x)$ has no zeros in D, then*

$$\bar{w}(f(x)-f(\gamma))>\bar{w}(f(x)).$$

(iii) *If $f(x)$ is in $K[x]$ and has canonical representation*

$$f(x)=\sum_{i=0}^{r}f_i(x)P(x)^i,$$

then

$$\bar{w}(f(x))=\inf_i\bar{w}(f_i(x)P(x)^i).$$

Proof. Let γ be any element of D. To prove (i) write

$$f(x)=a\prod(x-\alpha_i)=f(\gamma)\prod\left(\frac{x-\alpha_i}{\gamma-\alpha_i}\right),$$

which implies that

$$\bar{w}(f(x))=\bar{w}(f(\gamma))+\sum_i(\bar{w}(x-\alpha_i)-\bar{v}(\gamma-\alpha_i));$$

i.e.

$$\bar{w}(f(x))=\bar{w}(f(\gamma))+\sum_i(\min(\mu,\bar{v}(\alpha-\alpha_i))-\bar{v}(\gamma-\alpha_i)). \tag{4}$$

If $\alpha_i\in D$, then by the triangle law, we have

$$\mu-\bar{v}(\gamma-\alpha_i)\leqq\mu-\min(\bar{v}(\gamma-\alpha),\ \bar{v}(\alpha-\alpha_i))\leqq 0.$$

If $\alpha_i\notin D$ i.e. if $\bar{v}(\alpha-\alpha_i)<\mu$, then by the strong triangle law

$$\bar{v}(\gamma-\alpha_i)=\min(\bar{v}(\gamma-\alpha),\ \bar{v}(\alpha-\alpha_i));$$

consequently

$$\bar{v}(\alpha-\alpha_i)-\bar{v}(\gamma-\alpha_i)=0$$

in this case. It is now clear that (i) follows from (4).

To prove (ii), write

$$f(x)=a\prod(x-\alpha_i)=f(\gamma)\prod\left(\frac{x-\gamma}{\gamma-\alpha_i}+1\right).$$

It is given that $\bar{v}(\alpha-\alpha_i)<\mu$, therefore

$$\bar{w}(x-\gamma)-\bar{v}(\gamma-\alpha_i)=\mu-\bar{v}(\gamma-\alpha_i)=\mu-\bar{v}(\alpha-\alpha_i)>0$$

which shows that

$$\bar{w}\left(\frac{f(x)}{f(\gamma)}-1\right)>0.$$

Hence (ii) is proved.

To prove (iii) we use induction on the degree of the polynomial $f(x)\in K[x]$. If $\deg f(x)<n=\deg P(x)$, then the assertion is obvious. Assume now that $\deg f(x)\geqq n$ and that (iii) holds for polynomials in $K[x]$ of degree smaller than $\deg f(x)$. Set

$$Q(x)=\frac{f(x)-f_0(x)}{P(x)}=\sum_{i\geqq 1}f_i(x)P(x)^{i-1}.$$

By the induction hypothesis, we have

$$\bar{w}(Q(x)) \leqq \inf_{i \geqq 1} \bar{w}(f_i(x)P(x)^{i-1}). \tag{5}$$

Observe that the polynomial $f_0(x)$ being of degree less than n has no zeros in D, therefore by the second assertion of the lemma

$$\bar{w}(f_0(\beta)) = \bar{w}(f_0(x)).$$

It now follows from (i) that

$$\bar{w}(f(x)) \leqq \bar{w}(f(\beta)) = \bar{w}(f_0(\beta)) = \bar{w}(f_0(x)); \tag{6}$$

consequently

$$\bar{w}(Q(x)) \geqq \min\left(\bar{w}\left(\frac{f(x)}{P(x)}\right), \bar{w}\left(\frac{f_0(x)}{P(x)}\right)\right) = \bar{w}\left(\frac{f(x)}{P(x)}\right)$$

which together with (5) gives

$$\bar{w}(f(x)) \leqq \bar{w}(P(x).Q(x)) \leqq \min_{i \geqq 1} \bar{w}(f_i(x)P(x)^i). \tag{7}$$

Clearly (iii) is immediate from (6) and (7).

Note that for any polynomial $f(x)$ in $K[x]$ of degree less than n,

$$\bar{w}(g(x)) = \bar{v}(g(\alpha)) = \bar{v}(g(\beta))$$

holds by assertion (ii) of the lemma. It is easily verified that

$$\bar{w}(P(x)) = \sum_{i=1}^{n} \bar{w}(x - \beta_i) = \theta$$

where θ is as defined by (2). In view of these observations, Theorem 1.4 and Corollary 1.2 follow immediately from the last assertion of the lemma.

Remark 2.2. One can prove directly that $w_{\alpha\mu} = w$ defined by (3) is a valuation of $K(x)$ by extending the argument involved in [**2**, Theorem 1] or in [**1**, § **10.1, Lemma 1**] but the proof turns out to be very cumbersome.

3. Proof of Theorem 1.3

We shall keep the notations of the first section and shall denote the residue field of

$w=w_{\alpha\mu}$ by k. As before, we shall denote by $\bar{w}$ the prolongation of w to $\bar{K}(x)$ defined by inf. $\bar{v}$, α and μ. We shall regard k to be a subfield of the residue field of $\bar{w}$. For any element a in the valuation ring of $\bar{w}$, a^* will stand for its image in the residue field of $\bar{w}$.

To prove Theorem 1.3. we discuss two cases. Assume first that μ and hence θ is torsion mod G_0. Let s be the smallest positive integer such that $s\theta \in G_1$, say

$$s\theta = \bar{v}(q(\beta)) = w(q(x))$$

with $q(x)$ in $K[x]$ of degree less than n. We first show that the residue class $(P(x)^s/q(x))^* = t$ (say) is tr. over k_0. Suppose that t is algebraic over k_0 and that $y^m + a_1 y^{m-1} + \cdots + a_m$ is the minimal polynomial of t over k_0; here $v(a_i) \geqq 0$ and $v(a_m) = 0$. So if we write

$$F(x) = P(x)^{sm} + a_1 q(x) P(x)^{s(m-1)} + \cdots + a_m q(x)^m$$

then the supposition implies that

$$w(F(x)) > w(q(x)^m) = \bar{v}(q(\beta)^m) = \bar{v}(a_m q(\beta)^m),$$

i.e., we have

$$\bar{w}(F(x)) > \bar{w}(F(\beta))$$

which is impossible in view of Lemma 2.1(i). This contradiction proves the desired assertion.

If $h(x)$ is any polynomial in $K[x]$ of degree less than n with $\bar{v}(h(\beta)) = 0$, then by Lemma 2.1(ii)

$$\bar{w}(h(x) - h(\beta)) > 0;$$

this shows that k_1 and hence $k_1(t) \subseteq k$. To prove equality let $\xi^* = (f(x)/g(x))^*$ be an arbitrary non-zero element of k with $f(x)$ and $g(x)$ in $K[x]$ and let

$$f(x) = \sum_i f_i(x) P(x)^i, \; g(x) = \sum_j g_j(x) P(x)^j$$

be the canonical representations for $f(x)$ and $g(x)$ respectively. There exists a non-negative integer r and a polynomial $h(x)$ in $K[x]$ of degree less than n such that

$$w(h(x) P(x)^r) = w(f(x)) = w(g(x)).$$

Define elements ξ_1, ξ_2 of $K[x]$ by

$$\xi_1 = f(x)/h(x) P(x)^r, \; \xi_2 = g(x)/h(x) P(x)^r$$

then $\xi^* = \xi_1^*/\xi_2^*$. It is enough to show that ξ_i^* is in $k_1(t)$. Clearly

$$\xi_1^* = \sum{}'(f_i(x)P(x)^{i-r}/h(x))^*$$

where the sum $\sum'$ is carried over all those i for which $w(f_i(x)P(x)^i) = w(h(x)P(x)^r)$ holds. For each i in $\sum'$, $(i-r)\theta \in G_1$ and so $i-r$ is an integral multiple of s, say $i-r=sm_i$. In fact if $\varepsilon = f_i(x)P(x)^{sm}/h(x)$ with $w(\varepsilon)=0$, then since each of $f_i(x)$, $h(x)$, $q(x)$ has degree less than n, therefore in view of Lemma 2.1(ii) we deduce that

$$\varepsilon^* = (f_i(\beta)q(\beta)^m/h(\beta))^*.t^m$$

is in $k_1(t)$. Thus $\xi_1^* \in k_1(t)$. Similarly $\xi_2^* \in k_1(t)$.

The residue field in the other case can be easily determined by using the argument involved in the proof of [**2**, Theorem 2(ii)] or of [**1**, § 10.1, Prop. 1].

Remark 3.1. We show that the valuation $V_{P(x)}$ of [**2**, Theorem 1] is $w_{\alpha\mu}$ for suitable α in $\bar{K}$ and μ in the set of positive real numbers. In [**2**], $P(x)$ is a monic polynomial with co-efficients in the valuation ring of v such that the polynomial $P^*(x)$ obtained by replacing each co-efficient of $P(x)$ by its residue in k_0, is irreducible over k_0. Let α be a root of $P(x)$. If γ is any element of $\bar{K}$ with $\bar{v}(\gamma-\alpha)>0$, then

$$[K(\alpha):K] = [k_0(\alpha^*):k_0] = [k_0(\gamma^*):k_0] \leqq [K(\gamma):K].$$

Consequently in the case under consideration, the element 'β' may be taken to be α itself. We write $P(x) = a_1(x-\alpha) + \cdots + a_n(x-\alpha)^n$. In view of the fact that α^* is algebraic over k_0 of degree n, it is easily verified that for any polynomial $g(x) = \sum_i c_i x^i$ in $K[x]$ with degree $g(x) < n$, we must have

$$\bar{v}(g(\alpha)) = \min_i v(c_i).$$

It is now clear that the valuation $V_{P(x)}$ is nothing but the valuation $w_{\alpha\mu}$, where for a given positive real number θ involved in the definition of $V_{P(x)}$, the positive real number μ may be defined by the equation

$$\theta = \inf_i (\bar{v}(a_i) + i\mu)$$

i.e.

$$\mu = \sup_i ((\theta - \bar{v}(a_i))/i) > 0, \text{ since } \bar{v}(a_n) = 0.$$

Remark 3.2. Let $K_0(x)$ be a simple tr. extension of a field K_0, v_0 a valuation of K_0

and v be an extension of v_0 to $K_0(x)$. Let $k_0 \subseteq k$ and $G_0 \subseteq G$ denote respectively the residue fields and value and value groups of v_0 and v. The Ruled Residue Theorem conjectured by Nagata [3, Theorem 1] and proved by Ohm [5] asserts that k is either an algebraic extension of k_0 or it is a simple tr. extension of a finite extension of k_0. The analogous result for value groups may be stated as follows:

Either G/G_0 is a torsion group or there exists a subgroup G_1 of G containing G_0 with $[G_1:G_0]<\infty$ such that G is the direct sum of G_1 and an infinite cyclic group. To deduce this, suppose that G/G_0 is not a torsion group. Extend v to a valuation $\bar{v}$ (say) of $\bar{K}_0(x)$. Let $\bar{v}_0$ denote the restriction of $\bar{v}$ to the algebraic closure $\bar{K}_0$ of K_0 and let $\bar{G}_0 \subseteq \bar{G}$ denote respectively the value groups $\bar{v}_0$ and $\bar{v}$. Then $\bar{G}/\bar{G}_0$ is not a torsion group. So there exists α in $\bar{K}$ such that $\bar{v}(x-\alpha)=\mu$ (say) is not torsion mod $\bar{G}_0$. It can be easily verified that $\bar{v}$ is the valuation of $\bar{K}_0(x)$ defined by inf. $\bar{v}_0$, α and μ. The desired assertion now follows from Theorem 1.4 and Corollary 1.2.

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Department of Mathematics
Panjab University, Chandigarh
India—160014