Canad. Math. Bull. Vol. **52** (1), 2009 pp. 154–160

A Big Picard Theorem for Holomorphic Maps into Complex Projective Space

Yasheng Ye and Min Ru

Abstract. We prove a big Picard type extension theorem for holomorphic maps $f\colon X - A \to M$, where X is a complex manifold, A is an analytic subvariety of X, and M is the complement of the union of a set of hyperplanes in $\mathbb{P}^n$ but is not necessarily hyperbolically imbedded in $\mathbb{P}^n$.

1 Introduction and Statements

The classical big Picard theorem states that any holomorphic map f from the punctured disk $\triangle^*$ to the Riemann sphere $\mathbb{P}^1$ which omits three points can be extended to a holomorphic map $f\colon \triangle \to \mathbb{P}^1$. Through work of Kwack, Kobayashi, and Kiernan [4–6], the big Picard theorem has been generalized to showing that any holomorphic map $f\colon X - A \to M \subset Y$ can be extended to a meromorphic mapping $f\colon X \to Y$ provided that M is hyperbolically imbedded in Y, where X is a complex manifold, A is an analytic subvariety of X, and M and Y are complex spaces. Here, according to S. Kobayashi [5], a relatively compact open set M in a complex space Y is said to be *hyperbolically imbedded in Y* if

(i) M is Kobayashi hyperbolic, *i.e.,* the Kobayahsi pseudo-distance d_M is a proper distance;

(ii) whenever p_n and q_n are sequences in M converging to distinct boundary points, then $d_M(p_n, q_n)$ does not converge to 0.

The space $\mathbb{P}^1 - \{0, 1, \infty\}$ is, for example, hyperbolically imbedded in $\mathbb{P}^1$. More generally, according to the result of Dufresnoy, Fujimoto, and Green [1–3], if $H_1, \dots, H_{2n+1}$ are hyperplanes in general position in $\mathbb{P}^n$, then $M = \mathbb{P}^n - (H_1 \cup \cdots \cup H_{2n+1})$ is hyperbolically imbedded in $\mathbb{P}^n$. Hence the above mentioned result of Kwack, Kobayashi, and Kiernan holds when $M = \mathbb{P}^n - (H_1 \cup \cdots \cup H_{2n+1})$ and $Y = \mathbb{P}^n$, where $H_1, \dots, H_{2n+1}$ are hyperplanes in general position in $\mathbb{P}^n$.

The purpose of this paper is to study the case when M is the complement of the union of a set of hyperplanes in $\mathbb{P}^n$, but M is not necessarily hyperbolically imbedded in $\mathbb{P}^n$. Hence the theorem of Kwack, Kobayashi, and Kiernan does not apply. To see what M looks like, we recall the following result.

Theorem A **(Ru)** *Let $\mathcal{H}$ be a collection of hyperplanes in $\mathbb{P}^n$ and let $\mathcal{L}$ be a set of linear forms in $z_0, \dots, z_n$ that define the hyperplanes in $\mathcal{H}$. Denote by $|\mathcal{H}|$ the union of the hyperplanes in $\mathcal{H}$ and denote by $\mathcal{L}$ the vector space generated by the elements in $\mathcal{L}$ over*

Received by the editors July 18, 2006; revised January 25, 2007.

The first author is supported by the NNSF of China Approved No. 10671067, and the second author is supported in part by NSA grant H98230-07-1-0050.

AMS subject classification: Primary: 32H30.

https://doi.org/10.4153/CMB-2009-018-6 Published online by Cambridge University Press

$\mathbb{C}$. *Then* $\mathbb{P}^n - |\mathcal{H}|$ *is Brody hyperbolic (*i.e., *every holomorphic map* $f\colon \mathbb{C} \to \mathbb{P}^n - |\mathcal{H}|$ *must be constant) if and only if* $\mathcal{H}$ *satisfies the following conditions:*

(i) $\dim(\mathcal{L}) = n + 1$;

(ii) *for each proper non-empty subset* $\mathcal{L}_1$ *of* $\mathcal{L}$, *we have* $\mathcal{L} \cap (\mathcal{L}_1) \cap (\mathcal{L}\backslash(\mathcal{L}_1) \neq \varnothing$.

Example 1.1 *Let* $\mathcal{L} = \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,1,1)\}$. *Then* $\mathcal{L}$ *satisfies* (i) and (ii) in Theorem A. Note that the hyperplanes in $\mathbb{P}^2$ defined by these linear forms are not in general position.

It is shown in [8] that $\mathbb{P}^2 - |\mathcal{H}|$ is not hyperbolically imbedded in $\mathbb{P}^2$ when $\mathcal{H}$ consists of the hyperplanes from Example 1.1. So in general $M = \mathbb{P}^n - |\mathcal{H}|$ does not have to be hyperbolically imbedded in $\mathbb{P}^n$ if $\mathcal{H}$ satisfies (i) and (ii) in Theorem A. However, one of the results in this paper will show that the extension theorem still holds if $\mathcal{H}$ satisfies (i) and (ii) in Theorem A.

Next we recall a result of A. Levin [7] which generalizes Theorem A to the following setting.

***Theorem B* (Levin)** *Let* $\mathcal{H}$ *be a collection of hyperplanes in* $\mathbb{P}^n$ *and let* $\mathcal{L}$ *be a set of corresponding linear forms. Let* $m = \dim \bigcap_{H\in\mathcal{H}} H$. *Then there exists a holomorphic map* $f\colon \mathbb{C} \to \mathbb{P}^n - |\mathcal{H}|$ *with* $\dim f(\mathbb{C}) = d > m + 1$ *if and only if* $\mathcal{L}$ *satisfies the following condition: there exists a partition of* $\mathcal{L}$ *into* $d - m$ *nonempty pairwise disjoint subsets* $\mathcal{L}_j$, $\mathcal{L} = \bigcup_{j=1}^{d-m} \mathcal{L}_j$ *with* $\mathcal{L}_j \neq \varnothing$ *for all* j, *and* $\mathcal{L} \cap \sum_{j=1}^{d-m} (\mathcal{L}_j) \cap (\mathcal{L}\backslash\mathcal{L}_j) = \varnothing$.

In this paper, we prove an extension theorem which is motivated by Theorem A and Theorem B. To state our main theorem, we first introduce the following definition.

Definition 1.2 Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be a collection of hyperplanes in $\mathbb{P}^n$ and let $\mathcal{L}$ be a set of the corresponding linear forms. Let A be an analytic subvariety of a complex manifold X and let $f\colon X - A \to \mathbb{P}^n - |\mathcal{H}|$ be holomorphic. Consider the uniquely determined partition $\{1, \dots, q\} = I_1 \cup \cdots \cup I_s$ such that i and j are in the same class if and only if $L_i(f)/L_j(f)$ extends across A meromorphically. The integer s is called *the degree of irrationality of* f *with respect to* $\mathcal{H}$.

Note that, when $\dim(\mathcal{L}) = n+1$, the degree of irrationality of f with respect to $\mathcal{H}$ is equal to 1 if and only if $f\colon X - A \to \mathbb{P}^n - |\mathcal{H}|$ extends meromorphically across A.

We will prove the following result.

Main Theorem *Let* $\mathcal{H}$ *be a collection of hyperplanes in* $\mathbb{P}^n$ *and let* $\mathcal{L}$ *be a set of corresponding linear forms. Let* $m = \dim \bigcap_{H\in\mathcal{H}} H$. *Let* $d > m + 1$ *be an integer. Then for every complex manifold* X, *every proper analytic subvariety* A *of* X, *and every holomorphic map* $f\colon X - A \to \mathbb{P}^n - |\mathcal{H}|$, *the degree of irrationality of* f *with respect to* $\mathcal{H}$ *is less than* $d - m$ *if and only if* $\mathcal{H}$ *satisfies the following property: for every partition of* $\mathcal{L}$ *into* $d - m$ *nonempty pairwise disjoint subsets* $\mathcal{L}_j$, $\mathcal{L} = \bigcup_{j=1}^{d-m} \mathcal{L}_j$, *we have*

$$\mathcal{L} \cap \sum_{j=1}^{d-m} (\mathcal{L}_j) \cap (\mathcal{L}\backslash\mathcal{L}_j) \neq \varnothing. \tag{1.1}$$

https://doi.org/10.4153/CMB-2009-018-6 Published online by Cambridge University Press

Note that in the above theorems and elsewhere we define $\dim \varnothing = -1$.

Corollary 1.3 *Let $\mathcal{H}$ be a collection of hyperplanes in $\mathbb{P}^n$ and let $\mathcal{L}$ be a set of corresponding linear forms. Then for every complex manifold X, every proper analytic subvariety A of X, and every holomorphic map $f\colon X - A \to \mathbb{P}^n - |\mathcal{H}|$, f extends meromorphically across A if and only if $\mathcal{H}$ satisfies the following conditions:*

(i) $\dim(\mathcal{L}) = n + 1$;

(ii) *for each proper non-empty subset $\mathcal{L}_1$ of $\mathcal{L}$, we have $\mathcal{L} \cap (\mathcal{L}_1) \cap (\mathcal{L}\backslash(\mathcal{L}_1) \neq \varnothing$.*

In particular, for every holomorphic map $f\colon X - A \to \mathbb{P}^n$, if f omits $2n+1$ hyperplanes in general position, then f extends meromorphically across A.

To see how the Main Theorem implies Corollary 1.3, we first notice that the condition $\dim(\mathcal{L}) = n + 1$ implies that $\bigcap_{H\in\mathcal{H}} H = \varnothing$. Hence $m = -1$. On the other hand, it is clear that the assumption (ii) in Corollary 1.3 is the same as (1.1) with $d = 1$. Hence, Corollary 1.3 is the special case of the Main Theorem with $m = -1$ and $d = 1$.

Corollary 1.4 [M. Green [3]] Let X be a complex manifold and A be an analytic subvariety of X. Let $f\colon X-A \to \mathbb{P}^n$ be a holomorphic map omitting $n+k$ hyperplanes in general position, then the degree of irrationality of f is less than or equal to $[n/k]+1$.

We now show how the Main Theorem implies Corollary 1.4. Let $\mathcal{H} = \{H_1, \dots, H_{n+k}\}$, where $H_1, \dots, H_{n+k}$ are hyperplanes in general position. Then $m = \dim \bigcap_{H\in\mathcal{H}} H = -1$. We show that for every integer $d > n/k$ and for every partition of $\mathcal{L}$ into $d + 1$ nonempty disjoint subsets $\mathcal{L}_i$, (1.1) holds. In fact, using $d > n/k$, there must be a j_0 such that $\#\bigcup_{j\neq j_0} \mathcal{L}_j \geq \frac{d}{d+1}(n + k) > n$. Hence, by the "in general position" condition, we have $(\mathcal{L}\backslash\mathcal{L}_{j_0}) = \mathbb{C}^{n+1}$. Therefore

$$\mathcal{L} \cap (\mathcal{L}_{j_0}) \cap (\mathcal{L}\backslash\mathcal{L}_{j_0}) = \mathcal{L} \cap (\mathcal{L}_{j_0}) \neq \varnothing$$

which implies that

$$\mathcal{L} \cap \sum_{j=1}^{d+1} (\mathcal{L}_j) \cap (\mathcal{L}\backslash\mathcal{L}_j) \neq \varnothing.$$

Thus (1.1) is satisfied. By the Main Theorem, the degree of irrationality of f is less than or equal to $[n/k] + 1$.

2 Proof of the Main Theorem

We first recall the following well-known lemma (see, for instance, *The Borel Lemma for Punctured Domains* in [3, Page 56]).

Lemma 2.1 *Let $f_1, \dots, f_n$ be nowhere-vanishing holomorphic functions on $X - A$, where X is a complex manifold and A an analytic subvariety of X. If $f_1+\cdots+f_n = 1$ and the f_i are linearly independent on $X - A$, then all the f_i extend across A as meromorphic functions. Without the assumption of linear independence, then at least one of the f_i extends meromorphically across A.*

We are now ready to prove the Main Theorem.

Proof of the Main Theorem "$\Leftarrow$". Let $d > m + 1$ be an integer such that for every partition of $\mathcal{L}$ into $d - m$ non-empty pairwise disjoint subsets $\mathcal{L}_j$ we have

$$\mathcal{L} \cap \sum_{j=1}^{d-m} (\mathcal{L}_j) \cap (\mathcal{L} \backslash \mathcal{L}_j) \neq \varnothing. \tag{2.1}$$

We prove that, for every holomorphic map $f \colon X - A \to \mathbb{P}^n - |\mathcal{H}|$, the degree of irrationality of f with respect to $\mathcal{H}$ is less than $d - m$. If not, we assume that $f \colon X - A \to \mathbb{P}^n - |\mathcal{H}|$ has degree of irrationality $\geq d - m$. Let $\{I_1, \ldots, I_s\}$ be the set of equivalence classes of the elements of $\mathcal{L}$ under the equivalence relation defining the degree of irrationality of f. Let $\mathcal{L}_j = I_j$ for $1 \leq j < d - m$ and let $\mathcal{L}_{d-m} = \mathcal{L} \backslash \bigcup_{j=1}^{d-m-1} \mathcal{L}_j$. By assumption (see (2.1)),

$$\mathcal{L} \cap \sum_{j=1}^{d-m} (\mathcal{L}_j) \cap (\mathcal{L} \backslash \mathcal{L}_j) \neq \varnothing.$$

Thus, there is a linear form L in $\mathcal{L}$ and linearly independent linear forms L_i such that $L = \sum_i c_i L_i$ for non-zero constants c_i such that none of the L_i are in the same equivalence class as L. This contradicts Lemma 2.1, and hence the "$\Leftarrow$" is proven.

"$\Rightarrow$". Let $d > m + 1$ and assume $\mathcal{L}$ can be partitioned into $d - m$ pairwise disjoint non-empty subsets $\mathcal{L}_j$ such that

$$\mathcal{L} \cap \sum_{j=1}^{d-m} (\mathcal{L}_j) \cap (\mathcal{L} \backslash \mathcal{L}_j) = \varnothing. \tag{2.2}$$

We will construct a holomorphic map $f \colon \triangle^* \to \mathbb{P}^n - |\mathcal{H}|$ with degree of irrationality $s \geq d - m$, where $\triangle^*$ is the punctured unit disk in $\mathbb{C}$. This will contradict our assumption, and hence proves the "$\Rightarrow$" direction. To do so, we first prove the following claim.

Claim *There is a subspace $Y \subset \mathbb{P}^n$ such that $\dim Y = d$, $\#\mathcal{H}|_Y = d - m$, and the hyperplanes in $\mathcal{H}|_Y$ are linearly independent, where $\mathcal{H}|_Y$ is the set of hyperplanes which are the restriction of the hyperplanes in $\mathcal{H}$ to Y.*

The claim is contained in [7, Theorem 7]. We enclose a proof here for the sake of completeness. To construct such Y, let

$$U_0 = \sum_{j=1}^{d-m} (\mathcal{L}_j) \cap (\mathcal{L} \setminus (\mathcal{L}_j).$$

Obviously, since $(\mathcal{L}_j) \cap (\mathcal{L} \setminus \mathcal{L}_j) \subset U_0 \cap (\mathcal{L}_j)$ for all j, we have

$$U_0 = \sum_{j=1}^{d-m} U_0 \cap (\mathcal{L}_j). \tag{2.3}$$

https://doi.org/10.4153/CMB-2009-018-6 Published online by Cambridge University Press

We now construct inductively the vector spaces $U_i, 0 \leq i \leq d-m$, which satisfy the following four properties:

(1) $U_i \subset U_j$ for $i < j$;
(2) $\dim U_i \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$ for $i > 0$;
(3) $U_i \cap \mathcal{L} = \varnothing$,
(4) $U_i = \sum_{j=1}^{d-m} U_i \cap (\mathcal{L}_j)$.

First, by (2.2) and (2.3), U_0 satisfies (3) and (4). Suppose now that U_{i-1} has been constructed with properties (1), (2), (3), and (4). We now construct U_i. From the induction assumption, $U_{i-1} \cap \mathcal{L} = \varnothing$. Hence $U_{i-1} \cap (\mathcal{L}_i)$ is a proper subset of $(\mathcal{L}_i)$, *i.e.*, $\dim U_{i-1} \cap (\mathcal{L}_i) < \dim(\mathcal{L}_i)$. We distinguish two cases: $\dim U_{i-1} \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$ and $\dim U_{i-1} \cap (\mathcal{L}_i) \leq \dim(\mathcal{L}_i) - 2$. When $\dim U_{i-1} \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$, we let $U_i = U_{i-1}$. Then, by the induction assumption and the assumption that $\dim U_{i-1} \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$, we see that U_i satisfies (1), (2), (3) and (4). So we can assume that $\dim U_{i-1} \cap (\mathcal{L}_i) \leq \dim(\mathcal{L}_i) - 2$. Let $\{\mathbf{a}_1, \dots, \mathbf{a}_{t_i}\}$ be a basis for $U_{i-1} \cap (\mathcal{L}_i)$ (we take it as an empty set if $U_{i-1} \cap (\mathcal{L}_i) = \{0\}$) and expand it to form a basis $\{\mathbf{a}_1, \dots, \mathbf{a}_{t_i}, \mathbf{a}_{t_i+1}, \dots, \mathbf{a}_{r_i}\}$ for the space $(\mathcal{L}_i)$, where $r_i = \dim(\mathcal{L}_i)$. By our assumption, $r_i - t_i \geq 2$, and $(\mathbf{a}_1, \dots, \mathbf{a}_{t_i}) \cap \mathcal{L}_i = \varnothing$, where $(\mathbf{a}_1, \dots, \mathbf{a}_{t_i})$ means the the vector space generated by $\{\mathbf{a}_1, \dots, \mathbf{a}_{t_i}\}$ over $\mathbb{C}$. We then can easily choose non-zero constants $c_{t_i+1}, \dots, c_{r_i-1}$ such that, if we let $A_i = (\mathbf{a}_1, \dots, \mathbf{a}_{t_i}, \mathbf{a}_{t_i+1} - c_{t_i+1}\mathbf{a}_{r_i}, \dots, \mathbf{a}_{r_i-1} - c_{r_i-1}\mathbf{a}_{r_i})$, then $A_i \cap \mathcal{L}_i = \varnothing$, and $\dim A_i = \dim(\mathcal{L}_i) - 1$. Now we let $B_i = (\mathbf{a}_{t_i+1} - c_{t_i+1}\mathbf{a}_{r_i}, \dots, \mathbf{a}_{r_i-1} - c_{r_i-1}\mathbf{a}_{r_i})$ and let $U_i = (U_{i-1}, B_i)$. U_i is the vector space generated by the vectors in U_{i-1} and the vectors $\mathbf{a}_{t_i+1} - c_{t_i+1}\mathbf{a}_{r_i}, \dots, \mathbf{a}_{r_i-1} - c_{r_i-1}\mathbf{a}_{r_i}$. Then, from the above, we have $U_i \cap \mathcal{L}_i = \varnothing$, and $\dim U_i \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$. It remains to show that U_i satisfies properties (3) and (4). We first verify property (4). By induction assumption, we have

$$U_{i-1} = \sum_{j=1}^{d-m} U_{i-1} \cap (\mathcal{L}_j).$$

Hence,

$$\begin{aligned} U_i = (U_{i-1}, B_i) &= \sum_{j=1}^{d-m} U_{i-1} \cap (\mathcal{L}_j) + B_i \\ &\subset \sum_{j=1, j\neq i}^{d-m} U_i \cap (\mathcal{L}_j) + U_i \cap (\mathcal{L}_i) = \sum_{j=1}^{d-m} U_i \cap (\mathcal{L}_j) \subset U_i. \end{aligned}$$

Hence property (4) holds. To show $U_i \cap \mathcal{L} = \varnothing$, we assume that $L \in U_i \cap \mathcal{L}$. Since $U_i \cap \mathcal{L}_i = \varnothing$, we have $L \in \mathcal{L}_{i'}$ for some $i' \neq i$. Using $U_i = \sum_{j=1, j\neq i}^{d-m} U_{i-1} \cap (\mathcal{L}_j) + U_i \cap (\mathcal{L}_i)$, we may write $L = \sum_{j=1}^{d-m} u_j$ with $u_j \in U_{i-1} \cap (\mathcal{L}_j)$ for $j \neq i$ and $u_i \in U_i \cap (\mathcal{L}_i)$. Hence $L - u_{i'} = \sum_{j \neq i'} u_j \in (\mathcal{L} \setminus \mathcal{L}_{i'})$. That means $L - u_{i'} \in (\mathcal{L}_{i'}) \cap (\mathcal{L} \setminus \mathcal{L}_{i'}) \subset U_0 \subset U_{i-1}$. But $u_{i'} \in U_{i-1}$ which implies that $L \in U_{i-1}$. This contradicts the assumption that $U_{i-1} \cap \mathcal{L} = \varnothing$. Hence the property (3) also holds.

Let $U_0, \dots, U_{d-m}$ be the vector spaces as defined above. Let Y be the subspace of $\mathbb{P}^n$ such that $y \in Y$ if and only if $L(y) = 0$ for all $L \in U_{d-m}$. We show that Y satisfies

https://doi.org/10.4153/CMB-2009-018-6 Published online by Cambridge University Press

the conditions stated in the claim. Since $U_{d-m} \cap \mathcal{L} = \varnothing$, we have that $Y \not\subset |\mathcal{H}|$. Next we show that $\mathcal{L}|_Y$ is a linearly independent set. To do so, we first show that $\dim U_{d-m} \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$ for all i. In fact, from property (2), $\dim U_i \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$, and from (1) $U_i \subset U_{d-m}$. So $\dim(\mathcal{L}_i) - 1 \leq \dim U_{d-m} \cap (\mathcal{L}_i) \leq \dim(\mathcal{L}_i)$. But $\dim U_{d-m} \cap (\mathcal{L}_i)$ cannot be equal to $\dim(\mathcal{L}_i)$ because otherwise we would have $U_{d-m} \cap (\mathcal{L}_i) = (\mathcal{L}_i)$, which is impossible since $U_{d-m} \cap \mathcal{L}_i = \varnothing$. Hence $\dim U_{d-m} \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$ for all i. Therefore

$$\dim(U_{d-m} + (\mathcal{L}_i)) = \dim U_{d-m} + \dim(\mathcal{L}_i) - \dim U_{d-m} \cap (\mathcal{L}_i) = \dim U_{d-m} + 1.$$

Hence, $\dim(\mathcal{L}_i|_Y) = 1$. Let $\mathcal{H}_i$ be the set of the hyperplanes defined by the linear forms in $\mathcal{L}_i$. Then it implies that $\mathcal{H}_i|_Y$ consists of only a single hyperplane in Y. So $\mathcal{H}|_Y$ consists of at most $d - m$ hyperplanes. On the other hand, since $U_{d-m} \subset (\mathcal{L})$, the condition $\dim \bigcap_{H \in \mathcal{H}} H = m$ implies that $\dim \bigcap_{H \in \mathcal{H}|_Y} H = m$. This, together with the fact that $\dim Y = d$ (we will show it below) and the fact that $\mathcal{H}|_Y$ consists of at most $d - m$ hyperplanes, shows that $\mathcal{L}|_Y$ is a linearly independent set. Note that from $\dim(\mathcal{L}) \leq \#\mathcal{L}|_Y + n - d$, we also have $\#\mathcal{L}|_Y \geq d - m$. Hence, we in fact have $\#\mathcal{H}|_Y = d - m$. It remains to show that $\dim Y = d$, or equivalently, that $\dim U_{d-m} = n - d$. Repeatedly applying the dimension formula $\dim(U + V) = \dim U + \dim V - \dim U \cap V$, we get that

(2.4)
$$\dim \sum_{i=1}^{d-m} (\mathcal{L}_i) = \dim(\mathcal{L}) = n - m = \sum_{i=1}^{d-m} \dim(\mathcal{L}_i) - \sum_{j=1}^{d-m-1} \dim\Big((\mathcal{L}_{j+1}) \cap \sum_{i=1}^{j} (\mathcal{L}_i) \Big)$$

and

$$(2.5) \quad \dim U_{d-m} = \dim \sum_{i=1}^{d-m} ((\mathcal{L}_i) \cap U_{d-m}) = \sum_{i=1}^{d-m} \dim(\mathcal{L}_i) \cap U_{d-m} - \sum_{j=1}^{d-m-1} \dim\big((\mathcal{L}_{j+1}) \cap U_{d-m} \cap \sum_{i=1}^{j} U_{d-m} \cap (\mathcal{L}_i) \big).$$

We claim that

$$(2.6) \qquad (\mathcal{L}_{j+1}) \cap U_{d-m} \cap \sum_{i=1}^{j} U_{d-m} \cap (\mathcal{L}_i) = (\mathcal{L}_{j+1}) \cap \sum_{i=1}^{j} (\mathcal{L}_i).$$

In fact, let $u \in (\mathcal{L}_{j+1}) \cap \sum_{i=1}^{j} (\mathcal{L}_i)$. Then $u = \sum_{i=1}^{j} u_i$ where $u \in (\mathcal{L}_{j+1})$ and $u_i \in (\mathcal{L}_i)$. By the definition of U_0, we have $u \in U_0 \subset U_{d-m}$. Also $u_i = u - \sum_{k=1, k \neq i}^{j} u_k \in (\mathcal{L} \setminus \mathcal{L}_i)$, so all $u_i \in U_0 \subset U_{d-m}$. Hence

$$(\mathcal{L}_{j+1}) \cap \sum_{i=1}^{j} (\mathcal{L}_i) \subset (\mathcal{L}_{j+1}) \cap U_{d-m} \cap \sum_{i=1}^{j} U_{d-m} \cap (\mathcal{L}_i).$$

The other inclusion is obvious and hence (2.6) holds. Using $\dim U_{d-m} \cap (\mathcal{L}_i) = \dim(\mathcal{L}_i) - 1$, (2.6) and (2.4), the equation in (2.5) gives

$$\dim U_{d-m} = \sum_{i=1}^{d-m} \dim(\mathcal{L}_i) - \sum_{j=1}^{d-m-1} \dim((\mathcal{L}_{j+1}) \cap \sum_{i=1}^{j}(\mathcal{L}_i)) - (d-m)$$
$$= n - m - (d-m) = n - d.$$

This proves that $\dim Y = d$. Hence the claim is proved.

We now continue the proof of the Main Theorem. Let Y be the subspace in the claim. Then $\dim Y = d$, $\#\mathcal{H}|_Y = d - m$, and the hyperplanes in $\mathcal{H}|_Y$ are linearly independent. So, without loss of generality, we assume that $Y = \mathbb{P}^d$ and that $\mathcal{H}|_Y$ are the first $d - m$ coordinate hyperplanes $\{x_j = 0\}$ where $0 \le j \le d - m - 1$. Then,

$$f(z) = (1, e^{1/z}, e^{1/z^2}, \dots, e^{1/z^{d-m-1}}, 0, \dots, 0)$$

is a holomorphic map from $\triangle^*$ to $\mathbb{P}^d \subset \mathbb{P}^n$ omitting the hyperplanes in $\mathcal{H}$ which clearly has degree of irrationality $\ge d - m$. This proves the "$\Rightarrow$" direction. The proof of the Main Theorem is thus finished. ∎

Acknowledgement Yasheng Ye wishes to thank the Department of Mathematics at the University of Houston for its kind hospitality during which part of the work on this paper took place.

References

[1] J. Dufresnoy, *Théorie nouvelle des familles complexes normales.Applications à l'étude des fonctions algébroïdes.* Ann. Sci. École Norm. Sup. **61**(1944), 1–44.
[2] H. Fujimoto, *Extensions of the big Picard's theorem.* Tôhoku Math. J. **24**(1972), 415–422.
[3] M. L. Green, *Some Picard theorems for holomorphic maps to algebraic varieties.* Amer. J. Math. **97**(1975), 43–75.
[4] P. Kiernan, *Extensions of holomorphic maps.* Trans. Amer. Math. Soc. **172**(1972), 347–355.
[5] K. Kobayashi, *Hyperbolic manifolds and holomorphic mappings.* In: Pure and Applied Mathematics 2, Marcel Dekker, Inc., New York, 1970.
[6] M. H. Kwack, *Generalization of the big Picard theorem.* Ann. of Math. (2) **90**(1969), 9–22,
[7] A. Levin, *The dimension of integral points and holomorphic curves on the complements of hyperplanes.* math.NT/0601691, http://arxiv.org/pdf/math/0601691v1.
[8] M. Ru, *Geometric and arithmetic aspects of* $\mathbb{P}^n$ *minus hyperplanes.* Amer. J. Math. **117**(1995), no. 2, 307–321.

Department of Mathematics, University of Shanghai for Science and Technology, Shanghai 200093, P.R. China
e-mail: yashengye@yahoo.com.cn

Department of Mathematics, University of Houston, Houston, TX 77204
e-mail: minru@math.uh.edu

https://doi.org/10.4153/CMB-2009-018-6 Published online by Cambridge University Press