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A CONTINUITY CHARACTERIZATION OF ASPLUND SPACES

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Abstract

A Banach space is an Asplund space if every continuous gauge has a point where the subdifferential mapping is Hausdorff weak upper semi-continuous with weakly compact image. This contributes towards the solution of a problem posed by Godefroy, Montesinos and Zizler.

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1. Introduction

A continuous convex function ϕ on a nonempty open convex subset A of a Banach space $(X, \|\cdot\|)$ is said to be *strongly subdifferentiable* at $x \in A$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 \leq \phi(x+y) - \phi(x) - \phi'_{+}(x)(y) \leq \epsilon\|y\| \quad \text{for all } \|y\| < \delta,$$

and was the subject of an interesting study in [11]. The function ϕ is *Fréchet differentiable* at x if also $\phi'_{+}(x)(y)$ is linear in y.

A Banach space X is an *Asplund space* if every continuous convex function ϕ on a nonempty open convex subset A of X is Fréchet differentiable at the points of a dense G_δ subset of A. It is known that X is an Asplund space if X possesses an equivalent strongly subdifferentiable norm; see [5, Theorem 5.1, p. 68] and [9, Proposition 8, p. 64]. Further, if X is separable then it is an Asplund space if every equivalent norm has a nonzero point where it is strongly subdifferentiable [12, Theorem 1, p. 494]. It has remained an open question whether this last result can be extended to nonseparable spaces [12, Problem 6(v), p. 501]. Our aim here is to work towards such an extension.

Given a continuous convex function ϕ on a nonempty open convex subset A of a Banach space X, the *subdifferential* of ϕ at $x \in A$ is the nonempty weak* compact convex subset

$$\partial\phi(x) \equiv \{f \in X^* : f(y) \leq \phi'_{+}(x)(y) \text{ for all } y \in X\}.$$

The set-valued *subdifferential mapping* $x \mapsto \partial\phi(x)$ is always *Hausdorff weak* upper semi-continuous* on A; that is, given $x \in A$ and weak* open neighbourhood W of 0 in X^*, there exists $\delta > 0$ such that

$$\partial\phi(B(x;\delta)) \subseteq \partial\phi(x) + W$$

by [13, Proposition 2.5, p. 19]. Further, ϕ is strongly subdifferentiable at $x \in A$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\partial\phi(B(x;\delta)) \subseteq \partial\phi(x) + \epsilon B(X^*)$$

by [8, Theorem 3.2, p. 28]. We work with the slightly weaker property; we say that the subdifferential mapping $x \mapsto \partial\phi(x)$ is *Hausdorff weak upper semi-continuous* at $x \in A$ if, given a weak open neighbourhood V of 0 in X^*, there exists $\delta > 0$ such that

$$\partial\phi(B(x;\delta)) \subseteq \partial\phi(x) + V.$$

This has been studied in [1] and for subdifferentials of the norm in [10]. Contreras and Payá showed that X is an Asplund space if it possesses an equivalent norm whose subdifferential mapping is Hausdorff weak upper semi-continuous on its unit sphere [3, Theorem 1.2, p. 453]. Here we show that X is an Asplund space if every continuous gauge p on X has a point in its domain where the subdifferential mapping $x \mapsto \partial p(x)$ is Hausdorff weak upper semi-continuous with weakly compact image. (We note that every continuous gauge p on a Banach space X is always strongly differentiable at 0 so is always Hausdorff weak upper semi-continuous there but the subdifferential $\partial p(0)$ is not in general weakly compact.)

2. The density property

We explore the effect of Hausdorff weak upper semi-continuity on higher dual spaces. Consider a bounded closed convex set K with $0 \in \operatorname{int} K$ in a Banach space X. The *gauge* p of X defined by

$$p(x) \equiv \inf\{\lambda > 0 : x \in \lambda K\}$$

is a continuous positive sublinear functional on X. The *polar* of K is the subset K^0 of X^* defined by

$$K^0 \equiv \{f \in X^* : f(x) \leq 1 \text{ for all } x \in K\}$$

and is weak* compact convex and $0 \in \operatorname{int} K^0$. The gauge p^* of K^0 on X^* is continuous and weak* lower semi-continuous. We denote by K^{00} the polar of K^0 in X^{**} and by p^{**} the gauge of K^{00} on X^{**}, and note that $K^{00} = \overline{\hat{K}}^{\omega*}$ and $p^{**}|_{\hat{X}} = p$, [7, Lemma 3.1(i), p. 255].

We now characterize Hausdorff weak upper semi-continuity by a density property. Such a characterization for subdifferentials of norms was given in [6, Theorem 3.1, p. 103] and was proved more generally for subdifferentials of proper lower semi-continuous convex functions in [1, Theorem 3.1, p. 98]. Since our theorem in Section 3 concerns gauges of bounded closed convex sets we include a direct proof for subdifferentials of gauges.

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THEOREM 2.1. *Consider a Banach space X and a bounded closed convex subset K with $0 \in \operatorname{int} K$. The continuous gauge p of K has subdifferential mapping $x \mapsto \partial p(x)$ Hausdorff weak upper semi-continuous at $x \in X$ if and only if*

$$\widehat{\partial p(x)} \text{ is weak}^* \text{ dense in } \partial p^{**}(\hat{x}).$$

PROOF. Suppose that the density property does not hold. Then there exists $\mathfrak{F}_0 \in \partial p^{**}(\hat{x})$ which we can strongly separate from $\overline{\widehat{\partial p(x)}}^{\omega *}$ by a weak* continuous linear functional $F_0 \in S(X^{**})$. So there exists a weak* neighbourhood $N^*_{F_0}$ of 0 in X^{***} generated by F_0 such that

$$\mathfrak{F}_0 \notin \widehat{\partial p(x)} + N^*_{F_0}.$$

Since $\hat{K}^0$ is weak* dense in K^{000}, for each $n \in \mathbb{N}$ there exists $f_n \in K^0$ such that

$$|(f_n - \mathfrak{F}_0)(\hat{x})| < \frac{1}{n} \quad \text{and} \quad |(f_n - \mathfrak{F}_0)(F_0)| < \frac{1}{n}.$$

Since $\mathfrak{F}_0 \in \partial p^{**}(\hat{x})$ then $\mathfrak{F}_0(\hat{x}) = p^{**}(\hat{x})$ and $\mathfrak{F}_0(F) \le p^{**}(F)$ for all $F \in X^{**}$. Then $|f_n(\hat{x}) - p^{**}(\hat{x})| < 1/n$ and, since $f_n \in K^0$, $f_n(y) \le p(y)$ for all $y \in X$. So $f_n(\hat{x}) \ge p^{**}(\hat{x}) - 1/n = p(x) - 1/n$ and $f_n(y - x) \le p(y) - p(x) + 1/n$ for all $y \in X$. By the Brøndsted–Rockafellar theorem for each $n \in \mathbb{N}$ there exist $y_n \in X$ and $f_{y_n} \in \partial p(y_n)$ such that

$$\|f_n - f_{y_n}\| \le \frac{1}{\sqrt{n}} \quad \text{and} \quad \|x - y_n\| \le \frac{1}{\sqrt{n}}$$

by [13, Theorem 3.17, p. 48]. But if the subdifferential mapping $x \mapsto \partial p(x)$ is Hausdorff weak upper semi-continuous at x, then, for sufficiently large $n \in \mathbb{N}$,

$$f_{y_n} \in \partial p(x) + N_{F_0}$$

where N_{F_0} is the weak neighbourhood of 0 in X^*, the restriction of $N^*_{F_0}$ in X^{***}. However,

$$|(f_{y_n} - \mathfrak{F}_0)(F_0)| \le \|f_{y_n} - f_n\| + |(f_n - \mathfrak{F}_0)(F_0)| \le \frac{1}{\sqrt{n}} + \frac{1}{n},$$

which contradicts our original separation.

Conversely, suppose that

$$\widehat{\partial p(x)} \text{ is weak}^* \text{ dense in } \partial p^{**}(\hat{x}).$$

Consider a weak neighbourhood V of 0 in X^*. Now V is the restriction of a weak* neighbourhood V^* of 0 in X^{***}. Since the subdifferential mapping $F \mapsto \partial p^{**}(F)$ is Hausdorff weak* upper semi-continuous at $\hat{x} \in X^{**}$, there exists $\delta > 0$ such that

$$\partial p^{**}(B(\hat{x}, \delta)) \subseteq \partial p^{**}(\hat{x}) + \tfrac{1}{2} V^* \quad \text{but} \quad \partial p^{**}(\hat{x}) \subseteq \widehat{\partial p(x)} + \tfrac{1}{2} V^*$$

so $\partial p(B(x, \delta)) \subseteq \partial p(x) + V$. □

It is instructive to see how the density property has implications for the density of extreme points.

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COROLLARY 2.2. *A continuous gauge p on a Banach space X with subdifferential mapping $x \mapsto \partial p(x)$ Hausdorff weak upper semi-continuous at $x \in X$ satisfies:*

(i) $\partial p^{**}(\hat{x}) = \overline{\text{co ext } \widehat{\partial p(x)}}^{\omega*}$;

(ii) $\text{ext } \partial p^{**}(\hat{x}) \subseteq \overline{\text{ext } \widehat{\partial p(x)}}^{\omega*}$.

PROOF. Consider $\mathfrak{F}_0$ an extreme point of $\partial p^{**}(\hat{x})$. Choquet's theorem [4, p. 77], gives us that weak* slices of $\partial p^{**}(\hat{x})$, sets of the form

$$Sl(\partial p^{**}(\hat{x}),\ F,\ \delta) \equiv \{\mathfrak{F} \in \partial p^{**}(\hat{x}) : \mathfrak{F}(F) > \sup \partial p^{**}(\hat{x})(F) - \delta\},$$

containing $\mathfrak{F}_0$ form a weak* neighbourhood base for $\mathfrak{F}_0$. It then follows from Theorem 2.1 that there exists an element of $\widehat{\partial p(x)}$ in the slice and, moreover, an extreme point of $\widehat{\partial p(x)}$. Since by the Krein–Milman theorem we have that $\partial p^{**}(\hat{x}) = \overline{\text{co ext } \partial p^{**}(\hat{x})}^{\omega*}$, we deduce that

$$\partial p^{**}(\hat{x}) = \overline{\text{co ext } \widehat{\partial p(x)}}^{\omega*}.$$

But further, since $\widehat{\partial p(x)} \subseteq \partial p^{**}(\hat{x})$, it follows that

$$\text{ext } \partial p^{**}(\hat{x}) \subseteq \overline{\text{ext } \widehat{\partial p(x)}}^{\omega*}$$

by [4, Theorem 3.41, p. 78]. This concludes the proof. □

3. The continuity characterization

A Banach space X is an Asplund space if and only if its dual x^* has the Radon–Nikodým property [13, Theorem 5.7, p. 82]. We exploit the following characterization of the Radon–Nikodým property to establish our theorem.

PROPOSITION 3.1 [2, Corollary 3.76, (1) ⇔ (3), p. 67]. *A Banach space X has the Radon–Nikodým property if and only if every bounded closed convex subset K of X contains an extreme point of $\overline{\hat{K}}^{\omega^*}$.*

THEOREM 3.2. *A Banach space X is an Asplund space if every continuous gauge p on X has a point $x_0 \in X$ where the subdifferential mapping $x \mapsto \partial p(x)$ is Hausdorff weak upper semi-continuous and the subdifferential $\partial p(x_0)$ is weakly compact.*

PROOF. Consider K a bounded closed convex subset of X^*. We may assume that $0 \in K$. The support function p of K on X is

$$p(x) = \sup\{f(x) : f \in K\}.$$

Since K is bounded, p is continuous. Further, p is the gauge of the set

$$\{x \in X : p(x) \leq 1\} = K_0 \equiv \{x \in X : f(x) \leq 1 \text{ for all } f \in K\}$$

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which is a bounded closed convex subset of X and $0 \in \operatorname{int} K_0$. Now the polar of K_0 in X^*,

$$K_0^0 \equiv \{f \in X^* : f(x) \le 1 \text{ for all } x \in K_0\},$$

is a bounded weak* closed convex subset of X^* and $K_0^0 = \overline{K}^{\omega^*}$, by [4, Theorem 4.32, p. 119]. Consider p^* the gauge of K_0^0 on X^* and the polar of K_0^0 in X^{**},

$$K_0^{00} \equiv \{F \in X^{**} : F(f) \le 1 \text{ for all } f \in {K_0}^0\}.$$

Finally, consider p^{**} the gauge of $K_0^{00} = K^0$ on X^{**}. If the subdifferential mapping $x \mapsto \partial p(x)$ is Hausdorff weak upper semi-continuous at $x_0 \in X$, then, by Theorem 2.1,

$$\partial p^{**}(\hat{x_0}) = \overline{\widehat{\partial p(x_0)}}^{\omega^*}.$$

But if also $\partial p(x_0)$ is weakly compact in X^* then

$$\partial p^{**}(\hat{x_0}) = \widehat{\partial p(x_0)} \subseteq \hat{X}^*.$$

Now by the Krein–Milman theorem, $\partial p^{**}(\hat{x_0})$ has an extreme point, some $\hat{f_0} \in \widehat{\partial p(x_0)}$. However, $\partial p^{**}(\hat{x_0})$ is an extreme subset of $K_0^{000} = K^{00} = \overline{\hat{K}}^{\omega^*}$, so $\hat{f_0}$ is an extreme point of $\overline{\hat{K}}^{\omega^*}$. Suppose that $f_0 \notin K$. Then we can separate f_0 and K by a weakly closed hyperplane. Then we can separate $\hat{f_0}$ and $\overline{\hat{K}}^{\omega^*}$ by a weak* closed hyperplane. But this contradicts $\hat{f_0} \in \overline{\hat{K}}^{\omega^*}$. So then $\overline{\hat{K}}^{\omega^*}$ has an extreme point in K. By Proposition 3.1 we have that X^* has the Radon–Nikodým property and it follows that X is an Asplund space. □

4. Remarks

Our Theorem 3.2 goes some way towards an extension of the result of Godefroy *et al.* [12, Theorem 1, p. 494]. However, it is apparent that the weakly compact condition on the subdifferential, although satisfying the requirements of Proposition 3.1, is more stringent than is necessary. So, any advance with our line of argument requires us to explore further the relations given in Corollary 2.2.

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