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ON THE SPLITTING FIELD OF THE ALEXANDER POLYNOMIAL OF A PERIODIC KNOT

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We show that the Murasugi conditions for the Alexander polynomial of a cyclically periodic knot imply a modified form of the Burde-Trotter condition.

Let K be a knot in S^3 with Alexander polynomial $\Delta_K(t)$ and which is invariant under a rotation of prime power order $q = p^r$ about a disjoint axis A. Let $\overline{K}$ and $\overline{A}$ be the images of K and A in the orbit space $S^3/(Z/qZ) \cong S^3$. Murasugi showed that $\Delta_K(t) = \Delta_{\overline{K}}(t) \prod_{i=1}^{q-1} f(t, \zeta_q^i)$, where $f(t,u)$ is the 2-variable Alexander polynomial of $\overline{K} \cup \overline{A}$ and ζ_q is a primitive q-th root of unity [7]. In particular, $f(t, \zeta_q^i)$ is congruent modulo (p) to $f(t,1) = \big((t^\lambda - 1)/(t-1)\big)\Delta_{\overline{K}}(t)$, where $\lambda = link(K, A) = link(\overline{K}, \overline{A})$. On reducing the equation $\Delta_K(t) = \Delta_{\overline{K}}(t) \prod_{\sigma \in G} f(t)^\sigma$ modulo $(t-1, p)$ we get $1 = \lambda^{q-1}$ modulo (p), and so $(\lambda, p) = 1$. (This is also clear for topological reasons, since K is connected.) If moreover $\Delta_K(t) \equiv 1$ modulo (p) then $\lambda = 1$.

Trotter showed that if the commutator subgroup of the knot group is free and if Δ_K has no repeated roots then the q-th roots of unity are in $Split(\Delta_K/Q)$, the splitting field of Δ_K over Q [8]. Burde weakened these hypotheses to requiring that $\Delta_K(t) \not\equiv 1$ modulo (p) and that the second Alexander polynomial $\Delta_{2,K}(t)$ be trivial [1]. This was extended to all knots as a condition involving the higher Alexander polynomials: if $\Delta_K(t) \not\equiv 1$ modulo (p) but $\Delta_{n+1,K}(t) = 1$ then ζ_q has degree at most n over $Split(\Delta_K/Q)$ [6]. The higher Alexander polynomials are determined by (and determine) the structure of $H_1(X'; Q)$, the first homology group with rational coefficients of the infinite cyclic cover of the knot complement, as a module over the principal ideal domain $Q[t, t^{-1}]$. It follows easily from the elementary divisor theorem (that is, essentially the structure theorem for finitely generated modules over a PID) that $\Delta_{n+1,K}(t) = 1$ if Δ_K has no irreducible factor of multiplicity greater than n. Hence the Burde-Trotter condition implies that $[Q(\zeta_q) : Q(\zeta_q) \cap Split(\Delta_K/Q)] \leqslant m$, where m is the maximal multiplicity of irreducible factors of Δ_K. This condition involves only Δ_K and is closer to Trotter's original formulation.

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In this note we shall show that the Murasugi conditions as stated above imply this modified form of the Burde-Trotter condition. The Burde-Trotter and Murasugi conditions have been extended to knots in homology 3-spheres [**5**], and the present argument applies equally well to such knots. (Note that the formulation of the Murasugi conditions in [**3**] assumes that the Alexander polynomial of the axis $\overline{A}$ is trivial, as is always the case when the homology spheres involved are both S^3.) We also give simple examples to show that the Murasugi conditions do not imply the full Burde-Trotter condition involving the higher Alexander polynomials, and that the congruence alternative is needed in general.

THEOREM. *Let Δ be a polynomial in $Z[t]$ which satisfies the Murasugi conditions for some prime power $q = p^r$. Suppose Δ has irreducible factorisation $\Delta = \prod \delta_i^{e_i}$. Then either $\Delta \equiv 1$ modulo (p) or $[Q(\zeta_q) : Q(\zeta_q) \cap Split(\Delta/Q)] \leqslant m = \max\{e_i\}$.*

PROOF: Let $G = Gal(Q(\zeta_q)/Q)$. The Murasugi conditions assert that there is an integer λ, a knot polynomial $\widehat{\Delta}$ and a polynomial $f(t)$ in $Z(\zeta_q)[t]$ such that $\Delta = \widehat{\Delta} \prod_{\sigma\in G} f^\sigma$ and $f(t) \equiv \big((t^\lambda - 1)/(t-1)\big)\widehat{\Delta}$ modulo (p). We may assume that $\Delta \not\equiv 1$ modulo (p). Then f is nontrivial and so has a nontrivial irreducible factor h in $Q(\zeta_q)[t]$. Let $S = \{\sigma \in G \mid h^\sigma = h\}$, and let T be a set of coset representatives for S in G. Then $H = \prod_{\tau\in T} h^\tau$ is irreducible in $Z[t]$ and $H^{|S|} = \prod_{\sigma\in G} h^\sigma$ divides Δ, so $|S| \leqslant m$. Let $M = Q(\zeta_q)^S$ be the subfield of $Q(\zeta_q)$ fixed by S. Then M is generated over Q by the coefficients of h, and $[Q(\zeta_q) : M] = |S|$. Since the coefficients of h are elementary symmetric functions of the roots of h, which are among the roots of Δ, they are in $Split(\Delta/Q)$. The theorem follows easily. ∎

Let $f(t) = t^2 - 3t + 1$ be the Alexander polynomial of the figure eight knot. Then $\Delta = f^3$ satisfies the Murasugi conditions with $n = 3$ and $\lambda = 1$. It is the Alexander polynomial of the connected sum of three copies of the figure eight knot, which admits an obvious $Z/3Z$ symmetry. However the full Burde-Trotter condition implies that no knot with this Alexander polynomial and which admits such a cyclic symmetry can have a cyclic knot module. (In particular, no such knot can have unknotting number 1.) The polynomial $3t^2 - 5t + 3$ satisfies the Murasugi conditions with $q = 3$ and $\lambda = 1$, and hence is the Alexander polynomial of some knot with cyclic period 3, by Theorem 1.1 of [**3**]. As the splitting field for this quadratic is $Q\big(\sqrt{-11}\big)$, which does not contain $Q(\zeta_3)$, we see that in general the alternative "$\Delta_K(t) \equiv 1$ modulo (p)" is necessary. (This answers a question raised in the "Remark" on page 266 of [**2**].)

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