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Structure Theory of Totally Disconnected Locally Compact Groups via Graphs and Permutations

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Abstract. Willis's structure theory of totally disconnected locally compact groups is investigated in the context of permutation actions. This leads to new interpretations of the basic concepts in the theory and also to new proofs of the fundamental theorems and to several new results. The treatment of Willis's theory is self-contained and full proofs are given of all the fundamental results.

Introduction

Can there be such a thing as a general theory of totally disconnected locally compact groups? The class of totally disconnected locally compact groups includes all p-adic Lie groups and also all discrete groups, so the possibility of a general theory of any depth seems slim. But, the recent work of George Willis (see mainly [20] and [22]) has the potential to prove this premonition wrong. Willis's theory has already shown its worth in the solution to several long standing open problems (see [12], [16] and [21]) and ties in nicely with the theory of p-adic Lie groups and linear groups over local skew fields, as shown by Glöckner in [8] and [9].

This paper grew out of efforts to understand Willis's structure theory and then to apply it to the study of permutation groups, but what has emerged is an application of permutation group theory to give a new self-contained approach to Willis's theory. The final product includes different descriptions of the main concepts of the theory, new proofs of the fundamental results, and several new results that become visible in the permutation group theory setting.

Let us turn to the contents of this paper and go briefly over its main results and organisation. The approach to Willis's theory described here runs in many ways parallel to Willis's own development of the theory and many of the arguments here have direct counterparts there. The emphasis in this introduction is on what is new and different from Willis's original work.

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The objects we will be working with are a totally disconnected locally compact group G, a fixed element x in G and a compact open subgroup U of G. (The existence of a compact open subgroup is guaranteed by an old theorem of van Dantzig, see [6] and [10, Theorem 7.7].) The group G acts on the coset space $\Omega = G/U$ and it is this action that is fundamental in the approach presented here to Willis's theory. The first section contains background material from permutation group theory and graph theory and a description of how one can connect topology and permutation actions. In Section 1.4 we go over the basic notational set-up that is used in most of the results and their proofs.

The structure theory has two fundamental ingredients: tidy subgroups and the scale function. We start with the tidy subgroups.

Definition 1 ([**20**]) Let G be a totally disconnected locally compact topological group and x an element of G. For a compact open subgroup U of G define

$$U_+ = \bigcap_{i=0}^{\infty} x^i U x^{-i} \quad U_- = \bigcap_{i=0}^{\infty} x^{-i} U x^i,$$

and

$$U_{++} = \bigcup_{i=0}^{\infty} x^i U_+ x^{-i} \quad U_{--} = \bigcup_{i=0}^{\infty} x^{-i} U_- x^i.$$

The subgroup U is said to be *tidy* for x if

(T1) $$U = U_+U_- = U_-U_+$$

and

(T2) U_{++} and U_{--} are both closed in G.

This definition is investigated in Sections 2 and 3; condition T1 having centre-stage in Section 2. Condition T1 has several alternative formulations. The following has the distinction of not involving the action of G on the coset space $\Omega = G/U$.

Corollary 2.2 *Let G be a totally disconnected locally compact group, x an element of G and U a compact open subgroup of G. Then U satisfies condition T1 if and only if*

$$|U : U \cap x^{-1}Ux| = |U_+ : x^{-1}U_+x|.$$

That condition T1 implies the equality above is shown in [20, p. 354]; what is new is the converse. It also emerges, as in [20, Lemma 1], that it is easy to construct a compact open subgroup satisfying condition T1: start with any compact open subgroup V and there is a number m such that $U = \bigcap_{0\leq n\leq m} x^n V x^{-n}$ satisfies condition T1 (Corollary 2.4).

In the third section, condition T2 is combined with condition T1. The case when $\langle x \rangle$ has compact closure (then x is said to be *periodic*) has to be handled separately. It corresponds to the case where the orbits of x on Ω are finite. This is the "trivial"

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case in the theory and it is fully taken care of in Lemma 3.1. The main result of Section 3 is the following theorem that translates the definition of a tidy subgroup into permutation group theoretic terms.

Theorem 3.4 *Let G be a totally disconnected locally compact group, $x \in G$ a fixed element, U a compact open subgroup of G and $\Omega = G/U$. Set $\alpha_0 = U \in \Omega$ and note that $G_{\alpha_0} = U$. Set $\alpha_i = \alpha_0 x^i$. Suppose that the orbit of α_0 under x is infinite (that is, x is not periodic). Define a graph Γ_+ such that $V\Gamma_+ = \bigcup_{i\geq 0} \alpha_i U$ and $E\Gamma_+ = \bigcup_{i\geq 0}(\alpha_i, \alpha_{i+1})U$. Then U is tidy for x if and only if Γ_+ is a tree, every two vertices in Γ_+ have the same out-valency, and the in-valency of every vertex, except α_0, is 1 (that is, Γ_+ is a directed regular rooted tree such that all edges are directed away from the root α_0).*

This technical looking theorem can be used to deduce a new simple characterisation of tidy subgroups for a given element x.

Corollary 3.5 *Let G be a totally disconnected locally compact group, and x an element in G. A compact open subgroup U of G is tidy for x if and only if*

$$|U : U \cap x^{-n}Ux^n| = |U : U \cap x^{-1}Ux|^n,$$

for all integers $n \geq 0$.

In Section 4 it is shown how one can start with an arbitrary compact open subgroup V of G and "tidy it up" to produce a tidy subgroup for a given element x of G. The critical role in the construction is played by a graph theoretical result from [14]. It is interesting to note that this construction can, from the same raw material, produce a tidy subgroup different from the subgroup produced by Willis's constructions from [20] and [22]. An example of this is shown at the end of the section.

Now we turn to the scale function.

Definition 2 (**[22, Definition 2.2]**) Let G be a totally disconnected locally compact group. The *scale function* on G is defined for $x \in G$ by the formula

$$\mathbf{s}(x) = \min\{|V : V \cap x^{-1}Vx| : V \text{ an open and compact subgroup of } G\}.$$

Remark This definition is differs slightly from that in [22, Definition 2.2]: there the scale function $\mathbf{s}(x)$ is defined as the minimum value of $|xVx^{-1} : (xVx^{-1}) \cap V|$ for all compact open subgroups V. Clearly $|xVx^{-1} : (xVx^{-1}) \cap V| = |V : V \cap x^{-1}Vx|$, so the above definition is equivalent to the definition in [22]. The definition of $\mathbf{s}(x)$ shown here is not the same as the one used in [20], but in [22] it is shown that this definition is equivalent to the definition used in [20].

The most important properties of the scale function are:

(S0) $\mathbf{s}\colon G \to \mathbf{N}$ is continuous, where $\mathbf{N}$ (the set of natural numbers) has the discrete topology ([20, Corollary 4] and Corollary 7.3 below);

(S1) $\mathbf{s}(x) = 1 = \mathbf{s}(x^{-1})$ if and only if there is a compact open subgroup of G normalised by x. (See Corollary 5.4 below.)

(S2) $\mathbf{s}(x^n) = \mathbf{s}(x)^n$ for every positive integer n and every x in G ([20, Corollary 3], and Theorem 7.1 below).
(S3) Let $\Delta\colon G \to \mathbf{R}^+$ denote the modular function on G. Then $\Delta(x) = \mathbf{s}(x)/\mathbf{s}(x^{-1})$ for every x in G ([20, Corollary 1], and Theorem 5.2 below).

The scale function is closely related to tidy subgroups as will be shown in Section 6, but before exploring that relationship we try in Section 5 to work only with the definition of the scale function. We get a short direct proof of property S3 (Theorem 5.2 below) and a simple proof, which bypasses the theory of tidy subgroups, of the fact that if $x \in G$ and U is a compact open subgroup such that $|U : U \cap x^{-1}Ux| = \mathbf{s}(x)$ then $|U : U \cap xUx^{-1}| = \mathbf{s}(x^{-1})$, see Corollary 5.3 below. In [22] Willis asked for a such a proof. Property S1 can now be deduced directly from the definition of the scale function via Corollary 5.3.

The connection between tidy subgroups and the scale function is that, if U is a compact open subgroup of G then $|U : U \cap x^{-1}Ux| = \mathbf{s}(x)$ if and only if U is tidy for x ([22, Theorem 3.1], see Theorem 6.1 below). The proof of this uses the characterisation of tidy subgroups given in Theorem 3.4 and details from the construction of tidy subgroups.

In the seventh and final section of this paper the properties of the scale function are explored further, amongst other things we prove properties S0 and S2, see Corollary 7.3 and Theorem 7.1 below. The new results in Section 7 are an extension of [12, Lemma 1.7] and the following new description of the scale function.

Theorem 7.7 *Let V be a compact open subgroup of a totally disconnected group G. Then for $x \in G$,*

$$\mathbf{s}(x) = \lim_{n\to\infty} |V : V \cap x^{-n}Vx^n|^{1/n}.$$

Furthermore, $\mathbf{s}(x) = 1$ if and only if there is a constant C (depending on V) such that

$$|V : V \cap x^{-n}Vx^n| \le C$$

for all integers $n \ge 0$.

Remark Let G be a totally disconnected and locally compact group. The scope of this theory can be extended to treat continuous automorphisms of G, see [22]. For an automorphism φ of G we define

$$\mathbf{s}(\varphi) = \min\{|V : V \cap \varphi(V)| : V \text{ an open and compact subgroup of } G\}.$$

Then we define

$$U_+ = \bigcap_{i=0}^{\infty} \varphi^{-i}(U) \quad U_- = \bigcap_{i=0}^{\infty} \varphi^{i}U,$$

and

$$U_{++} = \bigcup_{i=0}^{\infty} \varphi^{-i}(U_+) \quad U_{--} = \bigcup_{i=0}^{\infty} \varphi^{i}(U_-).$$

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A compact open subgroup U is said to be tidy for φ if $U = U_+U_- = U_-U_+$, and both U_{++} and U_{--} are closed in G. The definition for a subgroup to be tidy for an element x in G corresponds to the case of it being tidy for the inner automorphism $\varphi_x(y) = x^{-1}yx$. The results and proofs in this paper are phrased in terms of subgroups tidy for elements of G and the scale function as defined on G. If we are interested in knowing about subgroups tidy for a continuous automorphism φ then we can form the semi-direct product H of G with $\langle\varphi\rangle$ and apply the results and methods in this paper to H (see [22, Section 2]).

1 Preliminaries

To explain the new approach to tidy subgroups and the scale function we need concepts from the theory of permutation groups and graph theory. We also need to connect the topology on G to a permutation action of G. We start with the permutation group theory in Section 1.1 and we link topology and permutation actions of G in Section 1.2. Graph theoretical terms are explained in Section 1.3. Finally, in Section 1.4 the notation that will be used in most of what follows is introduced.

1.1 Permutation Groups

Let G be a group acting on a set Ω. The group action will be written on the right, so the image of a point $\alpha \in \Omega$ under an element $g \in G$ is written as αg.

The action is said to be *transitive* if for every two points $\alpha, \beta \in \Omega$ there is some element $g \in G$ such that $\alpha g = \beta$. For a point $\alpha \in \Omega$, the subgroup

$$G_\alpha = \{g \in G : \alpha g = \alpha\}$$

is called the *stabiliser* in G of the point α. If Γ is a subset of Ω then the *pointwise stabiliser* $G_{(\Gamma)}$ of Γ is defined as the subgroup of all the elements in G that fix every element of Γ, that is,

$$G_{(\Gamma)} = \{g \in G : \gamma g = \gamma \text{ for all } \gamma \in \Gamma\} = \bigcap_{\gamma\in\Gamma} G_\gamma.$$

The *setwise stabiliser* $G_{\{\Gamma\}}$ of Γ is defined as the subgroup consisting of all elements of G that leave Γ invariant, that is,

$$G_{\{\Gamma\}} = \{g \in G : \Gamma g = \Gamma\}.$$

Suppose U is a subgroup of a group G. The group G acts on the set G/U of right cosets of U and this action is transitive. The image of a coset Uh under an element $g \in G$ is $U(hg)$. Conversely, if G acts transitively on some set Ω and α is a point in Ω then Ω can be identified with G/G_α. Here "identified" means that there is a bijective map $\theta\colon \Omega \to G/G_\alpha$ such that for every $\omega \in \Omega$ and every element $g \in G$ we get $\theta(\omega g) = \theta(\omega)g$, that is, θ gives an isomorphism of G-actions.

The orbits of stabilisers of points in Ω are called *suborbits*, that is, the suborbits are sets of the form βG_α where $\alpha, \beta \in \Omega$. Orbits of G on the set of ordered pairs of elements from Ω are called *orbitals*. When G is transitive on Ω one can, for a fixed point $\alpha \in \Omega$, identify the suborbits βG_α with the orbitals: the suborbit βG_α is identified with the orbital $(\alpha, \beta)G$. Define *the orbital graph* Γ for the orbital $(\alpha, \beta)G$ as the graph with vertex set Ω and set of directed edges $(\alpha, \beta)G$. (In Section 1.3 below there is a discussion of graph theoretical terms.) The action of G on its vertex set Ω induces an action of G as a group of automorphisms on the graph Γ, because if (α, β) is an edge in Γ then $(\alpha g, \beta g)$ is in the same orbital and therefore also an edge in Γ. The number of elements in a suborbit βG_α, often called the *length* of the suborbit, is given by the index $|G_\alpha : G_\alpha \cap G_\beta|$. In general the number of elements in the orbit αU of a subgroup U is equal to the index $|U : U \cap G_\alpha|$.

A *block of imprimitivity* for G is a subset Δ of Ω such that for every $g \in G$, either $\Delta g = \Delta$ or $\Delta \cap (\Delta g) = \varnothing$. The existence of a non-trivial proper block of imprimitivity Δ (*non-trivial* means that $|\Delta| > 1$ and *proper* means that $\Delta \neq \Omega$) is equivalent to the existence of a non-trivial proper G-invariant equivalence relation $\sim$ on Ω. When G acts transitively on Ω, the block Δ and its translates under G give the $\sim$-classes, and conversely if $\sim$ is a non-trivial proper G-invariant equivalence relation then the $\sim$-classes are non-trivial proper blocks of imprimitivity for G. When $\sim$ is a G-invariant equivalence relation on Ω then G permutes the $\sim$-classes and thus G acts on the set $\Omega/\sim$ of equivalence classes. If there is no non-trivial proper G-invariant equivalence relation on Ω we say that G acts primitively on Ω. In most (all?) books on permutation groups it is shown that, if G acts transitively on Ω then G acts primitively on Ω if and only if G_α is a maximal subgroup of G for every $\alpha \in \Omega$. Part of the proof of this is to show that if $G_\alpha < H < G$ then αH is a non-trivial proper block of imprimitivity.

Recent books covering this material (and of course more) are [3] and [7].

1.2 Permutation Topology

Let G be a group acting on a set Ω. The action of G on Ω can be used to introduce a topology on G. (The survey paper by Woess, [23], contains a detailed introduction to this topology.) The topology of a topological group is completely determined by a neighbourhood basis of the identity element. The *permutation topology* on G is defined by letting the pointwise stabilisers of finite sets form a neighbourhood basis of the identity, that is, a neighbourhood basis of the identity is given by the family of subgroups

$$\{G_{(\Phi)} : \Phi \text{ a finite subset of } \Omega\}.$$

Think of Ω as having the discrete topology and elements of G as maps $\Omega \to \Omega$. Note that the permutation topology is equal to the topology of pointwise convergence. Thus a sequence $\{g_i\}_{i \geq 0}$ of elements in G has an element g as a limit if and only if for every $\alpha \in \Omega$ there is a number N (depending on α) such that if $n \geq N$ then $\alpha g_n = \alpha g$.

Various properties of the action of G on Ω are reflected in properties of this topology on G. For instance, the group G is Hausdorff if and only if the action of G on Ω is faithful (*faithful* means that the only element of G that fixes all the points in Ω is the identity). If the action is faithful then G is totally disconnected.

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When G is a permutation group on Ω, that is, G acts faithfully on Ω, one can think of G as a subgroup of $\mathrm{Sym}(\Omega)$, the group of all permutations of Ω. We say that G is a *closed permutation group* if it is a closed subgroup of $\mathrm{Sym}(\Omega)$, where $\mathrm{Sym}(\Omega)$ has the permutation topology.

Let us now turn the tables and assume that G is a totally disconnected topological group and U a compact open subgroup of G. Define $\Omega = G/U$. Let $\alpha = U \in \Omega$, that is, α is equal to the coset U. Thus $G_\alpha = U$. Suppose $\Phi = \{\beta_1, \dots, \beta_n\}$ is a finite subset of Ω and $x_1, \dots, x_n$ are elements in G such that $\alpha x_i = \beta_i$. Then

$$G_{(\Phi)} = G_{\beta_1} \cap \cdots \cap G_{\beta_n} = (x_1^{-1} G_\alpha x_1) \cap \cdots \cap (x_n^{-1} G_\alpha x_n) = (x_1^{-1} U x_1) \cap \cdots \cap (x_n^{-1} U x_n).$$

Hence all the elements that form the chosen basis of neighbourhoods of the identity in the permutation topology are open in the topology on G. Therefore the permutation topology is contained in the topology on G. The permutation topology can be different from the topology on G: the permutation topology does not separate points in $K = \bigcap_{x \in G} g^{-1} U g$, the kernel of the action of G on Ω. But since the permutation topology is contained in the topology on G we see that if a sequence $\{g_i\}_{i \geq 0}$ of elements in G has an accumulation point g with respect to the topology on G then g is also an accumulation point of the sequence in the permutation topology on G. Note that for every $\beta \in \Omega$ the orbit $\beta G_\alpha = \beta U$ is finite. This is so because, if g is an element of G such that $\alpha g = \beta$ then $U \cap g^{-1} U g$ is an open subgroup of the compact subgroup U and thus

$$|\beta G_\alpha| = |G_\alpha : G_\alpha \cap G_\beta| = |U : U \cap g^{-1} U g| < \infty.$$

Therefore, all suborbits in the action of G on Ω are finite. Compactness has a very natural interpretation in the permutation topology as shown in the following lemma.

Lemma 1.1 (**[23, Lemma 2]**) *Let G be a totally disconnected locally compact group and U a compact open subgroup of G. Set $\Omega = G/U$. A subset A in G has compact closure in G if and only if the set αA is finite for all α in Ω.*

Furthermore, if A is a subset of G and αA is finite for some α in Ω then αA is finite for all α in Ω.

Proof Suppose first that A has compact closure. We can just as well assume that A is compact. Let α be a point in Ω. The cosets of the group G_α form an open covering of A and thus there is a finite subcovering $G_\alpha x_1, \dots, G_\alpha x_n$. If we choose this finite subcovering so that there is no redundancy, then $\alpha A = \{\alpha x_1, \dots, \alpha x_n\}$ and αA is finite.

On the other hand, assume that αA is finite, say $\alpha A = \{\alpha x_1, \dots, \alpha x_n\}$. Then $A \subseteq G_\alpha x_1 \cup \cdots \cup G_\alpha x_n$. The set $G_\alpha x_1 \cup \cdots \cup G_\alpha x_n$ is compact and thus the closure of A is compact.

The last statement of the lemma is now obvious. ■

Lemma 1.1 has a sort of a "dual": a subgroup H in G is cocompact (that is, G/H is compact) if and only if H has only finitely many orbits on Ω, see Lemma 7.5 below.

Turning back to permutation groups on Ω we notice that if G is a closed permutation group then G_α is also closed in $\mathrm{Sym}(\Omega)$. It is easy to show that if G is a closed permutation group and all the suborbits of G are finite then G_α is compact and G is a totally disconnected locally compact group (see [23, Lemma 1]).

1.3 Digraphs

In this section the basic graph theoretic terms needed are defined, in particular the concept of highly arc transitive digraphs is introduced. Highly arc transitive digraphs were first studied by Cameron, Praeger and Wormald in [5]. The terminology is useful when the conditions defining a tidy subgroup are translated into terms relating to group actions and results about highly arc transitive digraphs from [14] will be used when showing the existence of tidy subgroups in Section 4.

A digraph $\Gamma = (V\Gamma, E\Gamma)$ consists of a set $V\Gamma$ of *vertices* and a set $E\Gamma \subseteq V\Gamma \times V\Gamma$ of *edges*. An s-arc in a digraph Γ is a sequence $\alpha_0, \alpha_1, \ldots, \alpha_s$ of distinct vertices such that (α_i, α_{i+1}) is an edge in Γ for $i = 0, \ldots, s-1$ (also called a *directed path* of length s). A digraph Γ is said to be *s-arc transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on the set of s-arcs. A *highly arc transitive* digraph is a digraph that is s-arc transitive for every $s \geq 0$.

For a vertex α in Γ, define $\mathrm{in}_\Gamma(\alpha)$ as the set $\{\beta \in V\Gamma : (\beta, \alpha) \in E\Gamma\}$ and $\mathrm{out}_\Gamma(\alpha)$ as the set $\{\beta \in V\Gamma : (\alpha, \beta) \in E\Gamma\}$. The cardinality of $\mathrm{in}_\Gamma(\alpha)$ is called the *in-valency* of α and the cardinality of $\mathrm{out}_\Gamma(\alpha)$ is the *out-valency* of α. We say Γ is *locally finite* if both the out-valency and the in-valency of α are finite for every vertex α in Γ. A *directed line* in a digraph Γ is a sequence $\{\alpha_i\}_{i\in\mathbf{Z}}$ of distinct vertices such that (α_i, α_{i+1}) is an edge in Γ for every $i \in \mathbf{Z}$ (here $\mathbf{Z}$ denotes the set of integers). For a subset A of $V\Gamma$, the subgraph *spanned* by A is the subgraph Δ of $V\Gamma$ such that $V\Delta = A$ and $E\Delta$ consists of all edges (α, β) in Γ such that both α and β are in A.

The *set of descendants* of a vertex α in Γ is the set of all vertices β such that Γ contains a directed path from α to β. Denote this set by $\mathrm{desc}(\alpha)$. For $A \subseteq V\Gamma$ define

$$\mathrm{desc}(A) = \bigcup_{\alpha\in A} \mathrm{desc}(\alpha).$$

Several terms properly belonging to the world of undirected graphs will also be needed. If α and β are vertices in a digraph Γ we say that α and β are *adjacent* if (α, β) or (β, α) is an edge in Γ. A path in a digraph Γ is a sequence $\alpha_0, \alpha_1, \ldots, \alpha_s$ of distinct vertices such that α_i and α_{i+1} are adjacent for $i = 0, \ldots, s-1$. A digraph is said to be *connected* if for any two vertices there is always a path between them. A digraph is said to be a *tree* if it is connected and it has no non-trivial cycles, that is to say that there is no sequence of vertices $\alpha_0, \alpha_1, \ldots, \alpha_s$ with $s \geq 3$ such that $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are distinct but $\alpha_0 = \alpha_s$, and α_i and α_{i+1} are adjacent for $i = 0, \ldots, s-1$.

1.4 Notational Conventions

In what follows our environment will contain a totally disconnected locally compact group G with a specified element x. The conditions for a compact open subgroup U

to be tidy will be described in terms involving the action of G on the space G/U of right cosets of U in G. The same basic set-up will appear in most of the arguments and also in the results. Below the ingredients in this set-up are explained.

Set $\Omega = G/U$. Define α_0 as the point in Ω corresponding to the coset U. Thus U is equal to the stabiliser G_{α_0} in G of the point α_0. Define $\alpha_i = \alpha_0 x^i$. Once we start looking at the elements of Willis's theory it emerges that the case where the orbit of α under x is finite is the "trivial" case of the theory (see Lemma 3.1). Note that $G_{\alpha_i} = x^{-i} G_{\alpha_0} x^i = x^{-i} U x^i$. Define for $n < m$ the following subgroups of G:

$$U_{n,m} = \bigcap_{i=n}^{m} x^{-i} U x^i = \bigcap_{i=n}^{m} G_{\alpha_i};$$

$$U_{-\infty,m} = \bigcap_{-\infty}^{i=m} x^{-i} U x^i = \bigcap_{-\infty}^{i=m} G_{\alpha_i};$$

$$U_{n,\infty} = \bigcap_{i=n}^{\infty} x^{-i} U x^i = \bigcap_{i=n}^{\infty} G_{\alpha_i}.$$

Thus $U_{n,m}$ is the subgroup of the elements in G that fix all the points $\alpha_n, \alpha_{n+1}, \ldots,$ α_m, similarly $U_{-\infty,m}$ is the subgroup of all the elements in G that fix all the points $\alpha_m, \alpha_{m-1}, \ldots$, and $U_{n,\infty}$ is the subgroup of all the elements in G that fix all the points $\alpha_n, \alpha_{n+1}, \ldots$. The subgroup U is compact and therefore the subgroups defined above are also compact. Following Willis [22] we define

$$U_+ = \bigcap_{i=0}^{\infty} x^i U x^{-i} = U_{-\infty,0} \quad U_- = \bigcap_{i=0}^{\infty} x^{-i} U x^i = U_{0,\infty},$$

and

$$U_{++} = \bigcup_{i=0}^{\infty} x^i U_+ x^{-i} = \bigcup_{i=0}^{\infty} U_{-\infty,-i} \quad U_{--} = \bigcup_{i=0}^{\infty} x^{-i} U_- x^i = \bigcup_{i=0}^{\infty} U_{i,\infty}.$$

We define Γ as the orbital graph $\big(\Omega, (\alpha_0, \alpha_1)G\big)$. Then Γ_+ is defined as the subgraph of Γ with vertex set $V\Gamma_+ = \bigcup_{i\geq 0} \alpha_i G_{\alpha_0}$ and edge set $E\Gamma_+ = \bigcup_{i\geq 0} (\alpha_i, \alpha_{i+1}) G_{\alpha_0}$. Now assume that the orbit of α_0 under x is infinite. In the orbital graph Γ the set of vertices $\Lambda = \{\alpha_i\}_{i\in \mathbf{Z}}$ forms an infinite directed line. Informally the graph Γ_+ defined above is what we get if we keep the vertex α_0 fixed and look at the graph "traced" by images of the infinite directed path $\alpha_0, \alpha_1, \ldots$ under U.

In addition we have use for another subgraph Γ_{++} of Γ with vertex set $V\Gamma_{++} = \bigcup_{i\in \mathbf{Z}} \alpha_i U_{++}$ and edge set $E\Gamma_{++} = \bigcup_{i\in\mathbf{Z}} (\alpha_i, \alpha_{i+1}) U_{++}$. Informally this graph can be described as the graph "traced out" by the images of the line Λ if some infinite path of the form $\alpha_i, \alpha_{i-1}, \ldots$ is kept fixed and we let i vary.

In Sections 5 and 7 we will have several compact open subgroups of G and the action of G on their coset spaces simultaneously under the microscope. When needed we will use labels: so that $\Omega^{(V)}$ denotes the coset space G/V and $\alpha_0^{(V)}$ denotes the coset V in $\Omega^{(V)}$ such that $V = G_{\alpha_0^{(V)}}$ and then we get $\Gamma^{(V)}$ and so on.

2 Condition T1

In this section we explore the meaning of condition T1 that $U = U_+U_- = U_-U_+$. Theorem 2.1 gives the translation of condition T1 into terms involving the action of G on G/U. From Theorem 2.1 we deduce Corollary 2.2 where we get another formulation of condition T1, but this time without mentioning the action of G on G/U. The ideas in Theorem 2.1 are then employed, via Theorem 2.3, to construct subgroups satisfying condition T1 (Corollary 2.4).

Theorem 2.1 *Let G be a totally disconnected locally compact group, U a compact open subgroup of G and x an element of G. Set $\Omega = G/U$. Choose $\alpha_0 = U \in \Omega$. Set $\alpha_i = \alpha_0 x^i$. Define U_+ and U_- as in Section 1.4. The following are equivalent:*

(i) $\alpha_1 U = \alpha_1 U_+$;
(ii) *for every $g \in U$ there is an element $h_+ \in U_+$ such that $\alpha_i h_+ = \alpha_i g$ for all $i \geq 0$;*
(iii) *condition T1 holds, that is,*

$$U = U_+U_- = U_-U_+.$$

Remark Alternatively we can start with a closed permutation group G acting on a set Ω and such that all suborbits are finite. Then G is a totally disconnected locally compact group. If we choose some point $\alpha_0 \in \Omega$ then $U = G_{\alpha_0}$ is a compact open subgroup of G and Ω can be identified with the coset space G/U.

Proof Recall that U_+ is the subgroup of G consisting of all the elements that fix all the points $\alpha_0, \alpha_{-1}, \ldots$, and U_- fixes all the points $\alpha_0, \alpha_1, \ldots$.

(i) $\Rightarrow$ (ii): Assume that condition (i) above holds. Let g be an element in U. We use induction to find a sequence $\{h_n\}_{n\geq 1}$ in U_+ such that $\alpha_i g = \alpha_i h_n$ for $i = 0, \ldots, n$. By condition (i) there is an element h_1 in U_+ such that $\alpha_1 g = \alpha_1 h_1$. Assume now that we have found an element h_n in U_+ such that $\alpha_i g = \alpha_i h_n$ for $i = 0, \ldots, n$. Put $\beta_i = \alpha_i h_n$. Note that $\beta_i = \alpha_i$ for $i \leq 0$. Let V denote the stabiliser of the point $\beta_n = \alpha_n h_n = \alpha_n g$. Then $V = G_{\beta_n} = h_n^{-1} x^{-n} U x^n h_n$ and $G_{(\beta_n, \beta_{n-1}, \ldots)} = h_n^{-1} x^{-n} U_+ x^n h_n$, because U_+ is the subgroup of G fixing the points $\alpha_0, \alpha_{-1}, \ldots$ and $x^n h_n$ moves α_i to β_{i+n}. Condition (i) implies that $\beta_{n+1} V = \beta_{n+1} G_{(\beta_n, \beta_{n-1}, \ldots)}$. The two points β_{n+1} and $\alpha_{n+1} g$ are in the same V-orbit and thus, by (i), there is an element $h' \in G_{(\beta_n, \beta_{n-1}, \ldots)}$ such that $\alpha_{n+1} g = \beta_{n+1} h'$. Set $h_{n+1} = h_n h'$. Since h' fixes the points $\beta_n, \beta_{n-1}, \ldots$ we see that $\alpha_i g = \alpha_i h_{n+1}$ for $i = 0, \ldots, n, n+1$. Also note that h' is in U_+, and therefore h_{n+1} is also in U_+.

The sequence $\{h_n\}_{n\geq 1}$ is contained in the compact subgroup U_+ and has an accumulation point $h_+ \in U_+$. From the definition of the permutation topology it follows that $\alpha_i g = \alpha_i h_+$ for every $i \geq 0$.

(ii) $\Rightarrow$ (iii): Assume now that condition (ii) holds. Let $g \in U$. Find $h_+ \in U_+$ as described in (ii). Set $h_- = g h_+^{-1}$. Then $\alpha_i h_- = \alpha_i g h_+^{-1} = \alpha_i$ for every $i \geq 0$, and therefore $h_- \in U_-$. Hence $g = h_- h_+ \in U_- U_+$. We have shown that $U = U_- U_+$ and it follows that also $U = U_+ U_-$.

(iii) $\Rightarrow$ (i): Assume now that $U = U_+U_- = U_-U_+$. Take some point β in $\alpha_1 U$, say $\beta = \alpha_1 g$ where $g \in U$. Write $g = h_-h_+ \in U_-U_+$. Then, since h_- fixes α_1, we get that

$$\beta = \alpha_1 g = \alpha_1 h_-^{-1} g = \alpha_1 h_+ \in \alpha_1 U_+.$$

Obviously $\alpha_1 U_+ \subseteq \alpha_1 U$ and we conclude that $\alpha_1 U_+ = \alpha_1 U$. ■

The above theorem also leads to another characterisation of condition T1. Part of this result can be found in [20, p. 354].

Corollary 2.2 *Let G be a totally disconnected locally compact group, x an element of G and U a compact open subgroup of G. Then U satisfies condition T1 if and only if*

$$|U : U \cap x^{-1}Ux| = |U_+ : x^{-1}U_+x|.$$

Proof Use the notation in Theorem 2.1. First note that $|U : U \cap x^{-1}Ux| = |\alpha_1 U|$. Then note that $x^{-1}U_+x = U_+ \cap x^{-1}Ux = U_+ \cap G_{\alpha_1}$, and thus $|U_+ : x^{-1}U_+x| = |\alpha_1 U_+|$. Hence the condition that $|U : U \cap x^{-1}Ux| = |U_+ : x^{-1}U_+x|$ is equivalent to condition (i) in Theorem 2.1 and the corollary is established. ■

The next step is to use the above ideas to show that every compact open subgroup U contains a subgroup satisfying condition T1.

Theorem 2.3 *Adopt the notation explained in Section 1.4.*

(i) *There is some number $m > 0$ such that $\alpha_1 U_{-m,0} = \alpha_1 U_+$.*
(ii) *If $\alpha_1 U_{-m,0} = \alpha_1 U_+$ and $g \in U_{-m,0}$ then there is an element $h_+ \in U_+$ such that $\alpha_i g = \alpha_i h_+$ for $i \geq 0$.*
(iii) *If $\alpha_1 U_{-m,0} = \alpha_1 U_+$ then $U_{-m,0} = U_+U_{-m,\infty} = U_{-m,\infty}U_+$.*

Proof (i) The set $\alpha_1 U$ is finite. Clearly $\alpha_1 U_+ \subseteq \alpha_1 U_{-m,0} \subseteq \alpha_1 U_{-n,0}$ whenever $m \geq n \geq 0$. Suppose that $\alpha_1 U_+$ is a proper subset of $\alpha_1 U_{-m,0}$ for every $m \geq 0$. Then there is some point $\beta \in \Omega$ such that $\beta \in \alpha_1 U_{-m,0}$ for every $m \geq 0$, but $\beta \notin \alpha_1 U_+$. For every $n \geq 0$ there is an element $h_n \in U_{-n,0}$ such that $\alpha_1 h_n = \beta$. The sequence $\{h_n\}_{n \geq 0}$ is contained in U and, since U is compact, this sequence will have an accumulation point h_+. Clearly $h_+ \in U_+$ and $\alpha_1 h_+ = \beta$. Therefore $\beta \in \alpha_1 U_+$, contrary to our assumptions. We conclude that there must be a number m such that $\alpha_1 U_{-m,0} = \alpha_1 U_+$.

(ii) Use the same argument as proved the implication (i) $\Rightarrow$ (ii) in Theorem 2.1.

(iii) Here we use (ii) above and the same argument as proved the implication (ii) $\Rightarrow$ (iii) in Theorem 2.1. ■

The first of the following corollaries shows how the above analysis of condition T1 can be use to produce subgroups satisfying condition T1. In the second corollary an unexpected symmetry between U_+ and U_- is pointed out.

Corollary 2.4 (**[20, Lemma 1]**) *Let G be a totally disconnected locally compact topological group and x an element in G. Suppose U is a compact open subgroup of G. Then there exists an integer m such that if $V = \bigcap_{0 \le n \le m} x^n U x^{-n}$ then*

$$V = V_+ V_- = V_- V_+,$$

where $V_+ = \bigcap_{0 \le n} x^n V x^{-n}$ and $V_- = \bigcap_{0 \le n} x^{-n} V x^n$.

Proof Consider the action of G on $\Omega = G/U$. We choose m as in Theorem 2.3 (i) and set $V = U_{-m,0}$. Then $V_+ = U_+$ and $V_- = U_{-m,\infty}$ and the result follows. ∎

Corollary 2.5 *Use the same notation as in Theorem 2.3. If $\alpha_1 U_{-m,0} = \alpha_1 U_+$ then $\alpha_{-(m+1)} U_{-m,0} = \alpha_{-(m+1)} U_{-m,\infty}$. In particular, if $\alpha_1 U = \alpha_1 U_+$ then $\alpha_{-1} U = \alpha_{-1} U_-$.*

Proof By Theorem 2.3, the assumption $\alpha_1 U_{-m,0} = \alpha_1 U_+$ suffices to conclude that $U_{-m,0} = U_+ U_{-m,\infty}$. Take an element $g \in U_{-m,0}$ and write $g = h_+ h_-$, with $h_+ \in U_+$ and $h_- \in U_{-m,\infty}$. Then $\alpha_{-(m+1)} g = \alpha_{-(m+1)} h_+ h_- = \alpha_{-(m+1)} h_-$. Hence $\alpha_{-(m+1)} U_{-m,0} = \alpha_{-(m+1)} U_{-m,\infty}$. ∎

3 Tidy Subgroups

Now we turn our attention to the effect of condition T2 when it is combined with condition T1. The main result in this section is Theorem 3.4, which gives a new characterisation of tidy subgroups for a fixed element x in G.

It is necessary to distinguish between two cases according to whether the closure of $\langle x \rangle$ in G is compact or not. If the closure of $\langle x \rangle$ is compact then x is said to be a *periodic* element of G (some authors use the term *compact* element). In view of Lemma 1.1 we see that if U is a compact open subgroup of G and $\Omega = G/U$ then x is periodic if and only if every x-orbit in Ω is finite. In particular we see that if $\alpha_0 \in \Omega$ and $U = G_{\alpha_0}$ then there is a number N such that x^N fixes α_0, that is, $x^N \in U$. The following lemma takes complete care of the case when x is a periodic element of G.

Lemma 3.1 *Let G be a totally disconnected locally compact group and x a periodic element in G.*

(i) *A compact open subgroup U in G is tidy for x if and only if x normalises U.*
(ii) *There exists a compact open subgroup U in G that is tidy for x.*

Proof (i) Suppose U is tidy for x. Since x is periodic, there is a number N such that $x^N \in U$. Then clearly

$$U_+ = U \cap (xUx^{-1}) \cap \cdots \cap (x^{N-1} U x^{-(N-1)}) = U_-.$$

Since by assumption $U = U_+ U_-$ and $U_+ = U_-$ one concludes that $U = U_+ = U_-$ and that x normalises U. On the other hand, if x normalises U then one only needs to glance at the conditions for U to be tidy for x to see that they are satisfied. (From

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this we can also see that if x is periodic then a compact open subgroup U satisfies condition T1 if and only if U is tidy for x.)

(ii) Let V be some compact open subgroup of G. Let N be a number such that $x^N \in V$. The subgroup

$$U = V \cap (xVx^{-1}) \cap \cdots \cap (x^{N-1}Vx^{-(N-1)}),$$

is compact and open. It is clearly normalised by x and therefore tidy for x. ■

Lemma 3.2 (**[20, Lemma 3]**) *Let G be a totally disconnected locally compact group and $x \in G$. Suppose U is a compact open subgroup of G that satisfies condition T1. Recall from Section 1.4 that $U_{++} = \bigcup_{i=0}^{\infty} x^i U_+ x^{-i}$ and $U_{--} = \bigcup_{i=0}^{\infty} x^{-i} U_- x^i$. Then*

(i) *U_{++} is closed if and only if $U_{++} \cap U = U_+$ and, similarly, U_{--} is closed if and only if $U_{--} \cap U = U_-$;*
(ii) *U_{++} is closed if and only if U_{--} is closed.*

Proof (The proof of this lemma follows Willis's proof of [20, Lemma 3].) Consider the action of G on $\Omega = G/U$. Let α_0 denote the point in Ω corresponding to U and set $\alpha_i = \alpha_0 x^i$. By Lemma 3.1 we may assume that the orbit of α_0 under x is infinite. With this notation we see that

$$U_{++} = \{g \in G : \text{there is a number } n \text{ such that } \alpha_i g = \alpha_i \text{ for all } i \leq n\},$$

and

$$U_{--} = \{g \in G : \text{there is a number } n \text{ such that } \alpha_i g = \alpha_i \text{ for all } i \geq n\}.$$

The condition $U_{++} \cap U = U_+$, says that if $g \in U = G_{\alpha_0}$ and there is some number n such that g fixes α_i for all $i \leq n$, that is if $g \in U_{++}$, then g fixes α_i for all $i \leq 0$.

(i) Suppose U_{++} is closed. Clearly $U_+ \subseteq U_{++} \cap U$. Assume, seeking a contradiction, that U_+ is a proper subset of $U_{++} \cap U$ and that $g \in (U_{++} \cap U) \setminus U_+$. Since $g \in U$ we can apply Theorem 2.1, and find an element $h \in U_+$ such that $\alpha_i g = \alpha_i h$ for all $i \geq 0$. Then $gh^{-1} \in U_-$ and clearly $gh^{-1} \in (U_{++} \cap U) \setminus U_+$. Thus we may assume that $g \in U_-$. Since g is in U_{++} but not in U_+ there is some number $n < 0$ such that g fixes α_i for all $i \leq n$, but there is also a number l, between n and 0 such that g does not fix α_l. Define a sequence $\{h_i\}_{i \geq 1}$ by induction such that $h_1 = g$ and in general $h_{i+1} = x^{in} h_i x^{-in} g$. A schematic view of how the elements in the sequence $\{h_i\}_{i \geq 1}$ act on $\Lambda = \{\alpha_i\}_{i \in \mathbb{Z}}$ is shown in Figure 1. Thus h_i will fix all points α_k with $k \leq in$ and also fixes α_{jn} but does not fix the points α_{l+jn}, with j such that $0 \leq j \leq i-1$. All the elements in this sequence are contained in U. Because U is compact, this sequence will have an accumulation point h contained in U. But this element h is clearly not in U_{++}, a contradiction with the assumption that U_{++} is closed. Therefore $U_{++} \cap U = U_+$.

If $U_{++} \cap U = U_+$, then $U_{++} \cap U$ is closed. Since U is closed then, by [4, Proposition 2.4 in Chapter III] or [10, 5.37], U_{++} is closed.

The statement for U_{--} is proved in exactly the same way.

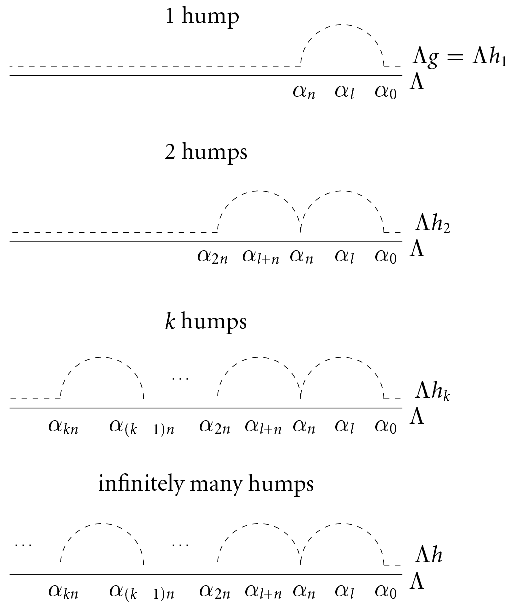

Figure 1

(ii) If U_{--} is not closed (that is, $U_{--} \cap U \neq U_-$) then there exists an element g that fixes α_i for all $i \leq 0$ and all $i \geq n$ for some n but there is a number l such that α_l is not fixed by g. We see that $x^n g x^{-n}$ is in $U_{++} \cap U$. As we saw in part (i), the existence of such an element leads to the conclusion that U_{++} is not closed. Hence, if U_{++} is closed then U_{--} must also be closed. By symmetry we can interchange U_{++} and U_{--}. ■

Theorem 2.1 and Lemma 3.2 combined give:

Corollary 3.3 *Let G be a totally disconnected topological group, x an element in G and U a compact open subgroup of G. Set $\Omega = G/U$ and use the terminology set up in Section 1.4. Then U is tidy for x if and only if*

$$\alpha_1 U_+ = \alpha_1 U$$

and

$$U_{-\infty,-m} \cap U = U_+, \quad \textit{for every } m \geq 0.$$

The notation and terms used in the following theorem are explained in Sections 1.3 and 1.4.

Theorem 3.4 *Let G be a totally disconnected locally compact group, $x \in G$ a fixed element, U a compact open subgroup of G and $\Omega = G/U$. Set $\alpha_0 = U \in \Omega$ and*

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note that $G_{\alpha_0} = U$. Set $\alpha_i = \alpha_0 x^i$. Suppose that the orbit of α_0 under x is infinite (that is, x is not periodic). Define a graph Γ_+ such that $V\Gamma_+ = \bigcup_{i\geq 0} \alpha_i U$ and $E\Gamma_+ = \bigcup_{i\geq 0}(\alpha_i, \alpha_{i+1})U$. Then U is tidy for x if and only if Γ_+ is a tree, every two vertices in Γ_+ have the same out-valency, and the in-valency of every vertex, except α_0, is 1 (that is, Γ_+ is a directed regular rooted tree such that all edges are directed away from the root α_0).

Remark A rooted directed tree with all edges directed away from the root is often called *arborescence* in the literature. Here we will not use that term but instead just talk about trees, or rooted trees, and specify any further properties.

Proof Assume first that U is tidy. Clearly the in- and out-valencies in Γ_+ are the same for all vertices in the orbit $\alpha_n G_{\alpha_0}$. Note also that Γ_+ is a connected graph.

Suppose $n > 0$ and $\mathrm{in}(\alpha_n) > 1$. Then there is an element $g \in G_{\alpha_0}$ such that $\alpha_n g = \alpha_n$ but $\alpha_{n-1} g \neq \alpha_{n-1}$. By Theorem 2.1 there is an element $h \in U_+$ such that $\alpha_i h = \alpha_i g$ for all $i \geq 0$. If $h' = x^n h x^{-n}$ then h' fixes α_0 and fixes α_i for all $i \leq -n$. Thus $h' \in U_{++} \cap U$. From Lemma 3.2 we know that $U_{++} \cap U = U_+$. So $h' \in U_+$ and h' therefore fixes α_i for all $i \leq 0$. But $h = x^{-n} h' x^n$ and h fixes the point α_i for all $i \leq n$, a contradiction. Thus every vertex in Γ_+, except α_0, has in-valency equal to 1.

From condition T1 we see that $\mathrm{out}(\alpha_0) = \alpha_1 G_{\alpha_0} = \alpha_1 U_+$ and we observe also that $\mathrm{out}(\alpha_n) = \alpha_{n+1}(G_{\alpha_0} \cap G_{\alpha_n})$. Considering the second condition in Theorem 2.1 we see that $\alpha_{n+1}(G_{\alpha_0} \cap G_{\alpha_n}) = \alpha_{n+1}(U_+ \cap G_{\alpha_n})$ and from condition T2 it follows, via Lemma 3.2, that $U_{++} \cap U = U_+$ and therefore $U_+ \cap G_{\alpha_n} = U_{-\infty,n}$. Hence, again applying T1,

$$\mathrm{out}(\alpha_n) = \alpha_{n+1} U_{-\infty,n} = \alpha_{n+1} G_{\alpha_n}.$$

Clearly $|\alpha_{n+1} G_{\alpha_n}| = |\alpha_1 G_{\alpha_0}|$. The out-valency of α_n is thus the same as the out-valency of α_0, and every two vertices in Γ_+ have the same out-valency. Since the in-valency is 1 and $V\Gamma_+$ is by assumption infinite we see that Γ_+ is a tree.

Assume now that Γ_+ is a tree and that all the vertices have the same out-valency and all vertices, except α_0, have in-valency equal to 1. First we note that since Γ_+ is a tree, then if a group element $g \in G_{\alpha_0}$ fixes some two vertices in Γ_+ then g must fix every vertex in the unique path between them. Hence $U \cap x^{-n} U x^n = U_{0,n}$ and $U \cap U_{n,\infty} = U_{0,\infty} = U_-$.

The assumption about the out-valency of vertices in Γ_+ implies that $|\alpha_{n+1} U_{0,n}| = |\alpha_1 G_{\alpha_0}|$ for every $n > 0$. We see that $x^n U_{0,n} x^{-n} = U_{-n,0}$ and that $\alpha_1 U_{-n,0} = \alpha_1 G_{\alpha_0}$ for every $n > 0$. Therefore $\alpha_1 G_{\alpha_0} = \alpha_1 U_+$. Hence condition T1 is satisfied.

By the first part of Lemma 3.2 we see that U_{--} is closed, and then, by the second part of Lemma 3.2, condition T2 is satisfied. ■

Example 1 (*cf.* [**20, Section 3**]) Let T_n denote the (undirected) regular tree of valency $n+1$. Set $G = \mathrm{Aut}(T_n)$. Putting the permutation topology on G makes G into a totally disconnected locally compact group. Let $\{\alpha_i\}_{i\in \mathbf{Z}}$ be a sequence of distinct vertices in T_n such that α_i is adjacent to α_{i+1} for all i. Let x be an element in G such that $\alpha_i x = \alpha_{i+1}$ for all i. Set $U = G_{\alpha_0}$. The graph Γ_+ constructed as described in Section 1.4 is a tree, indeed the same as T_n except that the edges have directions and

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all are directed away from α_0. The out-valency in Γ_+ of α_0 is $n + 1$ but the other vertices in Γ_+ have out-valency n. Theorem 3.4 says that U is not tidy for x. From condition T1 we see that α_{-1} is contained in $\alpha_1 U$ but is not contained in $\alpha_1 U_+$ and thus T1 fails. It is easy to show that $U = G_{\alpha_{-1}} \cap G_{\alpha_0}$ is tidy for x.

Corollary 3.5 *Let G be a totally disconnected locally compact group, and x an element in G. A compact open subgroup U of G is tidy for x if and only if*

$$|U : U \cap x^{-n}Ux^n| = |U : U \cap x^{-1}Ux|^n,$$

for all integers $n \geq 0$.

Proof Consider the permutation representation of G on $\Omega = G/U$. Use the notation explained above and in Section 1.4. By Lemma 3.1 it is clear that the corollary is true in the case when x is periodic, that is, when the orbit of α_0 under x is finite. It is thus safe to assume that the orbit of α_0 under x is infinite.

First suppose that U is tidy for x. Then the graph Γ_+ is a tree. The out-valency of Γ_+ is $d_+ = |G_{\alpha_0} : G_{\alpha_0} \cap G_{\alpha_1}| = |U : U \cap x^{-1}Ux|$ and the number of vertices in Γ_+ at distance n from α_0 (that is, in the orbit $\alpha_n G_{\alpha_0}$) is clearly $(d_+)^n$. Thus

$$|U : U \cap x^{-n}Ux^n| = |G_{\alpha_0} : G_{\alpha_0} \cap G_{\alpha_n}| = |\alpha_n G_{\alpha_0}| = (d_+)^n = |U : U \cap x^{-1}Ux|^n.$$

Suppose now that $|U : U \cap x^{-n}Ux^n| = |U : U \cap x^{-1}Ux|^n$ for all $n \geq 0$. Consider the graph Γ_+. The out-valency of α_0 is $d_+ = |G_{\alpha_0} : G_{\alpha_0} \cap G_{\alpha_1}| = |U : U \cap x^{-1}Ux|$. The out-valency of any vertex in Γ_+ can clearly not be greater than d_+. But the number of vertices at distance n from α_0 is by assumption

$$|\alpha_n G_{\alpha_0}| = |G_{\alpha_0} : G_{\alpha_0} \cap G_{\alpha_n}| = |U : U \cap x^{-n}Ux^n| = |U : U \cap x^{-1}Ux|^n = (d_+)^n.$$

This can only happen when Γ_+ is a tree and the in-valency of every vertex, except α_0, is 1, and every vertex in Γ_+ has out-valency equal to d_+. Thus, by Theorem 3.4, the subgroup U is tidy for x. ∎

From the definition of a tidy subgroup it is obvious that a subgroup that is tidy for x is also tidy for x^{-1}. The formula for the indices above gives an easy way to extend that useful observation.

Corollary 3.6 *Let G be a totally disconnected locally compact group, and x an element in G. If a compact open subgroup U of G is tidy for x then it is also tidy for x^n for all integers n.*

It is not true that a compact open subgroup U that is tidy for x^n is automatically tidy for x. A simple example would be when x is periodic and $x^n \in U$, but x does not normalise U.

Assume that $U = G_{\alpha_0}$ is tidy for x and that the orbits of x on $\Omega = G/U$ are infinite. Let Γ denote the orbital graph $\big(\Omega, (\alpha_0, \alpha_1)G\big)$. Every vertex in Γ has out-valency equal to $|\alpha_1 G_{\alpha_0}|$. This is the same as the out-valency of the vertices in Γ_+.

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From the definition we see that Γ_+ is a subgraph of Γ and, since the two graphs have the same out-valency, Γ_+ is the subgraph of Γ that is spanned by $V\Gamma_+$. Another way to describe Γ_+ is that Γ_+ is the subgraph spanned by the vertices in $\operatorname{desc}(\alpha_0)$.

Proposition 3.7 *Continue with the notation above and assume that U is tidy for x. The graph Γ_+ is the subgraph of $\big(\Omega, (\alpha_0, \alpha_1)G\big)$ spanned by $V\Gamma_+ = \bigcup_{i\geq 0} \alpha_i G_{\alpha_0}$.*

Furthermore, the graph Γ_{++} with vertex set $V\Gamma_{++} = \bigcup_{i\in\mathbf{Z}} \alpha_i U_{++}$ and edge set $E\Gamma_{++} = \bigcup_{i\in\mathbf{Z}} (\alpha_i, \alpha_{i+1})U_{++}$ is the subgraph of $\big(\Omega, (\alpha_0, \alpha_1)G\big)$ spanned by $\bigcup_{i\in\mathbf{Z}} \alpha_i U_{++}$. It follows from the fact that Γ_+ is a tree, that Γ_{++} is a tree.

Before turning to the matter of constructing tidy subgroups we show how the above observation can be used to deduce a condition for an element $g \in U$ to be in U_- or U_+. This lemma will be used in Section 6.

Lemma 3.8 (**[20, Lemma 9]**) *Let U be a compact open subgroup of a totally disconnected group G. Suppose that U is tidy for an element x in G. Then an element g of U belongs to U_- if and only if the family $\{x^n g x^{-n}\}_{n\geq 0}$ has an accumulation point. Similarly, $g \in U$ belongs to U_+ if and only if $\{x^{-n} g x^n\}_{n\geq 0}$ has an accumulation point.*

Proof We use the same notation as explained in Section 1.4.

If $g \in U_-$ then $\{x^n g x^{-n}\}_{n\geq 0} \subseteq U_-$. Since U_- is compact, the family $\{x^n g x^{-n}\}_{n\geq 0}$ must have an accumulation point.

Suppose now that $\{x^n g x^{-n}\}_{n\geq 0}$ has an accumulation point g_∞. Then there is an infinite sequence of numbers $0 \leq n_1 < n_2 < \cdots$ such that $\alpha_0(x^{n_i} g x^{-n_i}) = \alpha_0 g_\infty$ for all $i \geq 1$. Let m be the biggest number such that there is a directed path in Γ_{++} from α_m to $\alpha_0 g_\infty$. Note that g fixes α_0 so $x^{n_i} g x^{-n_i}$ fixes $\alpha_0 x^{-n_i} = \alpha_{-n_i}$. Thus there must be a directed path in Γ_{++} from α_{-n_1} to $\alpha_0 g_\infty$. Therefore the definition of m makes sense and $0 \geq m \geq -n_1$. The subgraph of Γ_{++} spanned by those vertices that can be reached by a directed path from α_{-n_i} is invariant under $x^{n_i} g x^{-n_i}$. This graph is a tree and contains α_0, α_m and $\alpha_0 g_\infty$. The path from α_{-n_i} to α_0 is mapped by $x^{-n_i} g x^{n_i}$ to a path from α_{-n_i} to $\alpha_0 g_\infty$. This means that $x^{n_i} g x^{-n_i}$ fixes α_m. Now we conclude that g fixes $\alpha_m x^{n_i} = \alpha_{m+n_i}$ for all $i \geq 0$. Hence there are arbitrarily large numbers i such that g fixes α_i, and because Γ_+ is a tree, g will fix α_i for all $i \geq 0$. So $g \in U_-$. The proof that if $\{x^{-n} g x^n\}_{n\geq 0}$ has an accumulation point if and only if $g \in U_+$ is exactly the same: we just replace x in all arguments with x^{-1}. ■

4 Construction of Tidy Subgroups

In this section it is shown that if G is a totally disconnected locally compact group and x is an element of G, then G contains a compact open subgroup that is tidy for x. Along the way we clarify the nature of tidy subgroups further.

By Theorem 3.4 we must find a compact open subgroup U that acts on a rooted tree such that all vertices have the same out-valency and all, except the root, have in-valency 1. The strategy is to show first that if V satisfies condition T1 then the graph Γ_{++} is highly arc transitive. From the results in [14] we learn that Γ_{++} is "tree-like" in a certain sense and that it is possible to construct a tree from Γ_{++}. There we have a tree and we can spot our tidy subgroup. It is interesting to note that the method of construction here can in some cases produce a different tidy subgroup than the constructions described by Willis in [20] and [22]. An example of this phenomenon is shown at the end of the section.

Theorem 4.1 (**[20, Theorem 1]**) *Let G be a totally disconnected locally compact group and x an element of G. There is a compact open subgroup of G that is tidy for x.*

Proof Let V be a compact open subgroup of G. By Corollary 2.4 there is some integer m such that $\bigcap_{0\leq n\leq m} x^n V x^{-n}$ satisfies condition T1. If necessary, replace V with $\bigcap_{0\leq n\leq m} x^n V x^{-n}$, and thus we may assume that V satisfies condition T1.

Consider the action of G on $\Omega = G/V$. Choose $\alpha_0 \in \Omega$ such that $V = G_{\alpha_0}$ and use the notation explained in Section 1.4. By Lemma 3.1 we only have to think about the case where the orbits of x are infinite. Now we make a short graph theoretical break in the proof of Theorem 4.1 and state and explain two results from [14]. Then we show how Theorem 4.2 can be applied to the graph Γ_{++}.

Theorem 4.2 (**[14, Theorem 1]**) *Let Γ be a locally finite highly arc transitive digraph. Suppose that there is a directed line $\Lambda = \{\alpha_i\}_{i\in \mathbf{Z}}$ in Γ such that $\Gamma = \mathrm{desc}(\Lambda)$. Then there exists a surjective homomorphism of digraphs $\varphi\colon \Gamma \to T$ where T is a directed tree with in-valency 1 and finite out-valency. The automorphism group of Γ has a natural action on T as a group of automorphisms such that $\varphi(\alpha g) = \varphi(\alpha)g$ for every $g \in \mathrm{Aut}(\Gamma)$ and every α in $V\Gamma$. The action of $\mathrm{Aut}(\Gamma)$ on T is highly arc transitive. Furthermore, the fibers of φ are finite and all have the same number of elements.*

Let us briefly review how T and φ are defined. One can make the set $\mathbf{Z}$ of integers into a digraph by saying that $(i, i+1)$ is an edge for all integers i. The first step in the proof of the above theorem is to show that there is a well defined map $\psi\colon V\Gamma \to \mathbf{Z}$ such that $\psi(\alpha_i) = i$ for all i, and if β is some vertex in Γ and there is a directed path from α_i to β of length k then $\psi(\beta) = i + k$, see [14, Lemma 4].

For an integer k, define $V_k = \psi^{-1}(k)$ and E_k as the set of all edges (α, β) in Γ such $\psi(\alpha) = k - 1$ and $\psi(\beta) = k$. The digraph $\Gamma \setminus E_k$ will have more than one infinite component. Define an equivalence relation on the vertex set $V\Gamma$ such that $\alpha \sim \beta$ if and only if α and β are both in V_k for some k and α and β are in the same component of $\Gamma \setminus E_k$. This equivalence relation is preserved under the action of $\mathrm{Aut}(\Gamma)$. The tree T in the theorem is the quotient digraph $\Gamma/\sim$. Two vertices a and b in T, that is two $\sim$-classes, are connected by an edge (a, b) in T if and only if there are vertices α, β in Γ such that $\alpha \in a$ and $\beta \in b$ and (α, β) is an edge in Γ. The natural quotient map $\varphi\colon \Gamma \to T$ is the homomorphism we seek.

The following result from [14] will be used in Section 5.

Proposition 4.3 (**[14, Lemma 5]**) *Let Γ be a locally finite highly arc transitive digraph. Denote by d^- the in-valency of Γ and by d^+ the out-valency of Γ. Suppose that there is a directed line $\Lambda = \{\alpha_i\}_{i\in \mathbf{Z}}$ such that $\Gamma = \mathrm{desc}(\Lambda)$. Then the out-valency t of the tree T described in Theorem 4.2 is given by $t = d^+/d^-$.*

Now we show that Theorem 4.2 can be applied to the graph Γ_{++}.

Lemma 4.4 *The graph Γ_{++} is highly arc transitive and* $\Gamma_{++} = \operatorname{desc}(\Lambda)$.

Proof of Lemma We show that the group $H = \langle V_{++}, x\rangle$ acts highly arc transitively on Γ_{++}. It is enough to show that if $\beta_0, \beta_1, \dots, \beta_n$ is some n-arc in Γ_{++} then there is some element $h \in H$ mapping the n-arc $\beta_0, \beta_1, \dots, \beta_n$ to the n-arc $\alpha_0, \alpha_1, \dots, \alpha_n$. First note that H acts vertex transitively on Γ_{++}. Hence there is an element $h' \in H$ such that $\beta_0 h' = \alpha_0$. Secondly we note that since V satisfies condition T1 then V_+ acts transitively on the set of n-arcs that start at α_0 (see condition (ii) in Theorem 2.1). Thus there is an element $h'' \in V_+ \subseteq H$ that maps the n-arc $\alpha_0, \beta_1 h', \dots, \beta_n h'$ to the n-arc $\alpha_0, \alpha_1, \dots, \alpha_n$. Whence $h = h'h'' \in H$ maps the n-arc $\beta_0, \beta_1, \dots, \beta_n$ to the n-arc $\alpha_0, \alpha_1, \dots, \alpha_n$.

Finally we have to prove that $\Gamma_{++} = \operatorname{desc}(\Lambda)$. Take some vertex β in Γ_{++}. Say $\beta = \alpha_m h$ where $h \in V_{++}$. There is a number $n \le m$ such that $\alpha_n h = \alpha_n$. Then h maps the directed path $\alpha_n, \alpha_{n+1}, \dots, \alpha_m$ to a directed path from α_n to β. Thus $\beta \in \operatorname{desc}(\alpha_n) \subseteq \operatorname{desc}(\Lambda)$. ∎

Proof of Theorem 4.1 Concluded Let $\varphi\colon \Gamma_{++} \to T$ be the homomorphism of digraphs described in Theorem 4.2. The fibers of φ define an equivalence relation $\sim$ on $V\Gamma_{++}$. This equivalence relation is preserved by $\operatorname{Aut}(\Gamma_{++})$, the classes are finite and all have the same size. Note that φ is the quotient map from Γ_{++} to $\Gamma_{++}/\sim$. Set $a_i = \varphi(\alpha_i)$ and $A_i = \varphi^{-1}(a_i)$. To summarise: α_i is a vertex in Γ_{++}, the image a_i of α_i under φ is a vertex in T and finally $A_i = \varphi^{-1}(a_i)$ is a set of vertices in Γ_{++}. Thus A_i is the $\sim$-class of α_i and $A_i = A_0 x^i$. Set $U = G_{\{A_0\}}$. Because U is open in the permutation topology it will also be open in G and thus also closed in G. Note also that $\alpha_0 U \subseteq A_0$. Thus $\alpha_0 U$ is finite and hence is U compact.

Since Γ_{++} is equal to the subgraph of the orbital graph $\Gamma = \big(\Omega, (\alpha_0, \alpha_1)G\big)$ spanned by $V\Gamma_{++}$, we see that the set of descendants of A_0 in Γ_{++} is equal to the set of descendants of A_0 in Γ. The set of descendants of A_0 is thus invariant under U. Define also T_+ as the subgraph in $T = \Gamma_{++}/\sim$ spanned by $\operatorname{desc}_T(a_0)$. The rooted tree T_+ is isomorphic to the graph one constructs from the action of G on G/U by taking as a vertex set $\bigcup_{i\ge 0} a_i G_{a_0}$ and edge set $\bigcup_{i\ge 0}(a_i, a_{i+1})G_{a_0}$. Since the graph T_+ is a tree and the conditions in Theorem 3.4 are satisfied we conclude that U is tidy for x. This finishes the proof of Theorem 4.1. ∎

The two following corollaries are of some independent interest.

Corollary 4.5 (**[20, Lemma 6]**) *Suppose G is a totally disconnected locally compact group, x an element in G and V a compact open subgroup of G. Then the group $L = \overline{V_{++} \cap V_{--}}$ is compact.*

Proof Continue with the same notation as above. By Theorem 2.3 there is a number m such that $U = \bigcap_{0\le n\le m} x^n V x^{-n}$ satisfies condition T1. Note that $V_{++} = U_{++}$ and $V_{--} = U_{--}$ and therefore $L = \overline{U_{++} \cap U_{--}}$. Think of G as acting on $\Omega = G/U$ and use the terminology in the proof of Theorem 4.1. An element $g \in G$ is in L precisely if there are numbers n and m such that g fixes α_i for all $i \le n$ and all $i \ge m$. When g acts on the tree T then g will fix all the vertices a_i where $i \le n$ and all $i \ge m$. But T is

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a tree so g must fix a_i for all i. In particular $a_0 g = a_0$ and thus also $A_0 g = A_0$. Hence A_0 is invariant under L. The set A_0 is finite so the L-orbits of the vertices of Γ_{++} that are in A_0 are finite. By Lemma 1.1 we conclude that L is compact. ∎

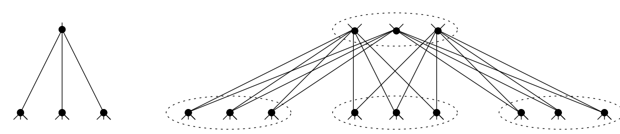
Let G be a permutation group acting transitively on a set Ω and L a subgroup of G. Denote the closures of G and L in the permutation topology with $\bar{G}$ and $\bar{L}$, respectively. Note that $\bar{L}$ is a subgroup of $\bar{G}$ and if $\alpha \in \Omega$ then $\alpha L = \alpha\bar{L}$. In the following corollary we may thus assume that G is closed in the permutation topology. Keeping Lemma 1.1 in mind the result is just a translation of Corollary 4.5 into the language of permutation groups.

Corollary 4.6 *Let G be a group acting transitively on a set Ω so that all suborbits of G are finite. Let x be an element of G and set $\alpha_i = \alpha_0 x^i$, where α_0 is some point in Ω. Then the subgroup of G defined by*

$$\{g \in G : \textit{there is a number } n \textit{ such that } \alpha_i g = \alpha_i \textit{ for all } i \textit{ such that } |i| \geq n\}$$

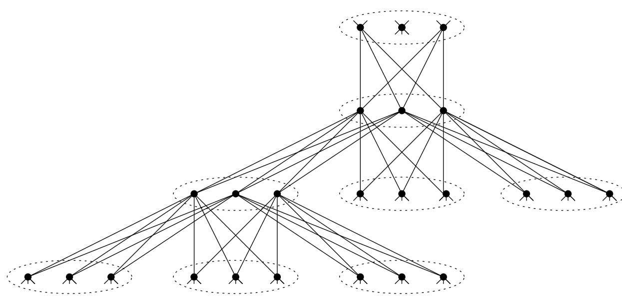
has only finite orbits.

The next example shows that the above construction can yield results different from the constructions that Willis gives in [20] and [22].



Part of T Corresponding part of Γ

Figure 2



Partial view of Γ

Figure 3

Example 2 We start by constructing a graph Γ. Take a directed tree T such that every vertex has in-valency 1 and every vertex has out-valency 3. We construct Γ from T by first replacing every vertex in T by a group of three distinct vertices and putting in

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edges so that to every subgraph in T of the type shown to the left on Figure 2 there will correspond a subgraph in Γ like the one shown to the right on Figure 2. On Figure 3 there is a view of a larger part of Γ. Each vertex has in-valency 2 and out-valency 6. Label the vertices in Γ as shown on Figure 3 and let $\{\alpha_i\}_{i\in\mathbf{Z}}$ be an infinite directed line in Γ. Set $G = \mathrm{Aut}(\Gamma)$. With the permutation topology, G is a totally disconnected locally compact group. It is easy to convince oneself that Γ is a highly arc transitive digraph. The map φ described in Theorem 4.2 maps a vertex in Γ to the corresponding vertex in T. Let x be an element in G such that $\alpha_i x = \alpha_{i+1}$. Set $V = G_{\alpha_0}$. Running V through the machinery in the construction above (note that V already satisfies T1) we produce the subgroup $U = G_{\{\alpha_0,\beta_0,\gamma_0\}}$ that is tidy for x. In the construction described in [22] we use $L = \overline{V_{++} \cap V_{--}}$, set $W = G_{\{\alpha_0 L\}}$ and W is tidy for x. Here we see that $W = G_{\{\alpha_0,\beta_0\}}$. The construction from [20] will give the same group. The reader may note that W and U are closely related in this example: $W = U \cap xUx^{-1}$ and $U = xWx^{-1}$.

5 The Scale Function

Definition 2 (**[22, Definition 2.2]**) Let G be a totally disconnected locally compact group. The *scale function* on G is defined for $x \in G$ by the formula

$$\mathbf{s}(x) = \min\{|V : V \cap x^{-1}Vx| : V \text{ an open and compact subgroup of } G\}.$$

As said in the introduction there is a close relationship between the scale function and tidy subgroups, but before developing that relationship we look at what can be deduce directly from the definition. We look here at properties S1 and S3 as stated in the Introduction.

The scale function is related to the modular function on G. First we review briefly the definition of the modular function and the Haar measure. Let G be a locally compact group. The (right-invariant) Haar measure μ on G is a non-zero Borel measure on G such that for all measurable subsets A and all group elements g in G we have $\mu(Ag) = \mu(A)$ (see [10, Chapter 4] and [11, Chapter III]). The Haar measure is unique up to a constant multiple. For an element $x \in G$ one defines μ_x with the formula $\mu_x(A) = \mu(xA)$ for every Borel set A. It is easy to show that μ_x is also a right-invariant Haar measure and thus there is a number $\Delta(x)$ such that $\mu_x(A) = \mu(xA) = \Delta(x)\mu(A)$ for every measurable subset A of G. It turns out that Δ is a continuous homomorphism of G into the multiplicative group of positive real numbers. The homomorphism Δ is called the *modular function* of G.

Lemma 5.1 (**[17, Lemma 1(iii)] and [19, Theorem 1]**) *Let G be a totally disconnected locally compact group and U a compact open subgroup of G. Set $\Omega = G/U$. Let $\alpha \in \Omega$ and $x \in G$. Set $\beta = \alpha x$. Then*

$$\Delta(x) = \frac{|\beta G_\alpha|}{|\alpha G_\beta|},$$

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and

$$\Delta(x) = \frac{|U : U \cap x^{-1}Ux|}{|U : U \cap xUx^{-1}|}.$$

Proof The proof is accomplished by direct calculation:

$$\begin{aligned} |\beta G_\alpha| &= |G_\alpha : G_\alpha \cap G_\beta| \\ &= \mu(G_\alpha)/\mu(G_\alpha \cap G_\beta) \\ &= \mu(xG_\beta x^{-1})/\mu(G_\alpha \cap G_\beta) \\ &= \mu(xG_\beta)/\mu(G_\alpha \cap G_\beta) \\ &= \Delta(x)\mu(G_\beta)/\mu(G_\alpha \cap G_\beta) \\ &= \Delta(x)|G_\beta : G_\alpha \cap G_\beta| \\ &= \Delta(x)|\alpha G_\beta|. \end{aligned}$$

Thus $\Delta(x) = |\beta G_\alpha|/|\alpha G_\beta|$.

To prove the second equation in the lemma we choose α such that $G_\alpha = U$. Then $G_\beta = x^{-1}G_\alpha x = x^{-1}Ux$. Hence

$$|\beta G_\alpha| = |G_\alpha : G_\alpha \cap G_\beta| = |U : U \cap x^{-1}Ux|,$$

and

$$|\alpha G_\beta| = |G_\beta : G_\alpha \cap G_\beta| = |x^{-1}Ux : U \cap x^{-1}Ux| = |U : U \cap xUx^{-1}|. \qquad \blacksquare$$

Theorem 5.2 (**[20, Corollary 1]**) *Let G be a totally disconnected locally compact group. Denote by Δ the modular function on G and by* $\mathbf{s}$ *the scale function on G. Then, for every $x \in G$,*

$$\Delta(x) = \frac{\mathbf{s}(x)}{\mathbf{s}(x^{-1})}.$$

Proof Let U_1 and U_2 be compact open subgroups of G such that

$$|U_1 : U_1 \cap x^{-1}U_1x| = \mathbf{s}(x) \quad \text{and} \quad |U_2 : U_2 \cap xU_2x^{-1}| = \mathbf{s}(x^{-1}).$$

Note that

$$|U_2 : U_2 \cap x^{-1}U_2x| \geq \mathbf{s}(x) \quad \text{and} \quad |U_1 : U_1 \cap xU_1x^{-1}| \geq \mathbf{s}(x^{-1}).$$

Now we use Lemma 5.1 and get

$$\frac{\mathbf{s}(x)}{\mathbf{s}(x^{-1})} \geq \frac{|U_1 : U_1 \cap x^{-1}U_1x|}{|U_1 : U_1 \cap xU_1x^{-1}|} = \Delta(x) = \frac{|U_2 : U_2 \cap x^{-1}U_2x|}{|U_2 : U_2 \cap xU_2x^{-1}|} \geq \frac{\mathbf{s}(x)}{\mathbf{s}(x^{-1})}.$$

Hence $\Delta(x) = \mathbf{s}(x)/\mathbf{s}(x^{-1})$. $\blacksquare$

From the above we see that $|U_1 : U_1 \cap xU_1x^{-1}| = \mathbf{s}(x^{-1})$. This approach thus yields a proof of the following corollary bypassing the theory of tidy subgroups. In [22] Willis asks for such a proof.

Corollary 5.3 ([**22, Corollary 3.11**]) *Let x be an element of a totally disconnected locally compact group G, and U a compact open subgroup of G. Then $|U : U \cap x^{-1}Ux|$ equals the minimum value $\mathbf{s}(x)$ if and only if $|U : U \cap xUx^{-1}|$ equals the minimum value $\mathbf{s}(x^{-1})$.*

Finally we prove property S1 of the scale function. As mentioned earlier this property can be easily deduced by using the connections between the scale function and tidy subgroups, but it can also be deduced more directly from the definition of the scale function by using Corollary 5.3.

Corollary 5.4 *Let G be a totally disconnected locally compact group. Then it is equivalent for an element x in G that $\mathbf{s}(x) = 1 = \mathbf{s}(x^{-1})$ and that x normalises some compact open subgroup of G.*

Proof Suppose that $\mathbf{s}(x) = 1 = \mathbf{s}(x^{-1})$. By Corollary 5.3 there is a compact open subgroup U of G such that both $|U : U \cap x^{-1}Ux| = \mathbf{s}(x) = 1$ and $|U : U \cap xUx^{-1}| = \mathbf{s}(x^{-1}) = 1$. And from these it is obvious that x must normalise U.

Conversely, if x normalises some compact open subgroup then it follows straight from the definition that $\mathbf{s}(x) = 1 = \mathbf{s}(x^{-1})$. ■

Remark Can property S2 be deduced directly from the definition of the scale function? If so, then that could possibly be used to simplify the proof of the connection between the scale function and tidy subgroups.

6 The Scale Function and Tidy Subgroups

Our aim in this section is to prove the link between tidy subgroups and the scale function.

Theorem 6.1 ([**22, Theorem 3.1**]) *Let G be a totally disconnected locally compact group and $x \in G$. Let U be a compact open subgroup of G. Then*

$$\mathbf{s}(x) = |U : U \cap x^{-1}Ux|$$

if and only if U is tidy for x.

Proof Before starting with the proof proper note that when x is periodic the theorem follows from Lemma 3.1. We will therefore assume that x is not periodic.

First we prove that, if $|U : U \cap x^{-1}Ux| = \mathbf{s}(x)$ then U is tidy for x. Consider the action of G on $\Omega = G/U$ and use the notation described in Section 1.4. Then $|U : U \cap x^{-1}Ux| = |\alpha_1 U|$. Looking at the definition of $\mathbf{s}(x)$, we see that for every $m \geq 0$ the following inequality holds

$$|\alpha_1 U| = |U : U \cap x^{-1}Ux| \leq |U_{-m,0} : U_{-m,0} \cap x^{-1}U_{-m,0}x| = |\alpha_1 U_{-m,0}|.$$

But, since $U_{-m,0} \leq U$ we know that $|\alpha_1 U_{-m,0}| \leq |\alpha_1 U|$ and hence $\alpha_1 U = \alpha_1 U_{-m,0}$ for every $m \geq 0$. Thus U must satisfy condition T1.

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Now we look at the latter half of the construction of a tidy subgroup in Section 4. Consider the graph Γ_{++} defined there. It has out-valency d^+ and in-valency d^-. Note that $d^+ = |\alpha_1 U| = |U : U \cap x^{-1}Ux|$. Denote the tidy subgroup constructed by U_0. In the tree T the in-valency is 1 and the out-valency is t, and $t = |U_0 : U_0 \cap x^{-1}U_0x|$. From [14] (see Proposition 4.3) we know that $t = d^+/d^-$, and we see that the assumption on U implies that $d^- = 1$. Thus Γ_+ is already a tree and it follows from Theorem 3.4 that U is tidy.

The technical part of the proof of the second half of Theorem 6.1 is taken care of in the proof of the following lemma whose proof we defer for the moment.

Lemma 6.2 (*cf.* [**20, Theorem 2**]) *Let G be a totally disconnected locally compact group and x an element of G. If $U^{(1)}$ and $U^{(2)}$ are compact open subgroups of G that are both tidy for $x \in G$, then*

$$|U^{(1)} : U^{(1)} \cap x^{-1}U^{(1)}x| = |U^{(2)} : U^{(2)} \cap x^{-1}U^{(2)}x|.$$

Conclusion of the Proof of Theorem 6.1 Assume for the moment that Lemma 6.2 is true. We have to show that if U is tidy then $|U : U \cap x^{-1}Ux|$ is equal to $\mathbf{s}(x)$. We have already shown that if V is a compact open subgroup such that $\mathbf{s}(x) = |V : V \cap x^{-1}Vx|$, then V is tidy for x. Using Lemma 6.2 we get

$$|U : U \cap x^{-1}Ux| = |V : V \cap x^{-1}Vx| = \mathbf{s}(x). \qquad \blacksquare$$

Now we return to Lemma 6.2. The proof is split up into a sequence of lemmas.

First, let us set up notation. Set $V = U^{(1)} \cap U^{(2)}$ and $\Omega = G/V$. We let α_0 denote the point in Ω such that $G_{\alpha_0} = V$. Set $\alpha_n = \alpha_0 x^n$. Construct Γ_+ in the usual fashion. The subgroups $U^{(1)}$ and $U^{(2)}$ both contain V and since $U^{(1)}$ and $U^{(2)}$ are both compact and V is open we see that V has finite index in both $U^{(1)}$ and $U^{(2)}$. For $i = 1, 2$ set $k_i = |U^{(i)} : V|$.

The following lemma is a restatement of some basic and well known facts about permutation groups, see Section 1.1.

Lemma 6.3 *The following holds with $i = 1, 2$. The orbit $A_0^{(i)} = \alpha_0 U^{(i)}$ is a finite block of imprimitivity in Ω for G, and $U^{(i)} = G_{\{A_0^{(i)}\}}$. If $\sim_i$ denotes the corresponding G-equivalence relation then the G-space $\Omega/\sim_i$ is isomorphic to the G-space $\Omega^{(i)} = G/U^{(i)}$. The size of the $\sim_i$-classes is equal to $k_i = |U^{(i)} : V|$.*

We make the following definitions for $i = 1, 2$. We consider the actions of G on $\Omega^{(i)} = G/U^{(i)} = \Omega/\sim_i$. Let $\alpha_0^{(i)}$ be a point in $\Omega^{(i)}$ such that $G_{\alpha_0^{(i)}} = U^{(i)}$. Furthermore, set $\alpha_n^{(i)} = \alpha_0^{(i)} x^n$. From these we define in the usual fashion graphs $\Gamma_+^{(i)}$. We get homomorphisms $\psi_i \colon \Gamma_+ \to \Gamma_+^{(i)}$. Note that $\Gamma_+^{(i)}$ is a trees, since $U^{(i)}$ is tidy for x. The out-valencies of vertices in $\Gamma_+^{(i)}$ are constant. Let $d_i = |U^{(i)} : U^{(i)} \cap x^{-1}U^{(i)}x|$ denote the out-valency of the vertices in $\Gamma_+^{(i)}$.

Lemma 6.4 *With the above notation, $k_i(d_i)^n \geq |\alpha_n V| \geq (d_i)^n/k_i$ for every $n \geq 1$ and $i = 1, 2$.*

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Proof Because $U^{(i)}$ is assumed to be tidy for x we know that $|\alpha_n^{(i)}U^{(i)}| = d_i^n$. But we identify the $\sim_i$-class $A_n^{(i)}$ in Ω with the point $\alpha_n^{(i)}$ in $\Omega^{(i)}$ and therefore the $\sim_i$-class $A_n^{(i)}$ in Ω must have precisely d_i^n distinct and disjoint images under $U^{(i)}$. From this we conclude that the orbit $\alpha_n U^{(i)}$ must have at least d_i^n elements (at least one element in each of the distinct images of $A_n^{(i)}$ under $U^{(i)}$) and at most $k_i d_i^n$ elements (the distinct images of $A_n^{(i)}$ under $U^{(i)}$ contain precisely $k_i d_i^n$ elements). Let $\{1, u_2^{(i)}, \ldots, u_{k_i}^{(i)}\}$ be a set of coset representatives of V in $U^{(i)}$. Then

$$\alpha_n U^{(i)} = \alpha_n V \cup \alpha_n(Vu_2^{(i)}) \cup \cdots \cup \alpha_n(Vu_{k_i}^{(i)}).$$

Whence

$$k_i d_i^n \geq |\alpha_n U^{(i)}| \geq |\alpha_n V| \geq \frac{|\alpha_n U^{(i)}|}{k_i} \geq \frac{d_i^n}{k_i}. \qquad \blacksquare$$

Lemma 6.5 *The graph Γ_+ is a tree.*

Proof The first step is to show that a $\sim_1$-class intersects a $\sim_2$-class in at most one point. It is enough to show that the $\sim_1$-class and $\sim_2$-class of α_0 intersect only in the point α_0. The two classes are the orbits $A_0^{(1)} = \alpha_0 U^{(1)}$ and $A_0^{(2)} = \alpha_0 U^{(2)}$. If $\beta \in A_0^{(1)} \cap A_0^{(2)}$ then there is some element $g \in G$ such that $\alpha_0 g = \beta$. Then g leaves both $A_0^{(1)}$ and $A_0^{(2)}$ invariant. But, $U^{(1)} = G_{\{A_0^{(1)}\}}$ and $U^{(2)} = G_{\{A_0^{(2)}\}}$, so $g \in U^{(1)} \cap U^{(2)} = V = G_{\alpha_0}$ and hence $\beta = \alpha_0 g = \alpha_0$.

Suppose now that Γ_+ is not a tree. Therefore some vertex in Γ_+ must have in-valency bigger than 1. Looking at the definition of Γ_+ we conclude that there must be two different directed paths $p_1 = \alpha_0, \beta_1, \beta_2, \ldots, \beta_n$ and $p_2 = \alpha_0, \gamma_1, \gamma_2, \ldots, \gamma_n$ with $\beta_n = \gamma_n$, and $\beta_i \neq \gamma_i$ for $i = 1, \ldots, n-1$. Applying the digraph homomorphism ψ_1 to the paths p_1 and p_2 we see that they have the same image in the tree $\Gamma_+^{(1)}$, that is $\psi_1(\beta_i) = \psi_1(\gamma_i)$ for $i = 1, \ldots, n$, and similarly $\psi_2(\beta_i) = \psi_2(\gamma_i)$ for $i = 1, \ldots, n$. Thus β_1 and γ_1 are distinct vertices in Γ_+ belonging to the same $\sim_1$-class (because $\psi_1(\beta_1) = \psi_1(\gamma_1)$), and they also belong to the same $\sim_2$-class. This contradicts the claim established above. We have a contradiction, and hence Γ_+ must be a tree. ■

Note that V acts on Γ_+ as a group of automorphisms. Thus all the vertices in the orbit $\alpha_n V$ have the same in-valency and the same out-valency. For $n \geq 0$ let t_n denote the out-valency of α_n. By the last lemma the in-valency of all the vertices in Γ_+ is 1.

Lemma 6.6 *The sequence $t_0, t_1, t_2, \ldots$ is decreasing and eventually settles to a constant value t. This number t is equal to both d_1 and d_2 and thus $d_1 = d_2$.*

Proof The out-valency of α_n is equal to the orbit size of α_{n+1} under $V_{\alpha_n} = V \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_n}$. Hence it is clear that t_n does not increase when n increases. The numbers $t_0, t_1, t_2, \ldots$ are all non-negative integers so there is a number t and a number N such that if $n \geq N$ then $t_n = t$.

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Since Γ_+ is a tree we see that

$$|\alpha_n V| = t_0 \cdot t_1 \cdot \cdots \cdot t_{n-1}.$$

Then we use that $t_n = t$ for all but finitely many values of n and therefore there is a constant l such that $t_0 \cdot t_1 \cdot \cdots \cdot t_{n-1} = lt^n$ for all n. By Lemma 6.4

$$k_i d_i^n \geq |\alpha_n V| = lt^n \geq d_i^n / k_i$$

for all $n \geq 0$ and $i = 1, 2$. This inequality can only hold if $t = d_i$. Therefore $d_1 = d_2$. ■

Proof of Lemma 6.2 By Lemma 6.6,

$$|U^{(1)} : U^{(1)} \cap x^{-1} U^{(1)} x| = d_1 = d_2 = |U^{(2)} : U^{(2)} \cap x^{-1} U^{(2)} x|. \qquad \blacksquare$$

This completes the proof of Theorem 6.1. We will continue with this thread of arguments and prove the following theorem, clarifying that the group V used in the above arguments is indeed tidy for x.

Theorem 6.7 (**[20, Lemma 10]**) *Let G be a totally disconnected locally compact group and x an element of G. If $U^{(1)}$ and $U^{(2)}$ are compact open subgroups of G that are both tidy for $x \in G$, then $U^{(1)} \cap U^{(2)}$ is tidy for x.*

Proof Adopt the same notation as above. Let N be a number so that the out-valency of α_n in Γ_+ is equal to t for all $n \geq N$. Let Δ denote the subtree of Γ_+ that is spanned by α_N and the vertices in Γ_+ that can be reached by directed paths from α_N. Similarly, define $\Delta^{(1)}$ as the subtree of $\Gamma_+^{(1)}$ spanned by $\alpha_N^{(1)}$ and the vertices in $\Gamma_+^{(1)}$ that can be reached from $\alpha_N^{(1)}$ by directed paths. Counting arguments like the ones applied in Lemma 6.4 show that when we think of the vertices in $\Delta^{(1)}$ as equivalence classes of points in Ω then each equivalence class contains precisely one vertex from Δ. Note also that from the way Γ_+ is defined we can conclude that $V_{\alpha_N} = V \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_N}$ acts transitively on the set of infinite directed paths starting at α_N. Because V is a closed subgroup and the valency of α_n is constant when $n \geq N$ we conclude that $V_+ \cap V \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_N}$ acts transitively on the set of infinite directed paths starting at α_N. Thus $V_+ \cap V \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_N}$ also acts transitively on the set of infinite directed paths in $\Gamma_+^{(1)}$ starting at $\alpha_N^{(1)}$. From this we conclude that V_+ acts transitively on the set of infinite directed paths in $\Gamma_+^{(1)}$ starting at $\alpha_0^{(1)}$.

Now take some element g from V. By the above we can find an element $g_+ \in V_+$ so that $\alpha_n^{(1)} g_+ = \alpha_n^{(1)} g$ for all $n \geq 0$. Then $g_- = g g_+^{-1}$ fixes $\alpha_n^{(1)}$ for all $n \geq 0$ so $g_- \in U_-^{(1)}$. Note also that g_- fixes α_0 so g_- is in V and thus in U_2. Because $g_- \in U_-^{(1)}$ we see that the family $\{x^n g_- x^{-n}\}_{n \geq 0}$ has an accumulation point. From Lemma 3.8 we conclude that $g_- \in U_-^{(2)}$. Then g_- fixes both $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$, and therefore the subsets $A_n^{(1)}$ and $A_n^{(2)}$ are invariant under g_- for all $n \geq 0$. In the proof of Lemma 6.5 we saw that $A_n^{(1)} \cap A_n^{(2)} = \{\alpha_n\}$ and therefore is α_n fixed by g_- for all $n \geq 0$. Thus g_- is in V_- and $g = g_+ g_- \in V_+ V_-$.

We have shown that V satisfies condition T1 and that therefore the out-valency of all the vertices in Γ_+ is the same. We have already proved that Γ_+ is a tree and we can conclude that V is tidy for x.

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7 More About the Scale Function

In this section we use the relationship between the scale function and tidy subgroups to get further information about the scale function.

Theorem 7.1 *Let G be a totally disconnected locally compact group. For every element $x \in G$ and every integer $n \geq 0$*

$$\mathbf{s}(x^n) = \mathbf{s}(x)^n.$$

Proof Let U be a compact open subgroup of U that is tidy for x. By Theorem 6.1

$$|U : U \cap x^{-1}Ux| = \mathbf{s}(x),$$

and by Corollary 3.5

$$|U : U \cap x^{-n}Ux^n| = |U : U \cap x^{-1}Ux|^n.$$

Corollary 3.6 says that since U is tidy for x it is also tidy for x^n. Thus, by combining Theorem 6.1, Corollary 3.5 and Corollary 3.6 we get

$$\mathbf{s}(x^n) = |U : U \cap x^{-n}Ux^n| = |U : U \cap x^{-1}Ux|^n = \mathbf{s}(x)^n. \qquad \blacksquare$$

The scale function does not in general behave well with respect to products. Let us continue with Example 1 where we left it in Section 3.

Example 1 continued From Theorem 7.1 we see that $\mathbf{s}(x) = n$. Indeed, if y is any element in G such that there is some sequence of distinct vertices $\{\beta_i\}_{i\in\mathbf{Z}}$ such that β_i is adjacent to β_{i+1} for all i and $\beta_i y = \beta_{i+k}$ for all i, then $\mathbf{s}(y) = n^k$. Let now g be an element in G_{α_0} such that $\alpha_i g = \alpha_{-i}$ for all i and let g' be an element in G_{α_1} such that $\alpha_i g' = \alpha_{-i+2}$. Set $h = gg'$ and note that $\alpha_i h = \alpha_{i+2}$ for all i. Both g and g' are periodic elements in G and therefore $\mathbf{s}(g) = 1 = \mathbf{s}(g')$ but $\mathbf{s}(gg') = \mathbf{s}(h) = n^2$.

Theorem 7.2 (**[20, see Theorem 3 and p. 357]**) *Let G be a totally disconnected locally compact group and x an element in G. If U is a compact open subgroup that is tidy for x then U is also tidy for every element in the double coset UxU. Furthermore, the scale function* $\mathbf{s}$ *is constant on UxU.*

Proof Once again we use the notation explained in Section 1.4. Take some element $y \in UxU$. Set $\beta_i = \alpha_0 y^i$. Then $\beta_1 = \alpha_0 y \in \alpha_1 U$ so $(\beta_0, \beta_1) = (\alpha_0, \alpha_0 y)$ is an edge in Γ_+. We see that $\beta_2 \in \alpha_2 U$ and in general $\beta_n \in \alpha_n U$. Thus

$$|U : U \cap y^{-1}Uy| = |\beta_1 U| = |\alpha_1 U| = |U : U \cap x^{-1}Ux|,$$

and, since $\beta_n U = \alpha_n U$, we get by using Corollary 3.5 that

$$\begin{aligned} |U : U \cap y^{-n}Uy^n| &= |\beta_n U| = |U : U \cap x^{-n}Ux^n| \\ &= |U : U \cap x^{-1}Ux|^n = |U : U \cap y^{-1}Uy|^n. \end{aligned}$$

Appealing again to Corollary 3.5 one concludes that U is tidy for y. By Theorem 6.1 we see that

$$\mathbf{s}(y) = |U : U \cap y^{-1}Uy| = |\alpha_1 U| = |U : U \cap x^{-1}Ux| = \mathbf{s}(x). \qquad \blacksquare$$

Corollary 7.3 *Let G be a totally disconnected locally compact group. Think of $\mathbf{N}$, the set of natural numbers, as having the discrete topology. The scale function $\mathbf{s}\colon G \to \mathbf{N}$ is continuous.*

Proof We have to show that if x is some element in G then there is an open neighbourhood of x such that on this neighbourhood $\mathbf{s}$ is constant. But by Theorem 7.2 UxU is precisely such a neighbourhood. ■

Before continuing our study of the general properties of the scale function let us make a brief digression and make some use of the above theorem and corollary.

Theorem 7.4 *Let G be a totally disconnected locally compact group. Suppose that G has a cocompact subgroup H such that $\mathbf{s}(h) = 1$ for all $h \in H$. Then $\mathbf{s}(x) = 1$ for all $x \in G$.*

If $\mathbf{s}(x) = 1$ for all $x \in G$ the we say that the group G is *uniscalar*. This result is an extension of [12, Lemma 1.7] where it is in addition assumed that H is normal. Before proving the theorem we must interpret the meaning of "cocompactness" in the permutation topology. The following lemma can be found in [15, Proposition 1] where it is stated for groups acting on trees.

Lemma 7.5 *Let G be a topological group acting transitively on a set Ω. Suppose the stabiliser G_α of a point $\alpha \in \Omega$ is both open and compact (for example, G is a closed permutation group acting transitively with finite suborbits). Then a subgroup H of G is cocompact if and only if H has only finitely many orbits on Ω.*

Remark The above lemma can be seen as a generalisation of the fact that if a group G acts transitively on a set Ω and H is a subgroup of G of finite index k then H has at most k orbits on Ω (see [3, Exercise 3(v)]).

Proof Suppose first that H is cocompact. This means that both the spaces of right and left cosets of H in G are compact (the map $Hx \mapsto x^{-1}H$ is a homeomorphism from the space of right cosets to the space of left cosets). Let X denote the set of left cosets of H in G. The quotient map $\pi\colon G \to X$ is open. The family of cosets $\{G_\alpha g\}_{g\in G}$ is an open covering of G and hence $\{\pi(G_\alpha g)\}_{g\in G}$ is an open covering of X. We are assuming that X is compact, so there is a finite subcovering $\pi(G_\alpha g_1), \dots, \pi(G_\alpha g_n)$ of X. Then $G = G_\alpha g_1 H \cup \cdots \cup G_\alpha g_n H$ and therefore $\Omega = (\alpha g_1)H \cup \cdots \cup (\alpha g_n)H$.

Suppose now that H has only finitely many orbits on Ω, say there are elements $g_1, \dots, g_n$ such that $\Omega = (\alpha g_1)H \cup \cdots \cup (\alpha g_n)H$. Then $G = G_\alpha g_1 H \cup \cdots \cup G_\alpha g_n H$ and $X = \pi(G_\alpha g_1) \cup \cdots \cup \pi(G_\alpha g_n)$. Each of the sets $\pi(G_\alpha g_i)$ is compact. We see that

X, the set of left cosets of H in G, is compact, because it can be written as a union of finitely many compact sets. ∎

Proof of Theorem 7.4 Suppose there is some element x in G with $\mathbf{s}(x) \neq 1$. Let U be a compact open subgroup that is tidy for x. Set $\Omega = G/U$ and adopt the usual notational set-up. Since H is cocompact it has only finitely many orbits on Ω and therefore there are some numbers i and j, with $i < j$, such that α_i and α_j are in the same H-orbit. If necessary, we can replace U with $x^{-i}Ux^i$ and renumber the points $\dots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots$, so we may assume that there is some element h in H such that $\alpha_0 h = \alpha_j$ with $j > 0$. Then $Ux^jU = UhU$. The subgroup U is tidy for x^j and $s(h) = s(x^j) = s(x)^j \neq 1$ by Theorem 7.2. We have reached a contradiction and the assumption that there is an element x in G with $\mathbf{s}(x) \neq 1$ must be wrong. ∎

The scale function can also be defined in terms of arbitrary compact open subgroup. First we need a preparatory lemma.

Lemma 7.6 *Let G be a totally disconnected locally compact group and $x \in G$. Then $\lim_{n\to\infty} |V : V \cap x^{-n}Vx^n|^{1/n}$ exists for every compact open subgroup V. If V and W are some compact open subgroups of G then*

$$\lim_{n\to\infty} |V : V \cap x^{-n}Vx^n|^{1/n} = \lim_{n\to\infty} |W : W \cap x^{-n}Wx^n|^{1/n}.$$

Proof For a compact open subgroup V of G we call the limit

$$\lim_{n\to\infty} |V : V \cap x^{-n}Vx^n|^{1/n}$$

the *index limit* of V.

First we assume that $W \le V$. Define $\Omega^{(V)} = G/V$ and $\Omega^{(W)} = G/W$. Let $\alpha_0^{(V)}$ be a point in $\Omega^{(V)}$ such that V is the stabiliser of $\alpha_0^{(V)}$ and similarly let $\alpha_0^{(W)}$ be a point in $\Omega^{(W)}$ such that W is the stabiliser of $\alpha_0^{(W)}$. Set $\alpha_i^{(V)} = \alpha_0^{(V)}x^i$ and $\alpha_i^{(W)} = \alpha_0^{(W)}x^i$. Then set $v_i = |\alpha_i^{(V)}V|$ and $w_i = |\alpha_i^{(W)}W|$. Note that

$$v_i = |V : V \cap x^{-i}Vx^i| \quad \text{and} \quad w_i = |W : W \cap x^{-i}Wx^i|.$$

Since $W \le V$, we can use V to define a G invariant equivalence relation $\sim$ on $\Omega^{(W)}$. The $\sim$-class of $\alpha_0^{(W)}$ is the set $\alpha_0^{(W)}V$. All the $\sim$-classes are finite and they all have the same number of elements $k = |V : W|$ and $\Omega^{(V)}$ can be identified with $\Omega^{(W)}/\sim$. Write $V = W \cup (Wg_2) \cup \dots \cup (Wg_k)$, where $g_2, \dots, g_k \in V$. From this we see that $w_i = |\alpha_i^{(W)}W| \ge |\alpha_i^{(W)}V|/k$. Since $|\alpha_i^{(W)}V| \ge |\alpha_i^{(V)}V| = v_i$ we conclude that $w_i \ge v_i/k$. Conversely, $\alpha_i^{(W)}W$ must be contained in the union of the $\sim$-classes contained in $\alpha_i^{(W)}V$. These $\sim$-classes correspond to the points in the set $\alpha_i^{(V)}V$ and $|\alpha_i^{(V)}V| = v_i$. Each $\sim$-class has k points and thus

$$w_i = |\alpha_i^{(W)}W| \le |\alpha_i^{(W)}V| \le k|\alpha_i^{(V)}V| = kv_i.$$

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We have now shown that $v_i/k \le w_i \le kv_i$ and therefore also $w_i/k \le v_i \le kw_i$. Thus, for every $n \ge 0$

$$(v_n/k)^{1/n} \le w_n^{1/n} \le (kv_n)^{1/n} \quad \text{and} \quad (w_n/k)^{1/n} \le v_n^{1/n} \le (kw_n)^{1/n}.$$

From these inequalities it is clear, by using the "squeeze theorem" of elementary Calculus, that if the limits

$$\lim_{n\to\infty} w_n^{1/n} \quad \text{and} \quad \lim_{n\to\infty} v_n^{1/n}$$

exist then they are equal and if one of the index limits exists then the other also does.

Assume now that U is tidy for x. From Corollary 3.5 we know that

$$|U : U \cap x^{-n}Ux^n| = |U : U \cap x^{-1}Ux|^n,$$

and thus (also using Theorem 6.1)

$$\lim_{n\to\infty} |U : U \cap x^{-n}Ux^n|^{1/n} = |U : U \cap x^{-1}Ux| = \mathbf{s}(x).$$

We now have at least one compact open subgroup, the tidy subgroup U, for which the index limit exists. Let now V be some compact open subgroup. Then $W = V \cap U$ is a compact open subgroup that is contained in U. Since the index limit for U exists then the index limit for W exists, by the above, and therefore, the index limit for V exists also. Thus the index limit exists for every compact open subgroup V.

Above we have been assuming that $W \le V$ and in that case we know that the two index limits are equal. In the general case we put $Z = V \cap W$. Then Z is a compact open subgroup of G and $Z \le V$ and $Z \le W$. Thus

$$\begin{aligned}\lim_{n\to\infty} |V : V \cap x^{-n}Vx^n|^{1/n} &= \lim_{n\to\infty} |Z : Z \cap x^{-n}Zx^n|^{1/n} \\ &= \lim_{n\to\infty} |W : W \cap x^{-n}Wx^n|^{1/n}.\end{aligned}$$

■

Theorem 7.7 *Let V be a compact open subgroup of a totally disconnected group G. Then for $x \in G$,*

$$\mathbf{s}(x) = \lim_{n\to\infty} |V : V \cap x^{-n}Vx^n|^{1/n}.$$

Furthermore, $\mathbf{s}(x) = 1$ if and only if there is a constant C such that

$$|V : V \cap x^{-n}Vx^n| \le C$$

for all integers $n \ge 0$.

Proof Let U be a compact open subgroup of G that is tidy for x. Lemma 7.6 gives

$$\lim_{n\to\infty} |V : V \cap x^{-n}Vx^n|^{1/n} = \lim_{n\to\infty} |U : U \cap x^{-n}Ux^n|^{1/n} = \mathbf{s}(x).$$

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Now suppose $\mathbf{s}(x) = 1$. If U is a compact open subgroup that is tidy for x then, by Corollary 3.5, $|U : U \cap x^{-n}Ux^n| = 1$ for all $n \geq 1$. The argument used in Lemma 7.6 now shows that there exists a constant C such that $|V : V \cap x^{-n}Vx^n| \leq C$ for all $n \geq 0$. For the reverse conclusion it is obvious, by the first part of the theorem, that the existence of such a constant C implies that $\mathbf{s}(x) = 1$. ■

Let G be a permutation group acting transitively on a set Ω. Assume that all suborbits are finite. The closure of G in the permutation topology is a totally disconnected locally compact group. The scale function on G is defined to be the restriction to G of the scale function on the closure of G. The following corollary is a translation into permutation group theoretic terms of Theorem 7.7.

Corollary 7.8 *Let G be a group acting transitively on a set Ω. Assume that all the suborbits (orbits of stabilisers of points) of G are finite. Let $\alpha_0 \in \Omega$ and $x \in G$. Set $\alpha_n = \alpha_0 x^n$.*

(i) *Then*

$$\mathbf{s}(x) = \lim_{n\to\infty} |\alpha_n G_{\alpha_0}|^{1/n}.$$

In particular the value of the limit is an integer.

(ii) $\mathbf{s}(x) = 1$ *if and only if there is some constant C such that $|\alpha_n G_{\alpha_0}| < C$ for all $n \geq 0$.*

Proof Replacing G with the closure of G in the permutation topology will not change the sizes of the suborbits. We can thus assume that G is closed in the permutation topology. Set $V = G_{\alpha_0}$. Note that V is a compact open subgroup of G. Now the result follows directly from Theorem 7.7. ■

Remark Let us continue with the notation in Corollary 7.8, but to simplify the exposition let us assume that G is a closed permutation group. We see that if there is a constant C such that $|\alpha_i G_{\alpha_0}| < C$ for all i then both $\mathbf{s}(x)$ and $\mathbf{s}(x^{-1})$ are equal to 1 and therefore, by property S1, there is some compact open subgroup U normalised by x. This is a special case of [1, Theorem 6(iii)], which in this context says that if K is a subgroup of G and there is a constant C such that $|\beta G_{\alpha_0}| < C$ for all β in the orbit $\alpha_0 K$ then there is some compact open subgroup of G that is normalised by K. The above quoted theorem of Bergman and Lenstra is an extension of the following theorem of Schlichting [18] (see also [1, Theorem 3]):

Suppose G is a closed permutation group acting transitively on a set Ω. If there is a constant C such that $|\beta G_\alpha| < C$ for all $\alpha, \beta \in \Omega$ then G has a normal compact open subgroup.

A natural question is to find additional conditions on G so that the assumption that every element in a totally disconnected locally compact group G normalises some compact open subgroup (that is, G is uniscalar) implies that G has a normal compact open subgroup.

Unfortunately the assumption that G is uniscalar seems to be a lot weaker than the assumptions in Schlichtings theorem. Even under the additional assumption that

G is compactly generated, uniscalar does not imply that there is a normal compact open subgroup. An example of a compactly generated uniscalar group that has no normal compact open subgroup was constructed by Bhattacharjee and Macpherson [2, Theorem 1.2] following earlier work of Kepert and Willis [13].

Let now Γ be a locally finite connected graph (undirected). The vertex set $V\Gamma$ of Γ carries a natural metric d such that for vertices α and β the distance $d(\alpha, \beta)$ is defined as the smallest possible number of edges in a path between α and β. For a vertex α_0 in Γ define $b_n = |\{\beta \in V\Gamma : d(\alpha_0, \beta) \leq n\}|$. If the group of automorphisms acts transitively on $V\Gamma$ then the value of b_n does not depend on the choice of α_0. We say Γ has subexponential growth if for every value of $a > 1$ there is a number N such that $b_n < a^n$ for all $n > N$. The proof of the following corollary is left to the reader.

Corollary 7.9 *Let Γ be a locally finite connected graph with subexponential growth. Set $G = \mathrm{Aut}(\Gamma)$ and assume that G acts transitively on $V\Gamma$. Then G is uniscalar.*

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