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Generalizations of the Poincaré Birkhoff fixed point theorem

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It is shown how George D. Birkhoff's proof of the Poincaré Birkhoff theorem can be modified using ideas of H. Poincaré to give a rather precise lower bound on the number of components of the set of periodic points of the annulus. Some open problems related to this theorem are discussed.

A twist homeomorphism $g : A \to A$ of the annulus A is one which can be lifted to a homeomorphism $\tilde{g} : \tilde{A} \to \tilde{A}$ of the universal cover of A, such that $\tilde{g}$ moves the two boundary components of $\tilde{A}$ in opposite directions. The Poincaré Birkhoff fixed point theorem states that an area preserving twist homeomorphism of the annulus has at least two fixed points. This was conjectured and proved in special cases by Poincaré [8]. The existence in general of one fixed point was proved by Birkhoff in 1912 [2] (he actually claimed two fixed points there, but as he himself later wrote, they might coincide). In 1925 he published a proof [3] which showed the existence of two distinct fixed points. Though the veracity of this paper has been doubted by many mathematicians and in fact the last few years have seen quite extensive efforts to correctly prove two distinct fixed points, Birkhoff's 1925 paper was in fact correct. In my talk at Stanford I attempted to persuade the audience of this fact by presenting a simplified version of Birkhoff's proof. Since this proof will appear

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elsewhere as a joint paper with Brown [4], I shall here stress some generalizations of the Poincaré Birkhoff theorem which were only briefly mentioned in the talk.

Poincaré's original motivation for studying area preserving twist homeomorphisms $g : A \to A$ was as holonomy normal to closed leaves of certain codimension 2 foliations. In this situation a fixed point of g determines a nearby closed leaf. This has applications to the restricted 3-body problem and elsewhere.

As Poincaré himself observed, a periodic orbit of g is as useful from this point of view as a fixed point; moreover for "most" $g : A \to A$, some power of g is a twist homeomorphism, so one can deduce existence of infinitely many periodic points of g in such situations. As we shall see, this remark actually allows one to give a rather precise lower bound on the number of components of the set of periodic orbits of g of period exactly n. This bound is asymptotic to a constant times $\varphi(n)$ (the Euler φ function). If it is non-zero it thus tends to infinity. The number of periodic points is of course n times the number of periodic orbits.

We talk in terms of area preserving maps throughout. Birkhoff in 1925 ([3]) replaces the condition "g area preserving" by "no open neighborhood U of a boundary component of A is contained in $g(U)$ as a non-dense subset". Using his finer techniques to deal with this condition everything in this article does go through under this weaker condition. Recall also that given diffeomorphic compact surfaces M_1, M_2 with volume forms ω_1, ω_2, there is a diffeomorphism $M_1 \to M_2$ mapping ω_1 to a constant multiple of ω_2, so the particular area we preserve is immaterial. This is true in any dimension, see [6] and [7].

The plan of this paper is as follows. After sketching the proof of two fixed points I describe how a simple modification also gives information on non-discrete fixed point sets. The above mentioned result on periodic points is also deduced from a simple observation about the proof.

In a final section I describe some questions which I believe (as a non-expert in this field) to be open.

1. Proof of the Poincaré Birkhoff theorem

We shall work in the universal cover

$$S = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1\}$$

of the annulus $A = \{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2+y^2 \le 2\}$. As covering map $\pi : S \to A$ we take $\pi(x, y) = \big((1+y)\cos 2\pi x,\ (1+y)\sin 2\pi x\big)$. We assume the standard area $dxdy$ in S and the induced area in A .

If $g : A \to A$ is an area preserving twist homeomorphism we can lift it to S and extend it in a trivial fashion to $\mathbb{R}^2$ to get a map

$$h : \mathbb{R}^2 \to \mathbb{R}^2 , \quad h|S \text{ area preserving,}$$

$$h(x, y) = \big(x-r_1(y), y\big) , \quad y \ge 1 ,$$

$$= \big(x+r_2(y), y\big) , \quad y \le 0 ,$$

$$h(x+1, y) = h(x, y) + (1, 0) ,$$

where r_1 and r_2 are positive functions. If F is a fixed point of h then so is $F + (n, 0)$ for any $n \in \mathbb{Z}$. This periodic family of fixed points projects to a single fixed point in A . We assume h has at most one such periodic family of fixed points and deduce a contradiction.

If P and Q are distinct points of $\mathbb{R}^2$ let $D(P, Q) = (Q-P)/\|Q-P\|$ be the "direction from P to Q ". If $\alpha : [t_0, t_1] \to \mathbb{R}^2 - \text{fix}(h)$ is any curve, define $\text{ind}_h\alpha$ to be the total rotation of the direction $D\big(P, h(P)\big)$ as P moves along the curve α , counted in turns and fractions of a turn. That is $\text{ind}_h\alpha = (1/2\pi)\int_\alpha d\theta$ where $\theta(x) = \arg D\big(P, h(P)\big)$. For a closed curve α , $\text{ind}_h\alpha \in \mathbb{Z}$.

We claim that under our given assumptions in fact $\text{ind}_h\alpha = 0$ for any closed curve α . Indeed we must check this just for the boundary curve β of a rectangle with vertices $\{(x\pm\frac{1}{2}, 0), (x\pm\frac{1}{2}, 1)\}$ containing at most one fixed point of h in its interior, since any closed curve in $\mathbb{R}^2 - \text{fix}(h)$ is homotopic to a combination of such closed curves β . But the indices of the two horizontal segments of β are zero while the indices of the two

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vertical segments cancel, since $D(P, h(P)) = D(P+(1, 0), h(P+(1, 0)))$, so our claim follows. This is in fact just a special case of the Lefschetz fixed point theorem for the annulus.

Denote

$$H_- = \{(x, y) \mid y \leq 0\},$$
$$H_+ = \{(x, y) \mid y \geq 0\}.$$

We shall construct a curve γ of index $\frac{1}{2}$ running from H_- to H_+ in $\mathbb{R}^2 - \text{fix}(h)$ and another of index $\frac{1}{2}$ running from H_+ to H_- in $\mathbb{R}^2 - \text{fix}(h)$. By connecting these curves in H_+ and H_- we form a closed curve α with $\text{ind}_h\alpha = 1$, giving the desired contradiction.

To construct γ we shall proceed as follows. We first give a continuous family $h_t : \mathbb{R}^2 \to \mathbb{R}^2$, $0 \leq t \leq 1$, of homeomorphisms which are close to h and satisfy $h_0 = h$, $h_t(x+1, y) = h(x, y) + (1, 0)$, $\text{fix}(h_t) = \text{fix}(h)$; and then find a simple curve $\delta : \mathbb{R} \to \mathbb{R}^2 - \text{fix}(h)$ such that

(a) δ is a *flow line* for h_1, that is $\delta(t+1) = h_1\delta(t)$ for all $t \in \mathbb{Z}$,

(b) $\delta(-\infty, 0] \subset H_-$; $\delta(r) \in H_+$ for some $r > 0$.

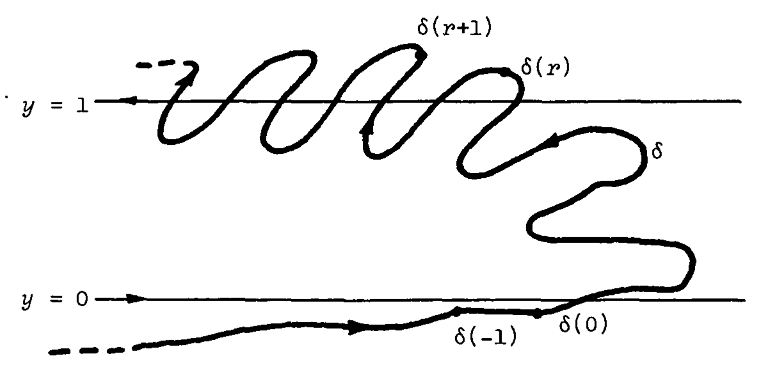

FIGURE 1

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Assuming the existence of the h_t and δ, if we set $\gamma = \delta|[-1, r]$, it is not hard to see that $\text{ind}_{h_1}\gamma$ is close to $\frac{1}{2}$ and to deduce that $\text{ind}_h\gamma = \frac{1}{2}$, as desired. One way of seeing this is as follows (see [4] for another way). $\text{ind}_{h_1}\gamma$ is an intrinsic invariant of $\delta|[-1, r+1]$, namely it is the rotation of $D(\delta(t), \delta(t+1))$ as t goes from -1 to r. A continuous deformation of the curve $\delta|[-1, r+1]$ through simple curves keeping $\delta(-1), \delta(0), \delta(r), \delta(r+1)$ fixed does not alter this intrinsic invariant. In this way we can deform $\delta|[-1, r+1]$ to a curve for which it is obvious that this intrinsic invariant is close to $\frac{1}{2}$, whence the same holds for $\text{ind}_{h_1}(\gamma)$. Now $\text{ind}_{h_t}(\gamma)$ is defined for each t, and modulo 1 its value can be read off from $D(\gamma(-1), h_t\gamma(-1))$ and $D(\gamma(r), h_t\gamma(r))$ and is seen to remain close to $\frac{1}{2}$. By continuity this holds without the modulo 1 restriction, so $\text{ind}_h(\gamma) = \text{ind}_{h_0}(\gamma)$ is close to $\frac{1}{2}$. Since evidently $\text{ind}_h(\gamma) \equiv \frac{1}{2} \pmod 1$, it follows that $\text{ind}_h(\gamma) = \frac{1}{2}$.

To construct the h_t and δ we proceed as follows. We can assume the fixpoints of h, if any exist, lie at integer values of x. Let $W = \{(x, y) \mid 1/4 \le x \le 3/4 \pmod 1\}$ and choose ε such that $\|P-h(P)\| > \varepsilon$ for $h(P) \in W$. Let $\lambda : \mathbf{R} \to \mathbf{R}$ be a non-negative continuous function which is periodic $(\lambda(x+1) = \lambda(x))$ and satisfies

$$\lambda(x) = 0 \quad \text{for} \quad -1/4 \le x \le 1/4 ,$$

$$0 < \lambda(x) < \varepsilon \quad \text{for} \quad 1/4 < x < 3/4 ,$$

and define for $0 \le t \le 1$,

$$T_t : \mathbf{R}^2 \to \mathbf{R}^2 , \quad T_t(x, y) = (x, y+t\lambda(x)) .$$

Then T_t is area preserving, and if we put $h_t = T_t h$, then $\text{fix}(h_t) = \text{fix}(h)$ for $0 \le t \le 1$. Now if the area (calculated in the "rolled up plane" $S^1 \times \mathbf{R} = \mathbf{R}^2/((x, y) \equiv (x+1, y))$) of $h_1(H_-) - H_-$ is m, then the area of $h_1^2(H_-) - H_-$ is $2m$, of $h_1^3(H_-) - H_-$ is $3m$, and

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so on, until ultimately $h_1^k(H_-)$ must intersect H_+. Choose $P \in H_-$ with $h_1^k(P) \in H_+$ and let $\delta : [-1, 0] \to \mathbb{R}^2$ be the straight line segment from $h_1^{-1}(P)$ to P. Extend this to a map $\delta : \mathbb{R} \to \mathbb{R}^2$ by requiring $\delta(t+k) = h_1^k \delta(t)$ for all $k \in \mathbb{Z}$. Then δ is a simple curve, since any self-intersection would recur periodically along δ and δ has monotonic x-component for negative t. δ is thus the desired flow line.

To find a curve from H_+ to H_- of index $\frac{1}{2}$ we use T_t^{-1} instead of T_t in the above, to give a different approximation h_1' to h and a flow line δ' for h_1' starting in H_+ and having non-empty intersection with H_-.

This completes our sketch of the proof of the Poincaré Birkhoff theorem.

REMARK 1.1. If h has isolated fixed points then the above proof actually shows that they cannot all have index 0 (the index of a fixed point is $\text{ind}_h \alpha$ for a small circle α encircling the fixed point in a positive direction). On the other hand the Lefschetz fixed point theorem implies that the sum of the fixed point indices is zero. Also Simon [9] has shown that an isolated fixed point of an area preserving homeomorphism in $\mathbb{R}^2$ has index less than or equal to 1 (under suitable additional assumptions this seems to go back to Poincaré and Birkhoff). Thus the set of fixed point indices for an area preserving twist homeomorphism with isolated fixed points on A is a set $\{n_i \mid i = 1, \ldots, r\}$ with $n_i \le 1$, $\sum n_i = 0$, $\sum n_i^2 > 0$. It is easy to check by examples that any such set actually occurs. Smooth area preserving flows give sufficient examples (see Proposition 4.4).

2. Nondiscrete fixed point sets

All that was needed to find the closed curve α of index 1 in the above proof was the area preserving periodic auxiliary transformation

$T_t : \mathbb{R}^2 \to \mathbb{R}^2$, $0 \le t \le 1$, satisfying $\|T_t(P)-P\| < \|h^{-1}(P)-P\|$ for all $P \notin \text{fix}(h)$ (whence $\text{fix}(T_t h) = \text{fix}(h)$) and satisfying $H_- \subset T_1(H_-)$, $T_1(H_+) \subset H_+$, both inclusions proper. To find such T_t it suffices that there be a curve connecting H_- and H_+ in $\mathbb{R}^2 - \text{fix}(h)$, for we can then define T_t via an area preserving flow defined in a tubular neighborhood of this curve ($\text{fix}(h)$ is closed, so such a neighborhood exists disjoint from $\text{fix}(h)$). Thus the proof actually shows the stronger result:

THEOREM 2.1. *If $g : A \to A$ is an area preserving twist homeomorphism then one of (a) and (b) holds:*

(a) $\text{fix}(g)$ separates the boundary components of A (that is, the components of ∂A lie in distinct components of $A - \text{fix}(g)$);

(b) $\text{fix}(g)$ has at least 2 components; in fact it has an open closed subset of nonzero index.

The index of an open closed subset X of $\text{fix}(g)$ is defined as the sum $\sum \text{ind}_h \alpha_i$ where the α_i are a collection of nullhomotopic closed curves in A bounding a closed neighborhood of X in $A - (\text{fix}(g)-X)$, oriented as this boundary. More generally the same definition applies to any closed subset X of A having a neighborhood N with $(N-X) \cap \text{fix}(g) = \emptyset$. If X is invariant and contractible, then by Simon's result (Remark 1.1), it has index less than or equal to 1 . This says a lot about possible fixed point configurations, but some questions remain open (see Section 4).

3. Periodic points

Let $g : A \to A$ be a homeomorphism of the annulus and $h : S \to S$ a lifting of g to the strip. Denote

$$h^n(x, 0) = (x_n, 0) , \quad h^n(x, 1) = (x'_n, 1) .$$

$$\left.\begin{aligned} a &= \lim\left((x'_n - x)/n\right) \\ b &= \lim\left((x_n - x)/n\right) \end{aligned}\right\} \text{ as } n \to \pm\infty .$$

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It is easily seen (and well known) that these limits exist and do not depend on x . Assume for convenience that $a \leq b$ by taking g^{-1} and h^{-1} instead of g and h if necessary. We call the open interval (a, b) the "twist of h ", denoted

$$\mathrm{TWIST}(h) = (a, b) .$$

The lifting h of g is only well defined up to multiplication by powers of the covering transformation

$$s : S \to S , \quad s(x, y) = (x+1, y) .$$

A different lifting $s^k h$ of g has twist $(a+k, b+k)$ instead of (a, b) , so up to translation by integers $\mathrm{TWIST}(h)$ is an invariant of g . We denote this equivalence class of intervals by

$$\mathrm{twist}(g) = [(a, b)] .$$

Observe that g is a twist homeomorphism if and only if $0 \in \mathrm{TWIST}(h)$ for some lifting h of g ; in other words if and only if $\mathrm{twist}(g) = [(a, b)]$ and $(a, b) \cap \mathbb{Z} \neq \emptyset$.

THEOREM 3.1. *(i) Let $g : A \to A$ be area preserving with $\mathrm{twist}(g) = [(a, b)]$. Then the number $f(n)$ of periodic points of g of period exactly n satisfies*

$$f(n) \geq 2n\varphi(n; a, b)$$

where

$$\varphi(n; a, b) = |\{q \in (na, nb) \mid \gcd(q, n) = 1\}| .$$

(ii) $\varphi(n; a, b)$ is asymptotic to $\varphi(n)|b-a|$, where $\varphi(n)$ is the Euler φ-function $\varphi(n) = \varphi(n; 0, 1)$.

Proof. Let $h : S \to S$ be a lifting of g with $\mathrm{TWIST}(h) = (a, b)$. Then h^n is a lifting of g^n and $\mathrm{TWIST}(h^n) = (na, nb)$. If $q \in (na, nb) \cap \mathbb{Z}$, then $s^{-q}h^n$ is a lifting of g^n which is a twist map of the strip, so either $s^{-q}h^n$ has infinitely many fixed points or by the proof in Section 1 it has fixed points P and Q of positive and negative index respectively. Let $\pi : S \to A$ be the covering map. The orbits $O_1 = \{g^i\pi(P)\}$ and $O_2 = \{g^i\pi(Q)\}$ are disjoint, since they consist of

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points of positive and negative fixed point index for g^n respectively. Note that every point of $\pi^{-1}O_1$ and $\pi^{-1}O_2$ is fixed by $s^{-q}h^n$, so the periodic orbits found this way for different values of q must be disjoint, since no point of S is fixed by two different liftings of g^n.

It remains to show that if $\gcd(q, n) = 1$, then $\pi(P)$ and $\pi(Q)$ have period exactly n, that is O_1 and O_2 each have exactly n points. But suppose $\pi(P)$ say had period d with d a proper divisor of n, $n = dm$. Then $g^d\pi(P) = \pi(P)$, so $s^r h^d P = P$ for some r, so $s^{rm}h^n P = (s^r h^d)^m P = P$, so $q = rm$ is not prime to n.

(ii). I am grateful to George Cooke for the following proof. Let

$$n = p_1^{a_1} \dots p_s^{a_s}$$

be the prime decomposition of n. Then $\varphi(n)$ can be calculated by a counting argument as

$$\varphi(n) = n - \sum_{i=1}^{s} \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \sum_{i<j<k} \frac{n}{p_i p_j p_k} + \dots .$$

Here the first term is the number of integers from 0 to $n-1$; the second term corresponds to removing for each i those integers divisible by p_i; integers divisible by two primes p_i and p_j have then been removed twice, so the third term replaces them; and so on. The same counting argument gives

$$\varphi(n;\, a, b) = \left[(nb-na) - \sum_{i=1}^{s} \frac{nb-na}{p_i} + \sum_{i<j} \frac{nb-na}{p_i p_j} - \dots\right] + \varepsilon(n)$$
$$= (b-a)\varphi(n) + \varepsilon(n) ,$$

where $\varepsilon(n)$ is an error term arising from the fact that the number of multiples of $p_i p_j \dots p_l$ in the interval (na, nb) is not exactly $(nb-na)/p_i p_j \dots p_l$ but differs by an absolute error of at most 1. Since the error in each term of the sum is at most 1,

$$|\varepsilon(n)| \le \left(1 + \sum_{i=1}^{s} 1 + \sum_{i<j}^{s} 1 + \dots\right) = 2^s .$$

Using the formula

$$\varphi(n) = \prod_{i=1}^{s} \left(p_i^{a_i} - p_i^{a_i - 1}\right) ,$$

it is easy to see that $2^s/\varphi(n)$ has limit zero as $n \to \infty$. Hence so does $\varepsilon(n)/\varphi(n)$, so the theorem is proved.

Using Theorem 2.1 instead of Poincaré Birkhoff we get the following minor improvement.

PROPOSITION 3.2. *The above theorem applies also to the number of components of the set* F_n *of periodic points of period exactly* n *if we convene that a component of* F_n *which separates the boundaries of* A *be counted with multiplicity* n .

THEOREM 3.3. *The bound in Theorem* 3.1 *is precise in the sense that for any* $a < b$ *and any* $\varepsilon > 0$ *there exists an area preserving homeomorphism* $g : A \to A$ *with* $\text{twist}(g) = [(a, b)]$ *such that the number* $f(n)$ *of periodic points of period exactly* n *satisfies*

$$2n\varphi(n;\, a,\, b) \le f(n) \le (1+\varepsilon)2n\varphi(n;\, a,\, b) .$$

We only sketch the proof. Consider the area preserving flow in A described by the following picture in the strip S (Figure 2); see also Proposition 4.4. Let $f_0 : A \to A$ be the homeomorphism obtained by integrating this flow at time $t = \delta$ for some small $\delta \neq 0$ and let $f_1 : A \to A$ be $f_1 = ff_0$, where f is just a rotation of A by $2\pi q/n$ for some q prime to n . Then f_1 has exactly two periodic orbits of period n . We call such an f_1 a "standard example".

Now let $h_0 : S \to S$ be the area preserving "shear map" $h_0(x, y) = \left(x+ay+b(1-y),\, y\right)$ and let $g_0 : A \to A$ be the induced map on the annulus. The periodic points of g_0 occur in circles parallel to the boundaries of A . Note that g_0 shows the bound in Proposition 3.2 to be exact. Given a circle of periodic points of g_0 we can break this circle into isolated periodic points by replacing a neighborhood of it by a "standard example".

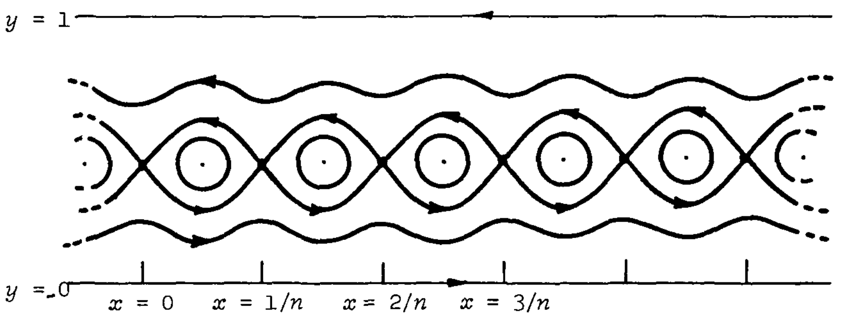

FIGURE 2

It is not hard to see that we can iteratively replace circles of periodic points by standard examples in such a way that the process approaches a limit g which is area preserving and has a discrete set of periodic points for each period. This process would lead to the equality $f(n) = 2n\varphi(n;\ a,\ b)$ except that during the iterative process new circles of periodic points appear around the isolated periodic points that are being created and these circles must be replaced by standard examples (actually generalized standard examples of the form $f_1 \times f_2 : A \times \mathbb{Z}/m \to A \times \mathbb{Z}/m$) later in the procedure, giving "extra" periodic points. If at the kth step of the iterative procedure we choose the parameter δ involved in the definition of the standard examples to be of order roughly $\varepsilon/2^k$ then the number of "extra" periodic points stays in bounds and the theorem follows.

REMARK 3.4. Call two periodic points P and Q of periods p and q "pseudo-close" if any curve α from P to Q is homotopic (relatively endpoints) in A to the curve $g^{pq}(\alpha)$ from P to Q. Clearly periodic orbits which are close in the intuitive sense are pseudo-close. In Theorem 3.1 let $p(n)$ be the number of periodic orbits of period n which are not pseudo-close to an orbit of lower period, counted modulo pseudo-closeness. Then the proof of Proposition 3.2 shows with little difficulty

$$\varphi(n;\ a,\ b) \le p(n) \le \varphi(n;\ c,\ d) ,$$

where $c = \min\big(x(h(P))-x(P)\big)$, $d = \max\big(x(h(P))-x(P)\big)$, over all $P \in S$, where $x(P)$ means x-coordinate and h is a lifting of g. This is the same lower bound as Theorem 3.1 and Proposition 3.2 except for a factor of

2 .

In fact Theorem 3.1 does not "see" periodic orbits which are close to each other, so the lower bound in Theorem 3.1 is too low in general. For example, the same construction as in the proof of Theorem 3.3 can also give g close to the "shear transformation" g_0 for which $f(n)$ increases exponentially.

4. Final remarks and problems

The essential step in the proof of the Poincaré Birkhoff theorem was finding a closed null-homotopic path α of index 1 with respect to the given homeomorphism. A natural question is whether this α can always be chosen to be a simple curve, considered as a curve in the annulus A . Simon's result quoted at the end of Section 1 shows that the answer is "yes" if the fixed point set is discrete; a general answer yes would tell one more about possible fixed point sets in general. Even better would be a positive answer to the following conjecture - among other things it would answer the above question, but also be a strengthening of Simon's result.

CONJECTURE. Let $h : X \to \mathbb{R}^2$ be an area preserving map from a compact submanifold-with-boundary $X \subset \mathbb{R}^2$ to $\mathbb{R}^2$ such that $\partial X \cap \mathrm{fix}(h) = \emptyset$. Orient ∂X as the boundary of X (Figure 3). Thus ∂X is a collection of closed curves α_i and we can define $\mathrm{ind}_h \partial X = \sum \mathrm{ind}_h \alpha_i$. We conjecture that when $\mathrm{ind}_h \partial X \geq 1$, then there exists an embedded disc $X_0 \subset X$ with $\partial X_0 \cap \mathrm{fix}(h) = \emptyset$ and $\mathrm{ind}_h \partial X_0 = 1$.

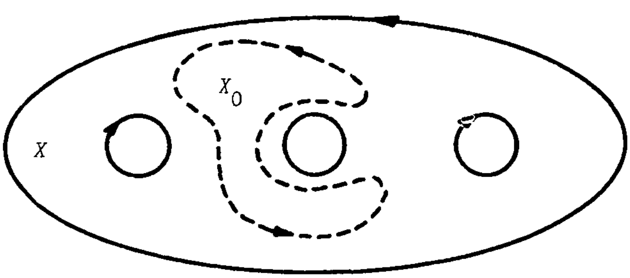

FIGURE 3

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This conjecture is true if fix(h) has finitely many components, or if h is embeddable in an area preserving flow.

Another natural question is in how far the Poincaré Birkhoff theorem extends to other domains. The following is one possible extension.

Let X be the sphere S^2 with $m > 2$ open discs removed and $g : X \to X$ a homeomorphism mapping each component of ∂X onto itself and with no fixed points on ∂X. Choose 2 boundary components S_1 and S_2 of X and fill in the remaining $(m-2)$ boundary components of X with discs to get an annulus A. Then g extends to a map $g' : A \to A$ of the annulus, and if g' is a twist homeomorphism of A we say that g is a *twist with respect to* S_1 *and* S_2.

PROBLEM 4.1. Suppose $g : X \to X$ is area preserving and is a twist with respect to any pair of its boundary components. Does it then follow that either fix(g) separates some boundary components of X from others or fix(g) contains a subset of positive index (and hence has at least two components - note that the Lefschetz fixed point theorem implies fix(g) $\neq \emptyset$)?

PROBLEM 4.2. More generally, can one improve the lower bound on the number $f(n)$ of periodic points of g of period exactly n beyond just maximizing the bound of Theorem 3.1 over all pairs of boundary components of X ? For instance if $m = 3$ there exists an invariant $(a_1, a_2, a_3) \in \mathbb{R}^2$ for g such that the twist of g with respect to its ith and jth boundary components is the pair $(-a_i, a_j)$ for any $1 \leq i < j \leq 3$. Thus optimizing Theorem 3.1 gives the lower bound $2cn\varphi(n)$ with $c = \max|a_i+a_j|$ for $f(n)$, yet one can show that if g is sufficiently close to a g_0 contained in an area preserving flow then $c = |a_1| + |a_2| + |a_3|$ gives a valid improvement. Is this improvement valid in general?

One may also ask if the growth rate $n\varphi(n)$ for $f(n)$ is in any sense generic, or at least an open condition under suitable smoothness conditions. It is probably much too small in general; see Remark 3.4.

This question is maybe related to understanding the following function. Let $h : S \to S$ be a lifting of the area preserving map $g : A \to A$ and for $P \in S$ put $\varphi(P) = \lim(\|h^n(P)-P\|/n)$ as $n \to \pm\infty$. By the ergodic theorem, φ is defined almost everywhere and is integrable. It can be thought of as a map $\varphi : A \to \mathbb{R}$ and is determined by g only up to translation by integers since it depends on the choice of lift h. Restricted to ∂A it gives the twist of g. Since φ is g-invariant, good qualities of φ mirror themselves in good qualities of g. For instance, continuity of φ would imply the existence of many invariant neighborhoods of the boundaries of A - a stability result - and would be related to the conclusion of Moser's beautiful stability theorem [5]. In fact with no smoothness condition on g, φ is more often than not constant almost everywhere by [7] and hence uninteresting, but under suitable conditions φ is interesting (for example, the conditions of Moser's theorem).

PROBLEM 4.3. When is φ continuous on all A ? Smooth on A ? Otherwise nice?

Finally we remark that for homeomorphisms embeddable in area preserving flows the answers to Problems 4.1 and 4.2 are "yes", Problem 4.3 has an easy answer, and in general most questions are answerable. This is due to the following proposition, which is classical.

PROPOSITION 4.4. *If* X *is a compact submanifold of* $\mathbb{R}^2$ *and* v *an area preserving vector field on* X *, then* $v = \left(-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)$ *for some* $f : X \to \mathbb{R}$ *. The flow lines of* v *thus follow contour lines of* f *, giving a good picture of the flow.*

Proof. The proposition is true locally by elementary calculus. If one tries to piece together local candidates for f, the only obstruction to succeeding globally is the existence of a closed curve having nonzero total flux across it, which contradicts area preservation, since any closed curve divides X into two pieces.

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