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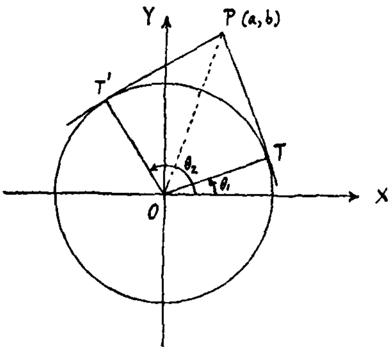
Experimental Determination of the Fundamental Solutions of the Equation $a\cos\theta + b\sin\theta = c$, with some general theorems on that Equation.

It is shown in this note how, by a simple geometrical construction, the two fundamental solutions of the equation $a\cos\theta + b\sin\theta = c$ (to which $\pm 2k\pi$ added gives, if required, the general solution) may be determined experimentally, using the protractor, without ambiguity of any kind. The method does not involve any laborious graph-tracing, and is at once simpler and more accurate than the geometrical discussions given in one or two text-books. (See, for example, Loney, Part I., page 141, § 130; Levett & Davison, page 138).

The reader may also be referred to the method of solution given by Dr J. Dougall on page 45, *Mathematical Notes*, No. 5.

§ 1. We suppose that c is positive, signs having been changed all along if, to begin with, the absolute term is negative. Draw the circle $x^2 + y^2 = c^2$, and plot the point $P(a, b)$. From P draw the tangents PT, PT' to the circle, and let $X\widehat{O}T = \theta_1$, $X\widehat{O}T' = \theta_2$.



Then θ_1 and θ_2, measured in the positive direction, are the fundamental solutions between 0 and 2π, and may readily be measured with the protractor to within half a degree, if the figure is at all carefully drawn. (The solutions are obviously imaginary when P lies within the circle, that is, when $a^2+b^2<c^2_1$).

Two general projection-proofs are offered : the second of these I had failed to notice, and I am indebted to my colleague, Dr R. J. T. Bell, for pointing it out to me. In respect of directness and brevity it is much to be preferred to the first, which, however, involving only projections of line-segments *on the coordinate axes*, may be found more suited to those students who, with little experience of projections, find difficulty in projecting segments of the axes on a given line.

(i) Projecting the sides of $\triangle\, OTP$ in order on OX and OY—

$$\left.\begin{aligned} OT\cos\theta_1+TP\cos(\theta_1+90^\circ)-a=0\\ OT\sin\theta_1+TP\sin(\theta_1+90^\circ)-b=0\end{aligned}\right\},$$

or,
$$\left.\begin{aligned} OT\cos\theta_1-TP\sin\theta_1-a=0\\ OT\sin\theta_1+TP\cos\theta_1-b=0\end{aligned}\right\},$$

whence, solving for OT,

$$OT=a\cos\theta_1+b\sin\theta_1=c.$$

Similarly, projecting on OX, OY, the sides of $\triangle\, OPT'$,

$$\left.\begin{aligned} OT'\cos\theta_2+T'P\cos(\theta_2-90^\circ)-a=0\\ OT'\sin\theta_2+T'P\sin(\theta_2-90^\circ)-b=0\end{aligned}\right\},$$

which, on simplifying and solving, give

$$OT'=a\cos\theta_2+b\sin\theta_2=c,$$

—the generality of the method of projections ensuring that these results are universally correct and not dependent on the particular figure drawn. Thus θ_1, θ_2 are two distinct (and therefore the fundamental) roots of the equation $a\cos\theta+b\sin\theta=c$.

(ii) In the second proof, let M be the projection of P on OX. Then $OM=a$, $MP=b$, and these segments make angles $-\theta_1$ and $-\theta_1+90^\circ$ respectively with OT. Also the sum of their projections on OT = projection of OP on OT,

i.e. $$a\cos(-\theta_1)+b\cos(-\theta_1+90^\circ)=c,$$

or, $$a\cos\theta_1+b\sin\theta_1=c,$$

and the same proof applies in the case of the angle θ_2.

§2. It may be of interest also to notice that this construction affords a method of removing the ambiguity which renders one of the methods of solution of the given equation practically worthless, besides giving a geometrical interpretation of that method.

The equation of TT', the chord of contact of the tangents from P, is

$$ax + by = c^2,$$

and hence the equation of the radii OT', OT'' is

$$\frac{x^2+y^2}{c^2} = \left(\frac{ax+by}{c^2}\right)^2,$$

or, $$(a^2-c^2)x^2 + 2ab\,xy + (b^2-c^2)y^2 = 0.$$

That is, $\tan\theta_1$ and $\tan\theta_2$ are the roots of the quadratic in m,

$$(b^2-c^2)m^2 + 2ab\,m + (a^2-c^2) = 0, \quad\ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

and can be calculated, the ambiguity as to the appropriate values of θ being then removed by consulting the figure.

This is exactly the solution, usually regarded as unsatisfactory, in which we write

$$a\cos\theta + b\sin\theta = c\sqrt{\cos^2\theta + \sin^2\theta},$$

or, on squaring and rearranging,

$$(b^2-c^2)\tan^2\theta + 2ab\tan\theta + (a^2-c^2) = 0. \quad\ldots\ldots\ldots\ldots (\beta)$$

It will be noted that equations (α) and (β) are identical.

§ 3. It will be almost needless to remark that even the expert calculator will find this circle-diagram of value as a rapid check on his results, a very rough sketch being quite sufficient for this purpose. Indeed, it need not even be *drawn*—a mental picture of the point (a, b) and the circle $x^2+y^2=c^2$ will in most cases be enough to give a rough idea of whereabouts the roots lie.

§ 4. Not the least surprising feature of the diagram is the almost ludicrous facility with which it affords proofs of the various general theorems on the roots of equations of the type $a\cos\theta + b\sin\theta = c$, and I have thought it worth while to append, in the form of a set of examples, a collection of these, some with solutions and some left to the reader's ingenuity. Most of them are taken from the examples in the various texts, and are familiar enough: in such cases I have indicated one only of the sources, while those without references are, so far as I know, new. Comparison of the present geometrical solutions with the usual analytical solutions—often of very considerable difficulty—will give convincing demonstration of the advantages of this method.

[It is understood in what follows that the coefficients in the equation have real finite values. This excludes the possibility of the two fundamental solutions differing by an integral multiple of π, for in that case the tangents would be parallel, and P at infinity.]

Ex. 1.—If the equation $a\cos\theta + b\sin\theta = c$ is satisfied by two values of θ that differ by a, then $c^2 = (a^2+b^2)\cos^2\dfrac{a}{2}$. (Levett & Davison, p. 157, 91.)

$$\textit{For} \qquad T\widehat{O}P = \frac{a}{2},$$

$$\text{and} \qquad \cos^2 T\widehat{O}P = \frac{OT^2}{OP^2} = \frac{c^2}{a^2+b^2}.$$

Ex. 2.—If a, β are distinct values of θ which satisfy the equation

$$a\cos\theta + b\sin\theta = c,$$

$$\text{then} \qquad \frac{\cos\frac{1}{2}(\beta - a)}{c} = \frac{\sin\frac{1}{2}(\beta + a)}{b} = \frac{\cos\frac{1}{2}(\beta + a)}{a}.$$

(*Cf.* solution given in Hobson, pp. 83, 84.)

$$\textit{For} \qquad \cos\tfrac{1}{2}(\beta - a) = \cos T\widehat{O}P = c/\sqrt{(a^2+b^2)},$$

$$\sin\tfrac{1}{2}(\beta + a) = \sin X\widehat{O}P = b/\sqrt{(a^2+b^2)},$$

$$\cos\tfrac{1}{2}(\beta + a) = \cos X\widehat{O}P = a/\sqrt{(a^2+b^2)}.$$

[Note that hence, or direct from the figure, $\tan\frac{1}{2}(a+\beta) = \dfrac{b}{a}$. That is, *the "subsidiary angle" (in that method of solution in which we express* $a\cos\theta + b\sin\theta$ *in the form* $R\cos\overline{\theta - \epsilon}$) *is the mean of the fundamental solutions.*]

Ex. 3.—If $a\cos\theta + b\sin\theta = c$, and $a\cos\phi + b\sin\phi = c$, prove that

$$\tan(\theta + \phi) = 2ab/(a^2 - b^2). \qquad \text{(John's, Camb., 1887.)}$$

Here $\theta + \phi = 2\,X\widehat{O}P = 2\tan^{-1}\dfrac{b}{a}$, whence the solution at once.

Ex. 4.—If θ_1 and θ_2 be two solutions of $a\cos\theta + b\sin\theta = c$, such that the cosines of θ_1 and θ_2 are not equal, then $\cos(\theta_1+\theta_2) = \dfrac{a^2 - b^2}{a^2+b^2}$

(Levett & Davison, p. 159, 103.)

$$\textit{For} \qquad \cos(\theta_1 + \theta_2) = 2\cos^2\frac{\theta_1+\theta_2}{2} - 1 = 2\cos^2 X\widehat{O}P - 1$$

$$= 2\,.\,\frac{a^2}{a^2+b^2} - 1.$$

Ex. 5.—In Ex. 4 prove also that

$$\text{(i)} \quad \cos(\theta_2 - \theta_1) = \frac{2c^2 - a^2 - b^2}{a^2+b^2};$$

$$\text{(ii)} \quad \sin(\theta_1 + \theta_2) = \frac{2ab}{a^2+b^2};$$

$$\text{(iii)} \quad \sin(\theta_2 - \theta_1) = \frac{2c\sqrt{(a^2+b^2-c^2)}}{a^2+b^2}.$$

Ex. 6.—And in Ex. 4 prove that

(i) $\cos\theta_1+\cos\theta_2=2ac/(a^2+b^2)$,

$\sin\theta_1+\sin\theta_2=2bc/(a^2+b^2)$.

(Todhunter & Hogg, p. 235, 46.)

(ii) $\cos\theta_1\cos\theta_2=(c^2-b^2)/(a^2+b^2)$,

$\sin\theta_1\sin\theta_2=(c^2-a^2)/(a^2+b^2)$.

(Davison, *Problem Papers*, p. 131, 7.)

These results follow at once on writing

$$\cos\theta_1+\cos\theta_2=2\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_2-\theta_1}{2}=2\cos\widehat{XOP}\,.\cos\widehat{TOP},$$

$$\cos\theta_1\cos\theta_2=\tfrac{1}{2}\{\cos\overline{\theta_1+\theta_2}+\cos\overline{\theta_2-\theta_1}\}=\cos^2\frac{\theta_1+\theta_2}{2}+\cos^2\frac{\theta_2-\theta_1}{2}-1,$$

and so on.

Ex. 7.—If $A\cos\theta+B\sin\theta-C$ vanish for $\theta=\alpha$, $\theta=\beta$, $\theta=\alpha+\beta$, respectively, then $A=C$. (Levett & Davison, p. 159, 101.)

For either α or β must be a multiple of 2π, so one of the tangents from P is perpendicular to OX, *i.e.* $A=x_P=C$. (Work out the case when $B=C$).

Ex. 8.—If the equations $a\cos\theta+b\sin\theta=c$, $l\cos\theta+m\sin\theta=n$, have a common solution α, then their other solutions β_1, β_2, respectively, are such that

$$\tan\beta_1/\tan\beta_2=(c^2-a^2)(n^2-m^2)/(n^2-l^2)(c^2-b^2).$$

Hint.—One of the tangents from (a, b) to $x^2+y^2=c^2$ is parallel to one of those from (l, m) to $x^2+y^2=n^2$, and on the same side of the centre. Use Ex. 6 (ii).

Ex. 9.—If $\sin\alpha$ and $\sin\beta$ be two values of $\sin\theta$ satisfying the equation $a\cos 2\theta+b\sin 2\theta=c$, then $\cos^2\alpha-\sin^2\beta=ac/(a^2+b^2)$.

(Levett & Davison, p. 153, 39.)

In our notation the theorem is $\cos^2\frac{\theta_1}{2}-\sin^2\frac{\theta_2}{2}=ac/(a^2+b^2)$.

And we have $\cos^2\frac{\theta_1}{2}-\sin^2\frac{\theta_2}{2}\equiv\frac{1}{2}(\cos\theta_1+\cos\theta_2)$. *Cf.* Ex. 6 (i).

Ex. 10.—If α and β be two angles both satisfying the equation

$$a\cos 2\theta+b\sin 2\theta=c,$$

then $$\cos^2\alpha+\cos^2\beta=(a^2+ac+b^2)/(a^2+b^2).$$

(Todhunter & Hogg, p. 236, 47.)

Ex. 11.—Prove that if one of the fundamental solutions of the equation

$$a\cos\theta+b\sin\theta=c,$$

which lie between 0° and 360°, is double the other, then

$$4c^3=(3c+a)(a^2+b^2).$$

For if $\theta_2 = 2\theta_1$, then $\frac{1}{2}(\theta_1+\theta_2) = \frac{3}{2}\theta_1$, $\frac{1}{2}(\theta_2-\theta_1) = \frac{1}{2}\theta_1$,
that is

$$X\widehat{O}P = 3 \,.\, T\widehat{O}P, \text{ and } \cos X\widehat{O}P = 4\cos^3 T\widehat{O}P - 3\cos T\widehat{O}P,$$

which, on substituting

$$\cos X\widehat{O}P = a/\sqrt{(a^2+b^2)}, \quad \cos T\widehat{O}P = c/\sqrt{(a^2+b^2)},$$

is the required condition. Note that that condition is simply the θ-eliminant of

$$a\cos\theta + b\sin\theta = c = a\cos 2\theta + b\sin 2\theta.$$

Ex. 12.—The solutions of $h\cos\theta + k\sin\theta = c$ are those of $a\cos\theta + b\sin\theta = c$ each increased by

(i) $\frac{\pi}{2}$ if $a-k=0$, $b+h=0$;

(ii) π if $a+h=0$, $b+k=0$;

(iii) $\frac{3\pi}{2}$ if $a+k=0$, $b-h=0$.

Ex. 13.—Eliminate θ and ϕ between $x\cos\theta + y\sin\theta = 2a$, $x\cos\phi + y\sin\phi = 2a$, and $2\cos\frac{1}{2}\theta\cos\frac{1}{2}\phi = 1$. (Carslaw, p. 231, 25, etc.)

We have
$$\begin{aligned} 2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\phi &= \cos\tfrac{1}{2}(\theta+\phi) + \cos\tfrac{1}{2}(\theta-\phi) \\ &= x/\sqrt{(x^2+y^2)} + 2a/\sqrt{(x^2+y^2)} = 1 \\ &\text{or, } y^2 = 4a(x+a). \end{aligned}$$

[*i.e.* If the point P moves so that $2\cos\frac{1}{2}\theta\cos\frac{1}{2}\phi = 1$, its locus is a parabola, focus at the origin, axis OX, and latus rectum equal to the diameter of the circle.]

Ex. 14.—If $a = b\cos\theta + c\sin\theta$, and $b = a\cos\theta + \delta\sin\theta$, then either

$$a^2 - b^2 = 0 \text{ and } \sin\theta = 0, \text{ or } a^2 + \delta^2 = b^2 + c^2 \text{ and } \theta = \tan^{-1}\frac{\delta}{a} + \tan^{-1}\frac{c}{b}.$$

(Queen's Coll., Belfast, 1869.)

(The second conditions are the conditions that one tangent from (b, c) to $x^2+y^2=a^2$ shall be parallel to one tangent from (a, δ) to $x^2+y^2=b^2$, and on the same side of the common centre, as is easily seen geometrically.)

Ex. 15.—If the equation $a\cos 4\phi + b\sin 4\phi = c$ has solutions ϕ_1, ϕ_2, ϕ_3, ϕ_4 not differing by multiples of π, prove that

$$\tan\phi_1\tan\phi_2\tan\phi_3\tan\phi_4 = 1, \text{ and } \sum \operatorname{cosec} 2\phi = 0.$$

(Carslaw, p. 229, 10.)

(*Hint.*—In our notation $\phi_1, \ldots \phi_4$ are respectively $\frac{1}{4}\theta_1$, $\frac{1}{4}\theta_2$,

$$\frac{\pi}{2} + \tfrac{1}{4}\theta_1, \text{ and } \frac{\pi}{2} + \tfrac{1}{4}\theta_2.)$$

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