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K-THEORETIC CLASSIFICATION FOR INDUCTIVE LIMIT Z_2 ACTIONS ON AF ALGEBRAS

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ABSTRACT. In this paper a K-theoretic classification is given of the C^*-dynamical systems $\lim_{\rightarrow}(A_n, \alpha_n, Z_2)$ where A_n is finite-dimensional. Corresponding to the trivial action is the K-theoretic classification for AF algebras obtained in [3] (also see [1]).

1. **Introduction.** Let G be a compact group, let $A = \lim_{\rightarrow} A_n$ be the inductive limit of a sequence of finite-dimensional C^*-algebras, and let $\alpha = \lim_{\rightarrow} \alpha_n$ be an inductive limit action of G on A, with α_n acting on A_n. Then one can form the C^*-algebra crossed product $A \times_\alpha G = \lim_{\rightarrow} A_n \times_{\alpha_n} G$. In [5], it was shown that if each α_n arises from a representation of G in the unitary group $U(A_n)$ of A_n, then the natural K-theory data of $A \times_\alpha G$ is a complete invariant for the C^*-dynamical system (A, α, G). In the case that A is unital, this data consists of the K-group $K_0(A \times_\alpha G)$, together with (i) the natural order structure, (ii) the special element coming from the projection obtained by averaging the canonical unitaries of the crossed product, and (iii) the natural module structure over representation ring $K_0(G)$ (also see [4], [2] and [6]).

In this note, we shall show that one has a similar K-theoretic classification result if one removes the restriction on α_n, but restricts attention to the group $Z/2Z$. Let us denote this group by Z_2. Since α_n may not be inner, the information contained in $K_0(A)$ may not be recovered completely from $K_0(A \times_\alpha Z_2)$. One needs to adjoin this, as well as the actions on the K-groups, to the invariant.

More precisely, let A be a unital C^*-algebra and let α be a Z_2 action on A. There is a canonical embedding of A into $A \times_\alpha Z_2$. There is also a dual action of $\hat{Z}_2$ on $A \times_\alpha Z_2$. The K-theory data we need here is the following:

(i) $(K_0(A), K_0(A)^+, [1_A], \alpha_*)$,

(ii) $(K_0(A \times_\alpha Z_2), K_0(A \times_\alpha Z_2)^+$, the special element, $\hat{\alpha}_*)$, and

(iii) the map $K_0(A) \longrightarrow K_0(A \times_\alpha Z_2)$.

2. **Order two actions on finite-dimensional C^*-algebras.** Let A be a finite-dimensional C^*-algebra and let α be an order two automorphism of A. Write $A = \oplus_{k=1}^m M_{n_k}$. Then $\alpha(M_{n_k})$ is another summand of A. So α can be decomposed into a direct sum of irreducible actions. Each such action β has the form (M_n, β) or $(M_n \oplus M_n, \beta)$. Let (A, α) be irreducible. There are two possible cases. We gather some simple facts.

CASE 1. $A = M_n$.

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In this case, α is determined by an order two unitary $V \in M_n$:

$$\alpha(a) = VaV^*, \quad a \in A.$$

V can be chosen to be diagonal. It is well known that $A \times_\alpha Z_2$ is isomorphic to $M_n \oplus M_n$, and we will use the identification:

$$a + bU_\alpha \longrightarrow (a + bV, a - bV^*), \quad a, b \in A,$$

where U_α is the canonical unitary in the crossed product.

If one identifies $K_0(A)$ with Z and identifies $K_0(A \times_\alpha Z_2)$ with Z^2, the map from $K_0(A)$ to $K_0(A \times_\alpha Z_2)$ sends x to (x, x). The special element in $K_0(A \times_\alpha Z_2)$ is (x, y) in Z^2 where x is the number of the positive eigenvalues of V and y is the number of the negative eigenvalues of V. α_* is of course trivial. In general, $\hat{\alpha}(a + bU_\alpha) = a - bU_\alpha$. Hence, $\hat{\alpha_*}$ permutes the two copies of Z in $K_0(A \times_\alpha Z_2)$.

CASE 2. $A = M_n \oplus M_n$.

There must exist a unitary $U \in M_n$ such that

$$\alpha(a, b) = (UbU^*, U^*aU), \quad (a, b) \in A$$

This action is equivalent to permuting the two components. In another word, we can take $U = 1$. The C^*-algebra crossed product is M_{2n} and the identification is

$$(a_1, a_2) + (b_1, b_2)U_\alpha \longrightarrow \begin{pmatrix} a_1 & b_1 \\ Ub_2U^* & Ua_2U^* \end{pmatrix}$$

Hence, $\hat{\alpha}$ can be identified with $\mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If one identifies $K_0(A)$ with Z^2 and identifies $K_0(A \times_\alpha Z_2)$ with Z, the map from $K_0(A)$ to $K_0(A \times_\alpha Z_2)$ will send (x, y) to $x + y$. The automorphism α_* is the permutation and $\hat{\alpha_*}$ is trivial. The special element is n.

3. **Existence.**

THEOREM 3.1. *Let (A, α, Z_2) and (B, β, Z_2) be two irreducible C^*-dynamical systems. Let F be an order preserving group homomorphism from $\big(K_0(A), [1_A], \alpha_*\big)$ to $\big(K_0(B), [1_B], \beta_*\big)$. Let ϕ be an order preserving group homomorphism from $\big(K_0(A \times_\alpha Z_2), \hat{\alpha_*}\big)$ to $\big(K_0(B \times_\beta Z_2), \hat{\beta_*}\big)$ preserving the special elements. Then there exists a morphism ψ from (A, α, Z_2) to (B, β, Z_2) such that $\psi_* = F$ and such that the extension of ψ to $A \times_\alpha Z_2$ induces ϕ.*

PROOF. There are four cases to be considered.

CASE 1. $A = M_k$ and $B = M_n$.

Let $U \in M_k$ and $V \in M_n$ be two unitaries determining α and β, respectively. Denote by $(\bar{x}, \bar{y})$ and (x, y) the number of positive eigenvalues and negative eigenvalues of U and V, respectively.

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Clearly, $F = n/k$. Since ϕ intertwines $\hat{\alpha_*}$ and $\hat{\beta_*}$, ϕ can be represented by $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. By the assumption on the special elements, one has

$$(a\bar{x} + b\bar{y}, b\bar{x} + a\bar{y}) = (x, y)$$

This gives

$$n = x + y = (a+b)(\bar{x} + \bar{y}) = (a+b)k.$$

Set $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_b = -I_k \in A$ and set $\epsilon_{b+1} = \cdots = \epsilon_{a+b} = I_k \in A$. Then $\epsilon = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_{a+b})$ is in B. Furthermore, ϵ commutes with $\mathrm{diag}(z, \ldots, z)$, $a+b$ copies of $z \in A$. Since $\mathrm{diag}(U, \ldots, U)$ has $(a+b)\bar{x}$ positive eigenvalues, $\mathrm{diag}(U, \ldots, U)\epsilon$ has

$$(a+b)\bar{x} - b\bar{x} + b\bar{y} = x$$

positive eigenvalues. Similarly, it has y negative eigenvalues.

Since V and $\mathrm{diag}(U, \ldots, U)\epsilon$ have the same eigenvalues, there exists a unitary $W \in M_n$ such that $W^*VW = \mathrm{diag}(U, \ldots, U)\epsilon$, and for $a \in A$,

$$\begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix}^* W^*VW \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} = \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix}^* W^*VW.$$

This is to say that the map ψ from A to B defined by:

$$\psi(a) = W \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} W^*, \quad a \in A,$$

intertwines α and β.

As we pointed out, $\psi_* = F$. Let $\tilde{\psi}$ be the map from $A \times_\alpha Z_2$ to $B \times_\beta Z_2$ induced by ψ. A direct computation shows that $\tilde{\psi}_* = \phi$.

CASE 2. $A = M_k \oplus M_k$ and $B = M_n$.

Let $U \in M_k$ and $V \in M_n$ be unitaries determining α and β, respectively. Identify $K_0(B)$ and $K_0(A \times_\alpha Z_2)$ with Z and identify $K_0(A)$ and $K_0(B \times_\beta Z_2)$ with Z^2. F can be identified with (a, b) and ψ can be identified with $\begin{pmatrix} c \\ d \end{pmatrix}$. Since F intertwines α_* and β_*, $a = b$. Since ϕ intertwines $\hat{\alpha}_*$ and $\hat{\beta}_*$, $c = d$. Let (x, y) denote the special element of $K_0(B \times_\beta Z_2)$ from Section 2 it follows that k is the special element of $K_0(A \times_\alpha Z_2)$. Hence,

$$\begin{pmatrix} c \\ d \end{pmatrix} k = \begin{pmatrix} x \\ y \end{pmatrix}.$$

This gives us $n = 2a = 2ck$, and $x = y$.

Consider the unitary N in M_n defined as follows:

$$N = \begin{bmatrix} & \begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix} \\ \begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix}^* & \end{bmatrix}$$

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Clearly, N has $\frac{n}{2}$ positive eigenvalues and $\frac{n}{2}$ negative eigenvalues. There exists a unitary $W \in M_n$ such that $W^* VW = N$.

For $(a_1, a_2) \in A$, set

$$\psi(a_1, a_2) = W \begin{bmatrix} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_1 \end{bmatrix} & \\ & \begin{bmatrix} a_2 & & \\ & \ddots & \\ & & a_2 \end{bmatrix} \end{bmatrix} W^*.$$

Then ψ is a morphism from (A, α, Z_2) to (B, β, Z_2).

It is clear that $\psi_* = F$. Let $\tilde{\psi}$ be the map from $A \times_\alpha Z_2$ to $B \times_\beta Z_2$ induced by ψ. $\tilde{\psi}_*$ maps Z^2 to Z and $\tilde{\psi}_*$ intertwines $\hat{\alpha}_*$ and $\hat{\beta}_*$. Hence, $\tilde{\psi}_*$ is necessarily ϕ.

CASE 3. $A = M_k$ and $B = M_n \oplus M_n$.

Let $U \in M_k$ and $V \in M_n$ determine α and β, respectively. Identify $K_0(A)$ and $K_0(B \times_\beta Z_2)$ with Z and identify $K_0(A \times_\alpha Z_2)$ and $K_0(B)$ with Z^2. One can write $F = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\phi = (c, d)$. Since these maps intertwine corresponding actions, $a = b$ and $c = d$. Again, denote the special elements of $K_0(A \times_\alpha Z_2)$ and $K_0(B \times_\beta Z_2)$ by $(\bar{x}, \bar{y})$ and n. Then one has

$$ck = c\bar{x} + c\bar{y} = n.$$

Define a map ψ from A to B as follows: for $\xi \in A$

$$\psi(\xi) = \big(\psi_1(\xi), \psi_2(\xi)\big)$$

where

$$\psi_1(\xi) = (\begin{bmatrix} \xi & & \\ & \ddots & \\ & & \xi \end{bmatrix}$$

$$\psi_2(\xi) = \mathrm{Ad}\, V^* \left(\begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix} \right) \psi_1(\xi)$$

Then it is readily checked that $\psi \circ \alpha = \beta \circ \psi$. It is also clear that $\psi_* = F$. With $\tilde{\psi}$ the map from $A \times_\alpha Z_2$ to $B \times_\beta Z_2$ induced by ξ, $\tilde{\psi}_*$ maps Z^2 to Z. Since $\tilde{\psi}_*$ intertwines $\hat{\alpha}_*$ and $\tilde{\beta}_*$, $\tilde{\psi}_* = \phi$.

CASE 4. $A = M_k \oplus M_k$ and $B = M_n \oplus M_n$.

Let $U \in M_k$ and $V \in M_n$ determine α and β, respectively. One can take U and V to be I_k and I_n, respectively. F can be written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since $F \circ \alpha_* = \beta_* \circ F$, we have $a = d$ and $b = c$.

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Define a map ψ from A to B by

$$\psi(a_1,a_2) = \big(\psi_1(a_1,a_2),\psi_2(a_1,a_2)\big)$$

where

$$\psi_1(a_1,a_2) = \begin{bmatrix} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_1 \end{bmatrix}_{d\times d} & \\ & \begin{bmatrix} a_2 & & \\ & \ddots & \\ & & a_2 \end{bmatrix}_{c\times c} \end{bmatrix}$$

$$\psi_2(a_1,a_2) = \begin{bmatrix} \begin{bmatrix} a_2 & & \\ & \ddots & \\ & & a_2 \end{bmatrix}_{d\times d} & \\ & \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_1 \end{bmatrix}_{c\times c} \end{bmatrix}$$

Then ψ is a morphism from (a,α) to (B,β) and ψ induces F and ϕ.

THEOREM 3.2. *Let (A,α,Z_2) and (B,β,Z_2) be two C^*-dynamical systems. Let F be an order preserving group homomorphism from $(K_0(A),[1_A],\alpha_*)$ to $(K_0(B),[1_B],\beta_*)$. Let ϕ be an order preserving group homomorphism from $(K_0(A\times_\alpha Z_2),\tilde{\alpha}_*)$ to $\big(K_0(B\times_\beta Z_2),\tilde{\beta}_*\big)$, mapping the special element to the special element. Assume further that the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \longrightarrow & K_0(A\times_\alpha Z_2) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ K_0(B) & \longrightarrow & K_0(B\times_\beta Z_2) \end{array}.$$

Then there exists a morphism ψ from (A,α,Z_2) to (B,β,Z_2) such that $\psi_* = F$ and such that the map induced by ψ from $A\times_\alpha Z_2$ to $B\times_\beta Z_2$ gives rise to ϕ.

PROOF. Clearly, we may assume that (B,β,Z_2) is irreducible and the unitary, say λ, implementing β is diagonal. Write $(A,\alpha,Z_2) = (A_1,\alpha_1,Z_2)\oplus(A_2,\alpha_2,Z_2)$. We will reduce the theorem to the case that $(A,\alpha,Z_2) = (A_i,\alpha_i,Z_2)$.

CASE 1. $B = M_n$.

Denote by (x,y) the special element of $K_0(B\times_\beta Z_2)$. Then

$$\phi = \Big(\Big[\frac{1+U_{\alpha_1}}{2}\Big],\Big[\frac{1+U_{\alpha_2}}{2}\Big]\Big) = (x,y).$$

Define (x_i,y_i), $i=1,2$, as follows:

$$(x_1,y_1) = \phi\Big(\Big[\frac{1+U_{\alpha_1}}{2}\Big],0\Big),$$

$$(x_2,y_2) = \phi\Big(0,\Big[\frac{1+U_{\alpha_2}}{2}\Big]\Big).$$

Let $P \in M_n$ be a diagonal projection such that $P\lambda P$ has x_1 positive eigenvalues and y_1 negative eigenvalues. Clearly, P commutes with λ. Hence, $\big(PBP, \mathrm{Ad}(P\lambda P), Z_2\big)$ is a dynamical system. Similarly, if one sets $Q = 1 - P$, then $\big(QBQ, \mathrm{Ad}(Q\lambda Q), Z_2\big)$ is also a dynamical system. Furthermore, the embedding of PBP into B, say i_P, and the embedding of QBQ into B, say i_Q, are equivariant. We shall denote the two dynamical systems by (PBP, β_1) and (QBQ, β_2), respectively.

It is clear that $K_0(PBP \times_{\beta_1} Z_2) = Z^2$. Furthermore, if one denotes the extension of i_P to $PBP \times_{\beta_1} Z_2$ by $\tilde{i}_P$, then for $a + U_{\beta_1} \in PBP \times_{\beta_1} Z_2$,

$$\tilde{i}_{P*}(a + bU_{\beta_1}) = i_P(a) + i_P(b)U_\beta.$$

So

$$\tilde{i}_{P*}([e]) = [e], \quad [e] \in K_0(PBP \times_{\beta_1} Z_2).$$

In other words, $\tilde{i}_{P*}$ is the identity map on Z^2. Similarly, one can define $\tilde{i}_Q$, and $\tilde{i}_{Q*}$ and they are the identity maps on Z^2. Hence, we have the following diagram:

$$\begin{array}{ccccc} K_0(PBP) & \cong & K_0(B) & \longleftarrow & K_0(A_1 \oplus 0) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(PBP \times_{\beta_1} Z_2) & \cong & K_0(B \times_\beta Z_2) & \longleftarrow & K_0(A_1 \oplus 0 \times_{\alpha_1 \oplus \alpha_2} Z_2) \end{array}.$$

Furthermore, the two rows in the diagram respect the corresponding actions, respectively. By this diagram, we can define two maps from $K_0(A_1 \oplus 0)$ to $K_0(PBP)$ and from $K_0(A_1 \oplus 0 \times_{\alpha_1 \oplus \alpha_2} Z_2)$ to $K_0(PBP \times_{\beta_1} Z_2)$, respectively. (PBP, β) was constructed to satisfy the conditions of the Theorem 3.2 for (A_1, α_1) and (PBP, β_1). So by applying the Theorem 3.1 and the induction, we obtain a unital $*$-homomorphism ψ_1 from (A_1, α_1) to (PBP, β_1) to realize the K-theory data associated with these two C^*-dynamical systems.

Similarly, there is a unital $*$-homomorphism ψ_2 from (A_2, α_2) to (QBQ, β_2) to realize the K-theory data associated with these two C^*-dynamical systems. Now $\psi = i_p \circ \psi_1 + i_Q \circ \psi_2$ is a desired map.

CASE 2. $B = M_n \oplus M_n$.

Using the same argument as above, one shows that the theorem can be reduced to the case that (A, α) is irreducible.

This completes the proof of the theorem.

4. **Uniqueness.**

THEOREM 4.1. *Let ϕ and ψ be two morphisms from (A, α, Z_2) to (B, β, Z_2). Suppose that both (A, α) and (B, β) are irreducible. Denote by $\tilde{\phi}$ and $\tilde{\psi}$ the maps from $A \times_\alpha Z_2$ to $B \times_\beta Z_2$ induced by ϕ and ψ, respectively. If $\phi_* = \psi_*$ and if $\tilde{\phi}_* = \tilde{\psi}_*$, then there exists a unitary $U \in B^\beta$, the fixed point subalgebra of B, such that $\phi = \mathrm{Ad}\, U \circ \psi$.*

PROOF. Again there are four cases to be considered.

CASE 1. $A = M_k$ and $B = M_n$.

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Let $U \in M_k$ and $V \in M_n$ be two unitaries that implement α and β, respectively. Let X and Y be two unitaries in B such that

$$\phi(a) = X \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} X^*, \quad a \in A,$$

$$\psi(a) = Y \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} Y^*, \quad a \in A.$$

Since both maps are equivariant, one has two order two unitaries

$$L = X^* V^* X \begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix},$$

$$N = Y^* V^* Y \begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix},$$

in the commutant of the following algebra:

$$\left\{ \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \in B \mid a \in A \right\} \cong 1_k \otimes M_{\frac{n}{k}}.$$

Let S and R be two unitaries in $1_k \otimes M_{\frac{n}{k}}$ such that

$$SLS^* = 1_k \otimes \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{\frac{n}{k}} \end{bmatrix},$$

$$RNR^* = 1_k \otimes \begin{bmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_{\frac{n}{k}} \end{bmatrix}.$$

Clearly, $\{\lambda_i\}_{i=1}^{\frac{n}{k}}$ and $\{\xi_i\}_{i=1}^{\frac{n}{k}}$ are ones or negative ones.

Identify $A \times_\alpha Z_2$ with $M_k \oplus M_k$ and identify $B \times_\beta Z_2$ with $M_k \oplus M_k$. Then for $a, b \in A$

$$\tilde{\phi}(a + bU, a - bU) = \big(\phi(a) + \phi(b)V, \phi(a) - \phi(b)V\big).$$

A direct computation shows that

$$\tilde{\phi}(I, 0) = XS^* \left(I + \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{\frac{n}{k}} \end{bmatrix}, I - \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{\frac{n}{k}} \end{bmatrix} \right) SX^*.$$

Write $\tilde{\phi_*} = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$. Then c and d are the ranks of

$$\frac{1}{2}\left(I + \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{\frac{n}{k}} \end{bmatrix} \right)$$

and

$$\frac{1}{2}\left(I-\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_{\frac{n}{k}}\end{bmatrix}\right)$$

respectively. Since $\tilde{\phi}_* = \tilde{\psi}_*$, this says that $\{\lambda_i\}_{i=1}^{\frac{n}{k}} = \{\xi_i\}_{i=1}^{\frac{n}{k}}$. So there exists $Z \in I_k \otimes M_{\frac{n}{k}}$ such that $ZLZ^* = N$. Then

$$V(XZ^*Y^*) = (XZ^*Y^*)V.$$

If one replaces Y by YZ, which will not change ψ, one has

$$\phi = \mathrm{Ad}(XY^*)\psi.$$

CASE 2. $A = M_k \oplus M_k$ and $B = M_n$.

Let $U \in M_k$ and $V \in M_n$ be two unitaries such that

$$\alpha(a_1,a_2) = (Ua_2U^*, U^*a_1U), \quad (a_1,a_2) \in A,$$
$$\beta(b) = VbV^*, \quad b \in B.$$

We may take $U = I_k$.

Let X and Y be two unitaries in B such that

$$\phi(x,y) = X\begin{bmatrix} x & & & & & \\ & \ddots & & & & \\ & & x & & & \\ & & & y & & \\ & & & & \ddots & \\ & & & & & y\end{bmatrix}X^*, \quad (x,y) \in A,$$

$$\psi(x,y) = Y\begin{bmatrix} x & & & & & \\ & \ddots & & & & \\ & & x & & & \\ & & & y & & \\ & & & & \ddots & \\ & & & & & y\end{bmatrix}Y^*, \quad (x,y) \in A.$$

From Section 3, we see that n is necessarily an even number and the numbers of copies for x and y should be the same, for ϕ and ψ.

Again, using the same computation as in Case 1, one can show that $L = X^*VX\begin{bmatrix} & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & \end{bmatrix}$ and $N = Y^*VY\begin{bmatrix} & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & \end{bmatrix}$ belong to the following commutant:

$$\left\{\begin{bmatrix} y & & & & & \\ & \ddots & & & & \\ & & y & & & \\ & & & x & & \\ & & & & \ddots & \\ & & & & & x\end{bmatrix} \,\middle|\, (x,y) \in A\right\} \cong I_k \otimes M_{\frac{n}{2k}} \oplus I_k \otimes M_{\frac{n}{2k}}.$$

Now L and N can be written as

$$L = \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & \\ & N_2 \end{bmatrix}.$$

This leads to the following equations:

$$X^* V X = \begin{bmatrix} & L_2 \\ L_1 & \end{bmatrix}$$
$$Y^* V Y = \begin{bmatrix} & N_2 \\ N_1 & \end{bmatrix}$$

Since V is of order two, $L_1 L_2 = I$ and $N_1 N_2 = I$. We have

$$\begin{bmatrix} N_1^* L_I & \\ & I \end{bmatrix} \begin{bmatrix} L_1^* N_I & \\ & I \end{bmatrix} = \begin{bmatrix} & N_1^* \\ N_1 & \end{bmatrix}.$$

So there exists $Z \in I_k \otimes M_{\frac{n}{2k}} \oplus I_k \otimes M_{\frac{n}{2k}}$ such that

$$Z X^* V X Z^* = Y^* V Y.$$

Finally, one obtains $\phi = \mathrm{Ad}(YZX^*) \circ \psi$ by replacing Y by YZ. Notice that YZX^* commutes with V.

CASE 3. $A = M_k$ and $B = M_n \oplus M_n$.

Let U and V be two unitaries determining α and β, respectively. We may assume that $V = I_n$. Let X_1 and X_2 be two unitaries in M_n such that

$$\phi(x) = \left(X_1 \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} X_1^*, X_2 \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} X_2^* \right), \quad x \in A,$$

and let Y_1 and Y_2 be two unitaries in M_n such that

$$\psi(x) = \left(Y_1 \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} Y_1{}^*, Y_2 \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} Y_2^* \right), \quad x \in A.$$

Since both maps intertwine α and β, one has that

$$L = Y_2{}^* Y_1 \begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix}$$
$$N = X_2{}^* X_1 \begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix}$$

belong to the commutant:

$$\left\{ \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} \in M_n \mid x \in A \right\}.$$

One computes that:

$$NL^* = X_2{}^* X_1 Y_1{}^* Y_2,$$

i.e.,

$$X_1 Y_1{}^* = X_2 N L^* Y_2{}^*.$$

If one replaces Y_2 by $Y_2 L N^*$, ψ will not be changed. On the other hand, one has

$$X_1 Y_1{}^* = X_2 Y_2{}^*.$$

This is exactly the condition for $(X_1 Y_1^*, X_2 Y_2^*)$ to commute with β. Clearly,

$$\phi = \mathrm{Ad}(X_1 Y_1{}^*, X_2 Y_2{}^*) \circ \psi.$$

CASE 4. $A = M_k \oplus M_k$ and $B = M_n \oplus M_n$.

Let U and V be two unitaries determining α and β, respectively. Since ϕ intertwines α and β, ϕ must have the following form: for $(x,y) \in A$,

$$\phi(x,y) = \big(\phi_1(x,y), \phi_2(x,y)\big)$$

where

$$\phi_1(x,y) = X \begin{bmatrix} \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix}_{c\times c} & \\ & \begin{bmatrix} y & & \\ & \ddots & \\ & & y \end{bmatrix}_{d\times d} \end{bmatrix} X^*$$

$$\phi_2(x,y) = X \begin{bmatrix} \begin{bmatrix} y & & \\ & \ddots & \\ & & y \end{bmatrix}_{c\times c} & \\ & \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix}_{d\times d} \end{bmatrix} X^*$$

Here X is a unitary in M_n and $\phi_* = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ if one identifies both $K_0(A)$ and $K_0(B)$ with Z^2. Similarly, there exists a unitary Y in M_n such that for $(x,y) \in A$,

$$\psi(x,y) = \big(\psi_1(x,y), \psi_2(x,y)\big)$$

where

$$\begin{bmatrix} \psi_1(x,y) = Y \begin{bmatrix} x & & \\ & \ddots & \\ & & xx \\ & \ddots & \\ & & x \end{bmatrix}_{c\times c} & \\ & \begin{bmatrix} y & & \\ & \ddots & \\ & & y \end{bmatrix}_{d\times d} \end{bmatrix} Y^*$$

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$$\psi_2(x,y) = Y \begin{bmatrix} \begin{bmatrix} y & & \\ & \ddots & \\ & & y \end{bmatrix}_{c\times c} & \\ & \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix}_{d\times d} \end{bmatrix} Y^*$$

Clearly,

$$\psi = \mathrm{Ad}(YX^*, YX^*) \circ \phi$$

and (YX^*, YX^*) is a fixed point of β.

This completes the proof of the theorem.

THEOREM 4.2. *Let ϕ and ψ be two morphism from (A, α, Z_2) to (B, β, Z_2) where each of which is a finite direct sum of irreducible C^*-dynamical systems. Denote by $\tilde{\phi}$ and $\tilde{\psi}$ the maps from $A \times_\alpha Z_2$ to $B \times_\beta Z_2$ induced by ϕ and ψ, respectively.*

If $\phi_* = \psi_*$ and if $\tilde{\phi}_* = \tilde{\psi}_*$, then there exists a unitary $U \in B^\beta$, the fixed point subalgebra of B, such that $\phi = \mathrm{Ad}\, U \circ \psi$.

PROOF. Clearly, we may assume that (B, β) is irreducible and the unitary V that implements β is diagonal. We will reduce the problem to the case that (A, α) is irreducible.

Let $P \in A^\alpha$ be a projection. Than $\phi(P) \in B^\beta$ and $\psi(P) \in B^\beta$. In our case, the crossed products and the fixed point subalgebras are stable isomorphic. So $\phi(P)$ and $\psi(P)$ have the same K-theory in $K_0(B^\beta)$. There is a unitary $X \in B^\beta$ so that $\mathrm{Ad}(X)\phi(P) = \psi(P)$.

Let $(A, \alpha) = \oplus(A_i, \alpha_i)$ where each (A_i, α_i) is irreducible. Let P_i be the identity of A_i. Then P_i is in the fixed point algebra. The same argument as above produces a unitary $Y \in B^\beta$ such that $\mathrm{Ad}(Y)\phi(P_i) = \psi(P_i)$ for all i. By considering $\mathrm{Ad}(Y)\phi$ and ψ from (A_i, α_i) to (P_iBP_i, β) we reduce the theorem to 4.1.

This completes the proof of the theorem.

5. **Classification.**

THEOREM. *Let $(A, \alpha, Z_2) = \varinjlim(A_n, \alpha_n, Z_2)$ and $(B, \beta, Z_2) = \varinjlim(B_n, \beta_n, Z_2)$ be two inductive limit C^*-dynamical systems, let F be an order preserving group isomorphism from $\big(K_0(A), \alpha_*\big)$ to $\big(K_0(B), \beta_*\big)$ mapping $[1_A]$ to $[1_B]$ and let ϕ be an order preserving group isomorphism from $\big(K_0(A \times_\alpha Z_2), \hat{\alpha}_*\big)$ to $\big(K_0(B \times_\beta Z_2), \hat{\beta}_*\big)$ mapping the special element to the special element. Suppose that the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \longrightarrow & K_0(A \times_\alpha Z_2) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ K_0(B) & \longrightarrow & K_0(B \times_\beta Z_2). \end{array}$$

Then there is an isomorphism ψ from (A, α) to (B, β) such that $\psi_* = F$ and such that the extension of a map from $A \times_\alpha Z_2$ to $B \times_\beta Z_2$ gives rise to the map ϕ.

PROOF. By passing to subsequences and changing notation, we may assume that we have the following intertwings:

$$\begin{tikzcd}
(K_0(A_1),\alpha_{1*}) \arrow[r] \arrow[d] & (K_0(A_2),\alpha_{2*}) \arrow[r] \arrow[d] & \cdots \arrow[r] & (K_0(A),\alpha_*) \arrow[d, Leftrightarrow] \\
(K_0(B_1),\beta_{1*}) \arrow[r] \arrow[ur] & (K_0(B_2),\beta_{2*}) \arrow[r] \arrow[ur] & \cdots \arrow[r] & (K_0(B),\beta_*)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(K_0(A_1 \times_{\alpha_1} Z_2),\hat{\alpha_1}_*) \arrow[r] \arrow[d] & (K_0(A_2 \times_{\alpha_2} Z_2),\hat{\alpha_2}_*) \arrow[r] \arrow[d] & \cdots \arrow[r] & (K_0(A \times_\alpha Z_2),\hat{\alpha}_*) \arrow[d, Leftrightarrow] \\
(K_0(B_1 \times_{\beta_1} Z_2),\hat{\beta_1}_*) \arrow[r] \arrow[ur] & (K_0(B_2 \times_{\beta_2} Z_2),\hat{\beta_2}_*) \arrow[r] \arrow[ur] & \cdots \arrow[r] & (K_0(B \times_\beta Z_2),\hat{\beta}_*)
\end{tikzcd}.$$

We would like to make these two intertwings compatible. Notice that we have:

$$\begin{tikzcd}
K_0(A_1) \arrow[r] \arrow[d] & K_0(A) \arrow[r, phantom, "\cong"] \arrow[d] & K_0(B) \arrow[d] & \cdots \arrow[l] & K_0(B_1) \arrow[l] \arrow[d] \\
K_0(A \times_{\alpha_1} Z_2) \arrow[r] & K_0(A \times_\alpha Z_2) \arrow[r, phantom, "\cong"] & K_0(B \times_\beta Z_2) & \cdots \arrow[l] & K_0(B_1 \times_{\beta_1} Z_2) \arrow[l]
\end{tikzcd}.$$

So there exists n such that

$$\begin{tikzcd}
K_0(A_1) \arrow[r] \arrow[d] & K_0(B_n) \arrow[d] \\
K_0(A \times_{\alpha_1} Z_2) \arrow[r] & K_0(B_n \times_{\beta_n} Z_2)
\end{tikzcd}$$

commutes. This shows that we can actually have two intertwing sequences satisfying

$$\begin{tikzcd}
K_0(A_n) \arrow[r] \arrow[d] & K_0(A_n \times_{\alpha_n} Z_2) \arrow[d] \\
K_0(B_n) \arrow[r] \arrow[d] & K_0(B_n \times_{\beta_n} Z_2) \arrow[d] \\
K_0(B_n) \arrow[r] \arrow[d] & K_0(B_n \times_{\beta_n} Z_2 \arrow[d] \\
K_0(An+1) \arrow[r] & K_0(A_{n+1} \times_{\alpha_{n+1}} Z_2).
\end{tikzcd}$$

We would like to point out that we may require these intertwings to preserve the special elements and the identity projections.

By Theorem 3.2, one may lift each map to the dynamical system. By Theorem 4.2, one can correct each up and down map by an inner automorphism commuting with the actions so the following diagram is commutative:

$$\begin{tikzcd}
(A_1,\alpha) \arrow[r] \arrow[d] & (A_2,\alpha_2) \arrow[r] \arrow[d] & \cdots \arrow[r] & (A,\alpha) \\
(B_1,\beta_1) \arrow[r] \arrow[ur] & (B_2,\beta_2) \arrow[r] & \cdots \arrow[r] & (B,\beta).
\end{tikzcd}$$

(*cf. e.g.*, [2].) Hence (A,α) and (B,β) must be isomorphic, and by an isomorphism giving rise to the given maps ϕ and F.

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