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A LATTICE-THEORETIC DESCRIPTION OF THE LATTICE OF HYPERINVARIANT SUBSPACES OF A LINEAR TRANSFORMATION

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1. Introduction. If A is a (linear) transformation acting on a (finite-dimensional, non-zero, complex) Hilbert space H the family of (linear) subspaces of H which are invariant under A is denoted by Lat A. The family of subspaces of H which are invariant under every transformation commuting with A is denoted by Hyperlat A. Since A commutes with itself we have Hyperlat $A \subseteq$ Lat A. Set-theoretic inclusion is an obvious partial order on both these families of subspaces. With this partial order each is a complete lattice; joins being (linear) spans and meets being set-theoretic intersections. Also, each has H as greatest element and the zero subspace (0) as least element. With this lattice structure being understood, Lat A (respectively Hyperlat A) is called the *lattice of invariant* (respectively, *hyperinvariant*) *subspaces of A*. A description of Hyperlat A is given in [**4**]. This description is partly linear-algebraic and partly lattice-theoretic. In the present paper this description, and some results of [**3**], are used to establish a purely lattice-theoretic description, in terms of Lat A, of Hyperlat A. That two transformations with the same invariant subspaces have the same hyperinvariant subspaces is an immediate consequence of this description.

2. Notation and preliminaries. A lattice L is the *direct product* of its sublattices $L_1, L_2, \ldots, L_m$ if each element a of L is uniquely expressible in the form $a = a_1 \vee a_2 \vee \ldots \vee a_m$ with $a_i \in L_i$ in such a way that the lattice operations in L can be performed "coordinate-wise". We write $L = \otimes_{i=1}^{m} L_i$. A lattice L is *reducible* if it is the direct product of two sublattices each having more than one element. Otherwise L is *irreducible.* An *isomorphism* (respectively, *anti-isomorphism*) of a lattice L_1 onto a lattice L_2 is a one-to-one mapping ν of L_1 onto L_2 with the property that $a \leqq b$ $(a, b \in L_1)$ implies $\nu(a) \leqq \nu(b)$ (respectively, $\nu(b) \leqq \nu(a)$) and conversely. An *automorphism* (respectively, *anti-automorphism*) of a lattice L is an isomorphism (respectively, anti-isomorphism) of L onto itself. Any automorphism ν of L satisfies $\nu(a \vee b) = \nu(a) \vee \nu(b)$ and $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$ $(a, b \in L)$. Even more is true. If $\bigvee_{\alpha \in \Omega} a_\alpha$ and $\bigwedge_{\alpha \in \Omega} a_\alpha$ both exist in L then $\bigvee_{\alpha \in \Omega} \nu(a_\alpha)$ and $\bigwedge_{\alpha \in \Omega} \nu(a_\alpha)$ both exist in L and $\nu(\bigvee_{\alpha \in \Omega} a_\alpha) = \bigvee_{\alpha \in \Omega} \nu(a_\alpha)$, $\nu(\bigwedge_{\alpha \in \Omega} a_\alpha) = \bigwedge_{\alpha \in \Omega} \nu(a_\alpha)$. The element a of L is *fixed* by the automorphism ν if $\nu(a) = a$. The set of elements of L which are fixed by every auto-

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morphism is denoted by Fix (L). An element b of L is *join-irreducible* if $b = x \vee y$ $(x, y \in L)$ implies $b = x$ or $b = y$. If c and d are elements of L, c *covers* d if $d < c$ and there is no element z of L satisfying $d < z < c$.

Throughout this paper all Hilbert spaces will be finite-dimensional, non-zero and complex. If A is a transformation acting on a Hilbert space H we denote by $\mathcal{N}(A)$ (respectively, $\mathcal{R}(A)$) the null space (respectively, range) of A. For any vector x of H, the subspace of H spanned by the vectors x, Ax, $A^2x, \ldots$ is denoted by $Z(x; A)$ and is called the *A-cyclic subspace generated by x*. A subspace M of H is *A-cyclic* if $M = Z(x; A)$ for some vector x. The transformation A is *cyclic* if H is A-cyclic.

3. A lattice-theoretic description of Hyperlat A. We give a purely lattice-theoretic description of the lattice of hyperinvariant subspaces of a linear transformation in terms of its lattice of invariant subspaces. First we consider the nilpotent case.

THEOREM 3.1. *If N is a nilpotent transformation acting on a (finite-dimensional) Hilbert space*, Hyperlat N = Fix (Lat N). *In other words, a subspace is hyperinvariant if and only if it is invariant and is fixed by every automorphism of* Lat N.

Proof. Let N act on space H. Let the minimum polynomial m_N of N be $m_N(z) = z^k$. Then $k \geqq 1$. The proof of the theorem is in several steps. In each step the first statement is proved.

(i) *The join-irreducible elements of* Lat N *are precisely the N-cyclic subspaces*. Let M be a join-irreducible element of Lat N. If $M = (0)$ then $M = Z(0; N)$. If $M \neq (0)$ let M_0 be the unique element of Lat N covered by M. Let x be a vector belonging to M but not M_0. Then $Z(x; N)$ belongs to Lat N and $M_0 \vee Z(x; N) = M$. Since M is join-irreducible it follows that $M = Z(x; N)$.

Let y be an arbitrary vector. Clearly $Z(y; N)$ is invariant under N. Let B be the transformation induced on $Z(y; N)$ by N. Since B is nilpotent and cyclic, Lat B is totally ordered [**3**]. Thus if $Z(y; N) = K \vee L$ $(K, L \in \text{Lat } N)$ then $K, L \in \text{Lat } B$ so either $K \subseteq L$ or $L \subseteq K$. It follows that $Z(y; N)$ is join-irreducible.

(ii) *Every automorphism of* Lat N *preserves the property of being N-cyclic.* This follows immediately from (i).

(iii) *Every automorphism of* Lat N *preserves dimension.* It is well-known that if M and L belong to Lat N and M covers L then dim M = dim $L + 1$. It follows that a chain $M_0 \subset M_1 \subset \ldots \subset M_n$ in Lat N is a maximal chain in Lat N if and only if dim $M_j = j$ and $M_n = H$. Every element of Lat N belongs to some maximal chain in Lat N and the image of a maximal chain under any automorphism is a maximal chain. The result follows.

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(iv) *The dimension of* $Z(x; N)$ *is the smallest non-negative integer* r *satisfying* $Z(x; N) \subseteq \mathcal{N}(N^r)$. The result is obviously true if $Z(x; N) = (0)$. If dim $Z(x; N) = r$ and $r \geqq 1$, the vectors $x, Nx, N^2x, \ldots, N^{r-1}x$ form a basis for $Z(x; N)$ and $N^r x = 0$, $N^{r-1}x \neq 0$ ([**5**, p. 228]). Thus $Z(x; N) \subseteq \mathcal{N}(N^r)$ and if $Z(x; N) \subseteq \mathcal{N}(N^s)$ then $N^s x = 0$ so $s \geqq r$.

(v) *Each of the subspaces* $\mathcal{N}(N^r)$ $(0 \leqq r \leqq k)$ *is fixed by every automorphism* ν *of* Lat N. This is proved by induction on r. Clearly it is true for $r = 0$. Suppose it is true for r. Since

$$\begin{aligned}\mathcal{N}(N^{r+1}) &= \mathcal{N}(N^r) \vee (\bigvee\{Z(x; N) : x \in \mathcal{N}(N^{r+1})\backslash\mathcal{N}(N^r)\}) \\ &= \mathcal{N}(N^r) \vee (\bigvee\{Z(x; N) : \dim Z(x; N) = r + 1\}) \quad \text{[by (iv)]}\end{aligned}$$

we have

$$\begin{aligned}\nu(\mathcal{N}(N^{r+1})) &= \mathcal{N}(N^r) \vee (\bigvee\{\nu(Z(x; N)) : \dim Z(x; N) = r + 1\}) \\ &= \mathcal{N}(N^{r+1}) \quad \text{[by (ii) and (iii)]}.\end{aligned}$$

(vi) *Each of the subspaces* $\mathcal{R}(N^r)$ $(0 \leqq r \leqq k)$ *is fixed by every automorphism* ν *of* Lat N. Notice that if ψ is an anti-automorphism of Lat N then $\psi^{-1} \circ \nu \circ \psi$ is an automorphism of Lat N. By (v), $\psi^{-1} \circ \nu \circ \psi$ fixes $\mathcal{N}(N^r)$ $(0 \leqq r \leqq k)$ so ν fixes $\psi(\mathcal{N}(N^r))$. We show that there is an anti-automorphism ψ of Lat N which maps $\mathcal{N}(N^r)$ onto $\mathcal{R}(N^r)$ $(0 \leqq r \leqq k)$. Now $m_N(z) = z^k$. There exist non-zero vectors $x_1, x_2, \ldots, x_t$ such that the N-cyclic subspaces $Z(x_j; N)$ are independent and span H i.e. $H = \sum_{j=1}^{t} \oplus Z(x_j; N)$ ([**5**, p. 223]). If dim $Z(x_j; N) = k_j$ then $x_j, Nx_j, \ldots, N^{k_j-1}x_j$ is a basis for $Z(x_j; N)$ and $N^{k_j}x_j = 0$. If $e_{j,i} = N^{i-1}x_j$ $(1 \leqq i \leqq k_j, 1 \leqq j \leqq t)$ then

$$\mathcal{B} = \{e_{11}, e_{12}, \ldots, e_{1k_1}; e_{21}, e_{22}, \ldots, e_{2k_2}; \ldots; e_{t1}, e_{t2}, \ldots, e_{tk_t}\}$$

is an ordered basis for H and

$$Ne_{j,i} = \begin{cases} e_{j,i+1} & i \neq k_j \\ 0 & i = k_j. \end{cases}$$

If

$$\mathcal{B}^* = \{f_{11}, f_{12}, \ldots, f_{1k_1}; f_{21}, f_{22}, \ldots, f_{2k_2}; \ldots; f_{t1}, f_{t2}, \ldots, f_{tk_t}\}$$

is the dual ordered basis and N^* is the adjoint of N we have

$$N^*f_{j,i} = \begin{cases} f_{j,i-1} & i \neq 1 \\ 0 & i = 1. \end{cases}$$

The transformation S defined by $Se_{j,i} = f_{j,k_j-i+1}$ $(1 \leqq i \leqq k_j, 1 \leqq j \leqq t)$ is invertible and $N^* = SNS^{-1}$. It follows that the mapping $M \to S^{-1}M$ $(M \in$ Lat $N^*)$ is an isomorphism of Lat N^* onto Lat N. With $K^\perp$ denoting the orthogonal complement of K, we therefore have that the mapping ψ: Lat $N \to$ Lat N defined by $\psi(K) = S^{-1}K^\perp$ is an anti-isomorphism of Lat N. The matrix representation of N relative to $\mathcal{B}$ is the same as the matrix representation of N^* relative to $\{Se_{11}, Se_{12}, \ldots, Se_{1k_1}; Se_{21}, Se_{22}, \ldots, Se_{2k_2}; \ldots; Se_{t1},$

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$Se_{t2}, \ldots, Se_{tk_t}\}$. It follows that $S^{-1}\mathscr{R}(N^{*r}) = \mathscr{R}(N^r)$ $(0 \leqq r \leqq k)$. Hence $\psi(\mathscr{N}(N^r)) = S^{-1}\mathscr{R}(N^{*r}) = \mathscr{R}(N^r)$ $(0 \leqq r \leqq k)$.

(vii) Hyperlat $N \subseteq$ Fix (Lat N). If ν is an automorphism of Lat N it fixes the subspaces $\mathscr{N}(N^r)$, $\mathscr{R}(N^r)$ $(0 \leqq r \leqq k)$ by (v) and (vi). So ν fixes every subspace of the form $\mathscr{N}(N^r) \cap \mathscr{R}(N^s)$ $(0 \leqq r, s \leqq k)$. Since every hyperinvariant subspace of N is a span of such subspaces ([**4**, Theorem 4.1]) ν fixes every hyperinvariant subspace of N.

(viii) Fix (Lat N) $\subseteq$ Hyperlat N. This follows immediately from the fact that if B is an invertible transformation commuting with N the mapping $M \rightarrow BM$ $(M \in \text{Lat } N)$ defines an automorphism of Lat N.

This completes the proof of the theorem.

We now turn to the general case. The result is presented as the following theorem.

THEOREM 3.2. *Let A be a transformation acting on a (finite-dimensional) Hilbert space. There exist sublattices $L_1, L_2, \ldots, L_m$ of* Lat A *each of which is irreducible and contains more than one element such that* Lat $A = \otimes_{i=1}^m L_i$. *With these properties the family of sublattices $\{L_i : 1 \leqq i \leqq m\}$ is unique, and* Hyperlat $A = \otimes_{i=1}^m$ Fix (L_i).

Proof. Let A act on space H. Let the minimum polynomial m_A of A have the factorization $m_A(z) = \prod_{i=1}^m (z - a_i)^{s_i}$ where the a_i are distinct complex numbers and the s_i are positive integers. The subspaces $W_i = \mathscr{N}((A - a_i)^{s_i})$ $(1 \leqq i \leqq m)$ are non-zero invariant subspaces of A and $H = \sum_{i=1}^m \oplus W_i$ ([**5**, p. 220]). If A_i is the transformation induced on W_i by A then $A_i - a_i$ is nilpotent ([**5**, p. 220]) and $L_i =$ Lat $(A_i - a_i)$ is an irreducible sublattice of Lat A with more than one element [**3**]. Also, Lat $A = \otimes_{i=1}^m L_i$ [**3**] and Hyperlat $A = \otimes_{i=1}^m$ Hyperlat $(A_i - a_i)$ [**4**]. By Theorem 3.1 we have Hyperlat $A = \otimes_{i=1}^m$ Fix (L_i). The uniqueness of the family $\{L_i : 1 \leqq i \leqq m\}$ follows from the work of Birkhoff ([**1**, p. 26; and **2**]). This completes the proof.

The following corollaries are immediate consequences of this description of the lattice of hyperinvariant subspaces of an arbitrary transformation.

COROLLARY 3.2.1. *If A and B act on the same space and* Lat $A =$ Lat B *then* Hyperlat $A =$ Hyperlat B.

COROLLARY 3.2.2. *If B_i acts on the space H_i $(i = 1, 2)$ and* Lat B_1 *is isomorphic to* Lat B_2 *then* Hyperlat B_1 *is isomorphic to* Hyperlat B_2.

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