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ON DELAY DIFFERENTIAL INEQUALITIES OF HIGHER ORDER

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ABSTRACT. Consider the nth order $(n \geq 1)$ delay differential inequalities $y^{(n)}(t)+(-1)^{n+1}[p^n+q(t)]y(t-n\tau)\leq 0$ (1) and $y^{(n)}(t)+(-1)^{n+1}[p^n+q(t)]y(t-n\tau)\geq 0$ (2) and the delay differential equation $y^{(n)}(t)+(-1)^{n+1}[p^n+q(t)]y(t-n\tau)=0$ (3), where $q(t)\geq 0$ is a continuous function and p, τ are positive constants. Under the condition $p\tau e>1$ we prove that when n is odd (1) has no eventually positive solutions, (2) has no eventually negative solutions, and (3) has only oscillatory solutions and when n is even (1) has no eventually negative bounded solutions, (2) has no eventually positive bounded solutions, and every bounded solution of (3) is oscillatory. The condition $p\tau e>1$ is sharp. The above results, which generalize previous results by Ladas and by Ladas and Stavroulakis for first order delay differential inequalities, are caused by the retarded argument and do not hold when $\tau=0$.

1. **Introduction.** In this paper we consider the nth order $(n\geq 1)$ delay differential inequalities

$$y^{(n)}(t)+(-1)^{n+1}[p^n+q(t)]y(t-n\tau)\leq 0 \tag{1}$$

and

$$y^{(n)}(t)+(-1)^{n+1}[p^n+q(t)]y(t-n\tau)\geq 0 \tag{2}$$

and the delay differential equation

$$y^{(n)}(t)+(-1)^{n+1}[p^n+q(t)]y(t-n\tau)=0, \tag{3}$$

where $q(t)\geq 0$ is continuous function for $t\in\mathbb{R}^+$ and p, τ are positive constants.

In the particular case where $n=1$ and $q(t)=0$, Ladas and Stavroulakis [4] obtained the following result:

The condition

$$p\tau e>1 \tag{4}$$

is necessary and sufficient so that: (1) *has no eventually positive solutions,* (2) *has no eventually negative solutions, and* (3) *has only oscillatory solutions.*

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Using this result we examine the oscillatory character of all solutions of the above mentioned relations (1), (2) and (3) when n is odd, and of the bounded solutions only when n is even. More precisely, if (4) holds we prove the following results:

(i) *when n is odd:* (1) *has no eventually positive solutions,* (2) *has no eventually negative solutions, and* (3) *has only oscillatory solutions*

(ii) *when n is even:* (1) *has no eventually negative bounded solutions,* (2) *has no eventually positive bounded solutions, and every bounded solution of* (3) *is oscillatory.*

As we shall explain in the following section our condition is the "best possible". The above results are caused by the retarded argument and are not valid when $\tau = 0$.

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros.

2. **Main results.** At first we consider the case where n is odd.

THEOREM 1. *Consider the delay differential inequality*

$$(1)' \qquad y^{(n)}(t)+[p^n+q(t)]y(t-n\tau)\le 0, \quad n \text{ odd}$$

where p, τ are positive constants and $q(t)\ge 0$ is a continuous function for $t\in\mathbb{R}^+$. Assume that

$$(4) \qquad p\tau e>1.$$

Then $(1)'$ *has no eventually positive solutions.*

Proof. Otherwise there exists a solution $y(t)$ of $(1)'$ such that for t_0 sufficiently large

$$y(t)>0, \qquad t>t_0.$$

Then $y(t-n\tau)>0$ for $t>t_0+n\tau$, and, from $(1)'$, $y^{(n)}(t)<0$ for $t>t_0+n\tau$. Now the conditions

$$y(t)>0 \quad \text{and} \quad y^{(n)}(t)<0 \quad \text{for} \quad t>t_0+n\tau$$

and the fact that n is odd imply that there exists an even integer l, $0\le l<n$ such that for $t>t_0+n\tau$

$$(5) \qquad \begin{cases} y^{(k)}(t)>0 & \text{for} \quad k=0,1,2,\ldots,l \\ \text{and} & \\ (-1)^k y^{(k)}(t)>0 & \text{for} \quad k=l+1,l+2,\ldots,n. \end{cases}$$

We claim that $l=0$, i.e. for $t>t_0+n\tau$

$$(6) \qquad (-1)^k y^{(k)}(t)>0 \quad \text{for} \quad k=0,1,2,\ldots,n.$$

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To prove it assume that $l>0$. Then integrating $(1)'$ $n-l$ times from t_1 to t for t_1 sufficiently large, we obtain

$$\begin{aligned} y^{(l)}(t) &\le \sum_{k=0}^{n-l-1} \frac{(t-t_1)^k}{k!} y^{(l+k)}(t_1) - \int_{t_1}^{t} \frac{(t-s)^{n-l-1}}{(n-l-1)!} [p^n+q(s)]y(s-n\tau)\,ds \\ &\le \sum_{k=0}^{n-l-1} \frac{(t-t_1)^k}{k!} y^{(l+k)}(t_1) - \frac{y(t_1-n\tau)}{(n-l-1)!} p^n \int_{t_1}^{t} (t-s)^{n-l-1}\,ds \\ &= \sum_{k=0}^{n-l-1} \frac{(t-t_1)^k}{k!} y^{(l+k)}(t_1) - c(t-t_1)^{n-l} \end{aligned}$$

where

$$c = \frac{y(t_1-n\tau)}{(n-l)!} p^n$$

is a positive constant. This implies that

$$y^{(l)}(t) \to -\infty \quad \text{as} \quad t \to +\infty$$

which contradicts (5) and proves (6).

Set

$$x(t) = y^{(n-1)}(t) - py^{(n-2)}(t-\tau) + p^2 y^{(n-3)}(t-2\tau) - \cdots + p^{n-1} y(t-(n-1)\tau).$$

Then, in view of (5)

$$x(t) > 0. \tag{7}$$

Observe that

$$x'(t) = y^{(n)}(t) - py^{(n-1)}(t-\tau) + p^2 y^{(n-2)}(t-2\tau) - \cdots + p^{n-1} y'(t-(n-1)\tau)$$

and therefore

$$x'(t) + px(t-\tau) = y^{(n)}(t) + p^n y(t-n\tau) \le -q(t)y(t-n\tau) \le 0.$$

That is

$$x'(t) + px(t-\tau) \le 0. \tag{8}$$

But $p\tau e > 1$ which implies that (8) has no eventually positive solutions. This contradicts (7) and the proof of Theorem 1 is complete.

THEOREM 2. *Consider the delay differential inequality*

$$y^{(n)}(t) + [p^n + q(t)]y(t-n\tau) \ge 0, \quad n \text{ odd} \tag{2$'$}$$

subject to the hypotheses of Theorem 1. *Then* $(2)'$ *has no eventually negative solutions.*

Proof. The result follows immediately from the observation that if $y(t)$ is a solution of $(2)'$ then $-y(t)$ is a solution of $(1)'$.

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From Theorems 1 and 2 it follows that when n is odd the delay differential equation (3) has no eventually positive or eventually negative solutions and therefore we are led to the following conclusion.

COROLLARY 1. *Consider the delay differential equation*

$$y^{(n)}(t)+[p^n+q(t)]y(t-n\tau)=0,\ n\ \textit{odd} \tag{3$'$}$$

subject to the hypotheses of Theorem 1. *Then every solution of* (3)′ *oscillates.*

Now we turn to the case where n is even.

THEOREM 3. *Consider the delay differential inequality*

$$y^{(n)}(t)-[p^n+q(t)]y(t-n\tau)\geq 0, \quad n\ \textit{even} \tag{2$''$}$$

subject to the hypotheses of Theorem 1. *Then* (2)″ *has no eventually positive bounded solutions.*

Proof. Otherwise there exists a bounded solution $y(t)$ of (2)″ such that for t_0 sufficiently large

$$y(t)>0, \qquad t>t_0.$$

Then $y(t-n\tau)>0$ for $t>t_0+n\tau$ and, from (2)″, $y^{(n)}(t)>0$ for $t>t_0+n\tau$. Since $y(t)$ is bounded, it follows that

$$y^{(n)}(t)>0,\ y^{(n-1)}(t)<0,\ldots,\ y''(t)>0,\ y'(t)<0,\ y(t)>0$$

i.e.

$$(-1)^k y^{(k)}(t)>0, \qquad k=0,1,2,\ldots,n.$$

Set

$$x(t)=y^{(n-1)}(t)-py^{(n-2)}(t-\tau)+p^2y^{(n-3)}(t-2\tau)-\cdots-p^{n-1}y(t-(n-1)\tau) \tag{9}$$

which for sufficiently large t is negative. Differentiating both sides of (9), we obtain

$$x'(t)=y^{(n)}(t)-py^{(n-1)}(t-\tau)+p^2y^{(n-2)}(t-2\tau)-\cdots-p^{n-1}y'(t-(n-1)\tau).$$

Observe that

$$x'(t)+px(t-\tau)=y^{(n)}(t)-p^ny(t-n\tau)\geq q(t)y(t-n\tau)\geq 0.$$

That is

$$x'(t)+px(t-\tau)\geq 0. \tag{10}$$

But $p\tau e>1$ which implies that (10) has no eventually negative solutions. This contradicts the fact that $x(t)$ is negative and the proof of Theorem 3 is complete.

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THEOREM 4. *Consider the delay differential inequality*

$$(1)'' \qquad y^{(n)}(t)-[p^n+q(t)]y(t-n\tau)\leq 0, \quad n \text{ even}$$

subject to the hypotheses of Theorem 1. *Then* (1)″ *has no eventually negative bounded solutions.*

Proof. The result follows immediately from the observation that if $y(t)$ is a bounded solution of (1)″ then $-y(t)$ is a bounded solution of (2)″.

From Theorems 3 and 4 it follows that when n is even the delay differential equation (3) has no eventually positive or eventually negative bounded solutions and therefore we are led to the following conclusion.

COROLLARY 2. *Consider the delay differential equation*

$$(3)'' \qquad y^{(n)}(t)-[p^n+q(t)]y(t-n\tau)=0, \quad n \text{ even}$$

subject to the hypotheses of Theorem 1. *Then every bounded solution of* (3)″ *oscillates.*

Next we show that when $q(t)=0$ that is, in the case of the delay differential inequalities

$$\text{(I)} \qquad y^{(n)}(t)+(-1)^{n+1}p^n y(t-n\tau)\leq 0$$

and

$$\text{(II)} \qquad y^{(n)}(t)+(-1)^{n+1}p^n y(t-n\tau)\geq 0$$

and the delay differential equation

$$\text{(III)} \qquad y^{(n)}(t)+(-1)^{n+1}p^n y(t-n\tau)=0$$

the condition

$$(4) \qquad p\tau e>1$$

is sharp. More precisely the following result holds:

THEOREM 5. *Assume that*

$$(11) \qquad p\tau e\leq 1.$$

Then for n odd: (I) *has eventually positive solutions,* (II) *has eventually negative solutions, and* (III) *has non-oscillatory solutions while for n even:* (I) *has eventually negative bounded solutions,* (II) *has eventually positive bounded solutions, and* (III) *has non-oscillatory bounded solutions.*

Proof. For n odd, one can show directly that

$$y(t)=e^{-(1/\tau)t} \text{ is a positive solution of (I)}$$

and

$$y(t) = -e^{-(1/\tau)t} \text{ is a negative solution of (II).}$$

Next we will prove that for n odd (III) has a non-oscillatory solution of the form $y(t) = e^{\lambda t}$ with $-1/\tau \leq \lambda < 0$. This follows from the observation that the characteristic equation of (III), namely,

$$F(\lambda) = \lambda^n + p^n e^{-\lambda n \tau}$$

satisfies the condition

$$F\left(-\frac{1}{\tau}\right)F(0) = \left(\frac{p}{\tau}\right)^n [(p\tau e)^n - 1] \leq 0.$$

In a similar way one can show that for n even,

$$y(t) = -e^{-(1/\tau)t} \text{ is a negative bounded solution of (I),}$$

$$y(t) = e^{-(1/\tau)t} \text{ is a positive bounded solution of (II),}$$

and that (III) has a non-oscillatory bounded solution of the form $e^{\lambda t}$ with $-(1/\tau) \leq \lambda < 0$.

In view of Theorems 1, 2, 3, 4, 5 and Corollaries 1, 2 we conclude that the following result is true.

COROLLARY 3. *The condition*

$$p\tau e > 1 \tag{4}$$

is necessary and sufficient so that:

(i) *when n is odd:* (I) *has no eventually positive solutions,* (II) *has no eventually negative solutions, and* (III) *has only oscillatory solutions*

(ii) *when n is even:* (I) *has no eventually negative bounded solutions,* (II) *has no eventually positive bounded solutions, and every bounded solution of* (III) *is oscillatory.*

EXAMPLE 1. The odd order delay differential inequality

$$y'''(t) + y(t-1) \leq 0$$

has the positive solution $y(t) = e^{-3t}$ and the inequality

$$y'''(t) + y(t-1) \geq 0$$

has the negative solution $y(t) = -e^{-3t}$.

On the other hand the even order delay differential inequality

$$y^{(4)}(t) - \tfrac{1}{81} y(t-4) \leq 0$$

has the negative bounded solution $y(t) = -e^{-t}$ and the inequality

$$y^{(4)}(t) - \tfrac{1}{81} y(t-4) \geq 0$$

has the positive bounded solution $y(t) = e^{-t}$.

As expected condition (4) is not satisfied for all these inequalities.

EXAMPLE 2. The odd order delay differential equation

$$y'''(t)+y\left(t-\frac{3\pi}{2}\right)=0$$

has the oscillatory solutions $y_1(t)=\sin t$ and $y_2(t)=\cos t$. Furthermore condition (4) is satisfied and therefore all solutions of this equation are oscillatory.

On the other hand the even order delay differential equation

$$y^{(4)}(t)-\frac{256}{e^4}\,y(t-1)=0$$

admits the non-oscillatory bounded solution $y(t)=e^{-4t}$. As expected condition (4) is not satisfied for this equation.

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