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Representation Stability of Power Sets and Square Free Polynomials

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Abstract. The symmetric group $\mathcal{S}_n$ acts on the power set $\mathcal{P}(n)$ and also on the set of square free polynomials in n variables. These two related representations are analyzed from the stability point of view. An application is given for the action of the symmetric group on the cohomology of the pure braid group.

1 Introduction

The symmetric group $\mathcal{S}_n$ acts naturally on the power set $\mathcal{P}(n)$ of the set $\underline{n} = \{1, 2, \dots, n\}$ as follows:

$$\text{if } \pi \in \mathcal{S}_n \text{ and } A \in \mathcal{P}(n), \text{ then } \pi \cdot A = \pi(A).$$

It is obvious that the orbits of this action are $\mathcal{P}_k(n) = \{A \subset \underline{n} \mid \operatorname{card}(A) = k\}$ for $k = 0, 1, \dots, n$. More interesting is the linear representation of the symmetric group on the linear space $L\mathcal{P}(n)$, the $\mathbb{Q}$-span of the power set: the $\mathcal{S}_n$-submodules $L\mathcal{P}_k(n)$ are not irreducible. We decompose them into irreducible $\mathcal{S}_n$-modules and we describe their bases using the isomorphic representation of $\mathcal{S}_n$ onto the quotient ring of square free polynomials in n variables

$$\mathcal{S}f(n) = \mathbb{Q}[x_1, x_2, \dots, x_n] \big/ \langle x_1^2, x_2^2, \dots, x_n^2 \rangle.$$

Next we analyze the sequences of these representations, $(\mathcal{P}(n))_{n\geq 0}$ and $(\mathcal{S}f(n))_{n\geq 0}$, and some related sequences from the stability point of view introduced by Church and Farb [CF] for the representation ring $R(\mathcal{S}_n)$. We define an analogue of this stability for the Burnside ring $\Omega(\mathcal{S}_n)$ and analyze the stability of the action of $\mathcal{S}_n$ on $\mathcal{P}(n)$; see Section 4.

For the irreducible $\mathcal{S}_n$-modules (in characteristic 0 these can be defined over $\mathbb{Q}$) we will use the standard notation: V_λ corresponds to the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t \geq 1)$ of n, and the stable notations of Church and Farb ([CF, C]) $V(\mu)_n = V_{(n-\sum \mu_i, \mu_1, \mu_2, \dots, \mu_s)}$ for $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_s \geq 1)$ satisfying the relation $n - \sum_{i=1}^{s} \mu_i \geq \mu_1$. Similarly, U_λ is the permutation module (see [J]) and $U(\mu)_n$ is the permutation module $U_{(n-\sum \mu_i, \mu_1, \mu_2, \dots, \mu_s)}$. See [FH, J, K] for references for the representation theory of $\mathcal{S}_n$.

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Following [CF, C], we say that a sequence

$$X_* = (X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \dots \longrightarrow X_n \xrightarrow{\varphi_n} X_{n+1} \longrightarrow \dots),$$

where X_n is an $\mathcal{S}_n$-module, is *consistent* if φ_n is $\mathcal{S}_n$-equivariant with respect to the natural inclusions $\mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1}$. The sequence is *injective* if φ_n is eventually injective and $\mathcal{S}_*$*-surjective* if for n large $\mathcal{S}_{n+1} \cdot \mathrm{Im}(\varphi_n) = X_{n+1}$. The sequence X_* is *representation stable* if it satisfies the above conditions and also, for any stable type $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ of $\mathcal{S}_n$ modules, the sequence $(c_{\mu,n})_n$ of multiplicities of $V(\mu)_n$ in X_n is eventually constant. The sequence is *uniformly representation stable* if there is a natural number N, independent of μ, such that for any μ and any $n \geq N$, $c_{\mu,n} = c_{\mu,N}$. We say that a consistent sequence is *monotone* if for each $\mathcal{S}_n$ submodule $U \cong V(\mu)_n^{\oplus c}$ in X_n, the $\mathcal{S}_{n+1}$-span of the image of U in X_{n+1} contains $V(\mu)_{n+1}^{\oplus c}$ as a submodule. See [CF, C] for other versions of representation stability.

In the Sections 2 and 3, using new geometric ideas we give a completely different proof of the next theorem: the decomposition is a classical result of Specht and representation stability are recent results of [CF, C, H].

Theorem A ([CF, C, H]) *The sequence of $\mathcal{S}_*$-modules $(L\mathcal{P}_k(n))_{n\geq 0}$ with*

$$L\mathcal{P}_k(n) = V(0)_n \oplus V(1)_n \oplus \cdots \oplus V(k)_n$$

(for $n \geq 2k$) is consistent, uniformly representation stable, and monotone.

For the proof we introduce an increasing $\mathcal{S}_n$-filtration; it will be used in Section 5 to describe an algorithm that will give bases of the irreducible $\mathcal{S}_n$ modules of the square free polynomials. The proof in [CF] relies on a result in [H].

Now we introduce the notion of *action stability* for a sequence X_n of $\mathcal{S}_n$-sets and maps $X_0 \longrightarrow^{\varphi_0} X_1 \longrightarrow^{\varphi_1} X_2 \longrightarrow^{\varphi_2} \cdots$. Here we define the really new notions, the obvious ones are defined in Section 4.

Definition 1.1 The transitive $\mathcal{S}_n$-set $\mathcal{S}_n/H$ is of the type $\lambda_* = (\lambda_1, \lambda_2, \dots, \lambda_t)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 1$, $\sum \lambda_i = n$, if the action of H on the set $\underline{n} = \{1, 2, \dots, n\}$ has t orbits and their cardinalities are $\lambda_1, \lambda_2, \dots, \lambda_t$. The $\mathcal{S}_n$-set $\mathcal{S}_n/H$ is of *stable type* $(\mu_*)_n = (\mu_1, \dots, \mu_s)_n$ if it is of type $(n - \sum_{i=1}^{s} \mu_i, \mu_1, \dots, \mu_s)$ (the same conditions are required: $n - \sum_{i=1}^{s} \mu_i \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \geq 1$).

For a given sequence $\mu_* = (\mu_1, \dots, \mu_s)$, $\mu_1 \geq \cdots \geq \mu_s \geq 1$, and an $\mathcal{S}_n$-set X_n, we denote by $\mu_*(X_n)$ the number of $\mathcal{S}_n$ orbits in X_n of stable type $(\mu_*)_n$ and by $X_n(\mu_*)$ the union of all these orbits.

Definition 1.2 A consistent sequence of $\mathcal{S}_n$-sets $(X_n, \varphi_n)_{n\geq 0}$ is *action stable* if for any sequence $\mu_* = (\mu_1, \dots, \mu_s)$ there is a natural number N_{μ_*} such that, for any $n \geq N_{\mu_*}$ the following conditions are satisfied:

(i) φ_n is injective and $\mathcal{S}_{n+1}$-surjective: $\mathcal{S}_{n+1} \cdot \varphi_n(X_n) = X_{n+1}$;
(ii) $\mu_*(X_n) = \mu_*(X_{n+1})$.

The sequence (X_n, φ_n) is *uniformly action stable* if it is action stable and one can take N_{μ_*} independent of μ_*.

The sequence is *strongly action stable* if it is action stable and we have, for $n \geq N_{\mu_*}$, the equality

(iii) $\mathcal{S}_{n+1} \cdot \varphi_n(X_n(\mu_*)) = X_{n+1}(\mu_*)$.

We will prove the following theorem.

Theorem B

(i) *The sequences* $(\mathcal{P}_k(n))_{n\geq 0}$ *are uniformly and strongly action stable.*

(ii) *The sequence* $(\mathcal{P}(n))_{n\geq 0}$ *is action stable.*

In the next section we transfer the results from $L\mathcal{P}_k(n)$ and $L\mathcal{P}(n)$ into the corresponding results for $\mathcal{S}f_k(n)$ and $\mathcal{S}f(n)$, the algebra of square free monomials. The Viète polynomials $\sigma_k^n = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_{i_1}x_{i_2}\cdots x_{i_k}$ give a basis for the invariant part $\mathcal{S}f(n)^{\mathcal{S}_n}$. Our Proposition 5.10 is a generalization of this classical result: we describe canonical bases for all the irreducible $\mathcal{S}_n$-submodules of the square free polynomial algebra. A different approach for the representation theory of nilpotent quotients of $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ is presented in [MWW].

In Section 6 we apply some of the previous results to find the irreducible $\mathcal{S}_n$-modules of the first graded components of the Arnold algebra, the cohomology algebra of the ordered configuration space of n points in the plane. The stable cases, $n \geq 4$ for the first decomposition and $n \geq 7$ for the second, are given by the following theorem.

Theorem C ([CF]) *The degree* 1 *and* 2 *components of the Arnold algebra decompose as*

$$\mathcal{A}^1(n) = V(0)_n \oplus V(1)_n \oplus V(2)_n,$$
$$\mathcal{A}^2(n) = 2V(1)_n \oplus 2V(2)_n \oplus 2V(1,1)_n \oplus V(3)_n \oplus 2V(2,1)_n \oplus V(3,1)_n.$$

These decompositions are given in [CF] *without proofs; a different proof, using the "two combinatorial types" contained in* $\mathcal{A}^2(n)$*, is presented in* [AAB].

The new contribution is the description of explicit bases of the irreducible $\mathcal{S}_n$ modules in the previous decompositions. We denote by $\{w_{ij}\}_{1\leq i<j\leq n}$ the canonical basis of degree one component of the Arnold algebra, $\mathcal{A}^1(n)$; we will also use the following notation:

$$\Omega^n = w_{12} + w_{13} + \cdots + w_{n-1,n},$$
$$\Omega^n_{ij} = \sum_{k\neq i,j} (w_{ik} - w_{jk}),$$
$$\Omega_{ijkl} = w_{il} - w_{ik} + w_{jk} - w_{jl}.$$

Theorem D *The following list gives bases of the three irreducible components of* $\mathcal{A}^1(n)$*:*

$$\mathcal{B}(n) = \{\Omega^n\},$$
$$\mathcal{B}(n-1,1) = \{\Omega^n_{12}, \Omega^n_{13}, \ldots, \Omega^n_{1n}\},$$

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$$\begin{aligned}\mathcal{B}(n-2,2) = \{&\Omega_{1234}, \Omega_{1324},\\ &\Omega_{1235}, \Omega_{1325}, \Omega_{1425},\\ &\Omega_{1236}, \Omega_{1326}, \Omega_{1426}, \Omega_{1526},\\ &\dots\\ &\Omega_{123n}, \Omega_{132n}, \Omega_{142n}, \dots, \Omega_{1,n-1,2,n}\}.\end{aligned}$$

The precise descriptions of these bases are used in [AAB] for cohomological computation of the Križ algebra, a model for the configuration space of n-points of a smooth complex projective variety.

To the list of computations present in [CF], we add the stable decomposition of the cubic part of the Arnold algebra.

Theorem E *For $n \geq 12$, the degree 3 component of the Arnold algebra decomposes as*

$$\begin{aligned}\mathcal{A}^3(n) \cong\; & 2V(1)_n \oplus 3V(2)_n \oplus 5V(1,1)_n \oplus 4V(3)_n \oplus 7V(2,1)_n \oplus 3V(1,1,1)_n\\ &\oplus V(4)_n \oplus 6V(3,1)_n \oplus 2V(2,2)_n \oplus 4V(2,1,1)_n \oplus 2V(4,1)_n\\ &\oplus 2V(3,2)_n \oplus 2V(3,1,1)_n \oplus V(2,2,1)_n \oplus V(4,1,1)_n \oplus V(3,3)_n.\end{aligned}$$

2 Canonical $\mathcal{S}_n$ Filtration on $L\mathcal{P}_k(n)$

For $0 \leq k \leq n$, we will define a *canonical filtration*

$$F_*L\mathcal{P}_k(n) : 0 < F_0L\mathcal{P}_k(n) \leq F_1L\mathcal{P}_k(n) \leq \cdots \leq F_kL\mathcal{P}_k(n) = L\mathcal{P}_k(n)$$

with $\mathcal{S}_n$-submodules as follows: for $A \in \mathcal{P}_i(n)$, $0 \leq i \leq k$, denote by $\sigma_k^n(A)$ the element of $L\mathcal{P}_k(n)$ given by

$$\sigma_k^n(A) = \sum_{B \in \mathcal{P}_{k-i}(\underline{n} \setminus A)} A \sqcup B$$

and define the $\mathcal{S}_n$-submodule $F_iL\mathcal{P}_k(n)$ as the span $\mathbb{Q}\langle \sigma_k^n(A) \mid \operatorname{card}(A) = i \rangle$.

Example 2.1

$$\begin{aligned}F_0L\mathcal{P}_2(4) &= \mathbb{Q}\big\langle \{1,2\} + \{1,3\} + \{1,4\} + \{2,3\} + \{2,4\} + \{3,4\} \big\rangle,\\ F_1L\mathcal{P}_2(4) &= \mathbb{Q}\big\langle \{1,2\} + \{1,3\} + \{1,4\}, \{1,2\} + \{2,3\} + \{2,4\},\\ &\qquad \{1,3\} + \{2,3\} + \{3,4\}, \{1,4\} + \{2,4\} + \{3,4\} \big\rangle,\\ F_2L\mathcal{P}_2(4) &= \mathbb{Q}\big\langle \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \big\rangle.\end{aligned}$$

In this example, $0 < F_0L\mathcal{P}_2(4) < F_1L\mathcal{P}_2(4) < F_2L\mathcal{P}_2(4) = L\mathcal{P}_2(4)$.

Example 2.2

$$
\begin{aligned}
F_0L\mathcal{P}_3(4) &= \mathbb{Q}\big\langle \{1,2,3\}+\{1,2,4\}+\{1,3,4\}+\{2,3,4\}\big\rangle,\\
F_1L\mathcal{P}_3(4) &= \mathbb{Q}\big\langle \{1,2,3\}+\{1,2,4\}+\{1,3,4\}, \{1,2,3\}+\{1,2,4\}+\{2,3,4\},\\
&\qquad \{1,2,3\}+\{1,3,4\}+\{2,3,4\}, \{1,2,4\}+\{1,3,4\}+\{2,3,4\}\big\rangle,\\
F_2L\mathcal{P}_3(4) &= \mathbb{Q}\big\langle \{1,2,3\}+\{1,2,4\}, \{1,2,3\}+\{1,3,4\}, \{1,2,4\}+\{1,3,4\},\\
&\qquad \{1,2,3\}+\{2,3,4\}, \{1,2,4\}+\{2,3,4\}, \{1,3,4\}+\{2,3,4\}\big\rangle\\
F_3L\mathcal{P}_3(4) &= \mathbb{Q}\big\langle \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\big\rangle.
\end{aligned}
$$

In this example, $0 < F_0L\mathcal{P}_3(4) < F_1L\mathcal{P}_3(4) = F_2L\mathcal{P}_3(4) = F_3L\mathcal{P}_3(4) = L\mathcal{P}_3(4)$.

Lemma 2.3 *For any $0 \le k \le n$, the sequence $\{F_iL\mathcal{P}_k(n)\}_{0\le i\le k}$ is an increasing filtration of $\mathcal{S}_n$-submodules.*

Proof The group $\mathcal{S}_n$ permutes the generators of F_i: $\pi \cdot \sigma_k^n(A) = \sigma_k^n(\pi(A))$. The inclusion $F_i \le F_{i+1}$, $i \le k-1$, is a consequence of the equality

$$(k-i)\sigma_k^n(A) = \sum_{b\notin A} \sigma_k^n(A \sqcup \{b\}). \qquad ■$$

Lemma 2.4 *For any $\frac{n}{2} \le k \le n$, we have*

$$F_{n-k}L\mathcal{P}_k(n) = F_{n-k+1}L\mathcal{P}_k(n) = \cdots = F_kL\mathcal{P}_k(n) = L\mathcal{P}_k(n).$$

Proof In order to prove that $L\mathcal{P}_k(n) \le F_{n-k}L\mathcal{P}_k(n)$, we will find, for any subset $A \in \mathcal{P}_k(n)$, rational numbers $\{c_i\}_{0\le i\le n-k}$ such that

$$A = \sum_{i=0}^{n-k} c_i s_i(A), \text{ where } s_i(A) = \sum_{\substack{B\in\mathcal{P}_{n-k}(n)\\ |A\cap B|=i}} \sigma_k^n(B).$$

This is done by (decreasing) induction on i. In the right-hand side, the set A is contained only in $s_{n-k}(A)$, $\binom{k}{n-k}$ times, and this gives the first coefficient

$$c_{n-k} = \frac{(n-k)!(2k-n)!}{k!}.$$

A k-set $D \in \mathcal{P}_k(n) \setminus \{A\}$ has an intersection with A of cardinality $|A \cap D| = i$, where $2k-n \le i \le k-1$. Let us denote by μ_j the number of appearances of D in the sum $s_j(A)$. It is clear that $\mu_{n-2k+i} \ge 1$ (take $B = E \sqcup (D \setminus A)$, where $E \subset A \cap D$ has the cardinality $n-2k+i \le i$) and also that $\mu_j = 0$ if $j \le n-2k+i-1$ (any set in $\sigma_k^n(B)$, $|B \cap A| \le n-2k+i-1$, $|B| = n-k$, contains at least $k-i+1$ elements in the complement of A). Looking for the coefficients of D in the equation $A = \sum_{i=0}^{n-k} c_i s_i(A)$ we find

$$0 = c_{n-k}\mu_{n-k} + c_{n-k-1}\mu_{n-k-1} + \cdots + c_{n-2k+i}\mu_{n-2k+i},$$

and this gives a solution $c_{n-2k+i} \in \mathbb{Q}$. The equation $A = \sum c_i s_i(A)$ is symmetric in k-sets D with $|A \cap D| = i$, so the solution c_{n-2k+i} does not depend on D. ■

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We introduce two natural linear maps:

$$\sqcup n\colon L\mathcal{P}_{k-1}(n-1) \to L\mathcal{P}_k(n), \qquad A \mapsto A \sqcup \{n\},$$

$$\mathrm{res}\colon L\mathcal{P}_k(n) \to L\mathcal{P}_k(n-1), \qquad A \mapsto \begin{cases} A & \text{if } n \notin A, \\ 0 & \text{if } n \in A. \end{cases}$$

A *semi-exact sequence* of vector spaces (or a chain complex) is a sequence of linear morphisms $(V_i \to^{f_i} V_{i+1})_{i\in I}$ in which $f_i \circ f_{i-1} = 0$ for any $i \in I$.

Lemma 2.5 *For any $0 \le i \le k \le n$, the following sequence is semi-exact:*

$$S_{n,k,i}\colon 0 \to F_{i-1}L\mathcal{P}_{k-1}(n-1) \xrightarrow{\sqcup n} F_iL\mathcal{P}_k(n) \xrightarrow{\mathrm{res}} F_iL\mathcal{P}_k(n-1) \to 0;$$

the map $\sqcup n$ is injective and the map res *is surjective. In particular,*

$$\dim F_iL\mathcal{P}_k(n) \ge \dim F_{i-1}L\mathcal{P}_{k-1}(n-1) + \dim F_iL\mathcal{P}_k(n-1).$$

Proof The map $\sqcup n : L\mathcal{P}_{k-1}(n-1) \to L\mathcal{P}_k(n)$ is injective and its restriction to F_{i-1} takes values in F_i:

$$\sqcup n(\sigma^{n-1}_{k-1}(A)) = \sigma^n_k(A \sqcup \{n\}).$$

The restriction of the second map is well defined and surjective:

$$\mathrm{res}(\sigma^n_k(A)) = \begin{cases} \sigma^{n-1}_k(A) & \text{if } n \notin A, \\ 0 & \text{if } n \in A. \end{cases}$$

It is obvious that $\mathrm{res} \circ (\sqcup n) = 0$, but in general $\ker(\mathrm{res})$ is bigger than $\mathrm{Im}(\sqcup n)$. ■

Now we compute the dimension of $F_iL\mathcal{P}_k(n)$, describe a basis of this space, and we show that the filtration F_i is strictly increasing, with the exception described in Lemma 2.4.

Proposition 2.6 *For any $n \ge 1$ we have:*

(B_n) *for any $0 \le k \le n$ and $0 \le i \le \min(k, n-k)$ or $i = k$, the set $\{\sigma^n_k(A)\}_{A\in\mathcal{P}_i(n)}$ is a basis of $F_iL\mathcal{P}_k(n)$;*

(D_n) *for any $0 \le k \le n$,*

$$\dim F_iL\mathcal{P}_k(n) = \begin{cases} \binom{n}{i} & \text{for } 0 \le i \le \min(k, n-k), \\ \binom{n}{k} & \text{for } n-k \le i \le k; \end{cases}$$

(E_n) *the sequence $S_{n,k,i}$ is exact with the unique exception $k > \frac{n}{2}$ and $i = n-k$;*

(F_n) *for any $0 \le k \le n$ the filtration $\{F_iL\mathcal{P}_k(n)\}_{0\le i\le\min(k,n-k)}$ is strictly increasing.*

Proof The implications $(B_n) \Leftrightarrow (D_n) \Rightarrow (F_n)$ are obvious as are the statements $F_kL\mathcal{P}_k(n) = L\mathcal{P}_k(n)$ (from definition) and, for $k > \frac{n}{2}$, the equality $F_{n-k}L\mathcal{P}_k(n) = L\mathcal{P}_k(n)$ (from Lemma 2.4). We will show, by induction on n, that $(D_{n-1}) \Rightarrow (D_n)$ and (E_n).

For $n = 1$ we have the equalities

$$F_0L\mathcal{P}_0(1) = \mathbb{Q}\langle\varnothing\rangle, \quad F_0L\mathcal{P}_1(1) = F_1L\mathcal{P}_1(1) = \mathbb{Q}\langle\{1\}\rangle,$$

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two exact sequences

$$S_{1,0,0}\colon 0 \to 0 \to \mathbb{Q}\langle\varnothing\rangle \to \mathbb{Q}\langle\varnothing\rangle \to 0,$$
$$S_{1,1,1}\colon 0 \to \mathbb{Q}\langle\varnothing\rangle \to \mathbb{Q}\langle\{1\}\rangle \to 0 \to 0,$$

and one semi exact, but not exact:

$$S_{1,1,0}\colon 0 \to 0 \to \mathbb{Q}\langle\{1\}\rangle \to 0 \to 0.$$

Now we suppose that the dimension formula is correct for $n-1$, and we compute the dimension of $F_iL\mathcal{P}_k(n)$ and check the exactness of $S_{n,k,i}$ by cases, according to "small values" of k, *i.e.*, $k \le \frac{n}{2}$, and "large values", $k > \frac{n}{2}$. We have to analyze eight cases because k small (or large) for the central term in $S_{n,k,i}$ does not imply $k-1$ small (or large) in the first term or k small (or large) in the last term. In fact, there are only two proofs: a simple one, when $i = \min(k, n-k)$, in which case we use Lemma 2.4: $\dim F_iL\mathcal{P}_k(n) = \binom{n}{i}$, and the other cases, where a sequence of inequalities gives the dimension of $F_iL\mathcal{P}_k(n)$ and the exactness of $S_{n,k,i}$.

Case 1: $0 \le i \le k < \frac{n}{2}$. This implies $i-1 \le k-1 \le \frac{n-1}{2}$, $i \le k \le \frac{n-1}{2}$, and, from the semi exact sequence

$$0 \to F_{i-1}L\mathcal{P}_{k-1}(n-1) \to F_iL\mathcal{P}_k(n) \to F_iL\mathcal{P}_k(n-1) \to 0,$$

we obtain

$$\binom{n}{i} \ge \dim F_iL\mathcal{P}_k(n) \ge \dim F_{i-1}L\mathcal{P}_{k-1}(n-1) + \dim F_iL\mathcal{P}_k(n-1)$$
$$= \binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i},$$

hence the expected dimension of $F_iL\mathcal{P}_k(n)$ and the exactness of the sequence.

Case 2: $0 \le i < k = \frac{n}{2}$. In this case, $i-1 \le k-1 \le \frac{n-1}{2}$, $i \le \frac{n-1}{2} - k = \min(k, \frac{n-1}{2} - k)$, and we obtain the same sequence of inequalities as in the previous case.

Case 3: $i = k = \frac{n}{2}$. This is obvious: $F_kL\mathcal{P}_k(n) = L\mathcal{P}_k(n)$, and the sequence is exact.

Case 4: $\frac{n+1}{2} < k \le n$, $i \le n-k-1$. This implies $i-1 \le (n-1)-(k-1) = \min(n-k, k-1)$, $i \le (n-1)-k = \min(n-1-k, k)$, and the same sequence of inequalities gives the correct dimension and the exactness.

Case 5: $\frac{n+1}{2} < k = n$, $i = n-k$. From Lemma 2.4, $\dim F_{n-k}L\mathcal{P}_k(n) = \binom{n}{k}$, which is strictly bigger than the sum of the two other dimensions:

$$\binom{n-1}{n-k-1} + \binom{n-1}{k} = \binom{n-1}{k} + \binom{n-1}{k}.$$

The sequence is not exact.

Case 6: $k = \frac{n+1}{2}$, $0 \le i \le k-2$. In this case, $i-1 \le k-1 \le \frac{n-1}{2}$, $i \le \frac{n-1}{2} - k = \min(\frac{n-1}{2} - k, k)$, and, as in the case 1, we have $\dim F_iL\mathcal{P}_k(n) = \binom{n}{i}$ and exactness.

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Case 7: $k = \frac{n+1}{2}$, $i = k-1$. By Lemma 2.4, $F_{k-1}L\mathcal{P}_k(2k-1) = L\mathcal{P}_k(2k-1)$, a space of dimension $\binom{2k-1}{k-1} = \binom{2k-1}{k}$, strictly bigger than the sum of dimensions of the other two terms: $\binom{2k-2}{k-2} + \binom{2k-2}{k}$. The sequence is not exact.

Case 8: $k = \frac{n+1}{2}$, $i = k$. Again this is simple. $F_kL\mathcal{P}_k(n) = L\mathcal{P}_k(n)$, and the counting of dimensions gives the exactness of $S_{n,k,i}$. ■

From now on we can assume $k \leq \lfloor \frac{n}{2} \rfloor$, because of the next obvious proposition.

Proposition 2.7 (i) *For any k, $0 \leq k \leq n$, the complementary map C is $\mathcal{S}_n$-equivariant:*

$$C\colon \mathcal{P}_k(n) \to \mathcal{P}_{n-k}(n), \quad A \mapsto \underline{n} \setminus A.$$

(ii) *The $\mathcal{S}_n$ representations $L\mathcal{P}_k(n)$ and $L\mathcal{P}_{n-k}(n)$ are isomorphic.*

Lemma 2.8 (i) *The $\mathcal{S}_n$-module $F_0L\mathcal{P}_k(n)$ is trivial.*
(ii) *For $0 \leq i \leq k \leq \frac{n}{2}$, the $\mathcal{S}_n$ representations $F_iL\mathcal{P}_k(n)$ and $L\mathcal{P}_i(n)$ are isomorphic.*

Proof (i) The space $F_0L\mathcal{P}_k(n)$ is generated by the invariant element

$$\sigma_k^n = \sigma_k^n(\varnothing) = \sum_{A \in \mathcal{P}_k(n)} A.$$

Using Proposition 2.6 the map $\varphi(\sigma_k^n(A)) = A$ is well defined; the maps

$$\varphi\colon F_iL\mathcal{P}_k(n) \leftrightarrows L\mathcal{P}_i(n) : \psi,$$

where $\psi(A) = \sigma_k^n(A)$, are $\mathcal{S}_n$-equivariant and inverse to each other. ■

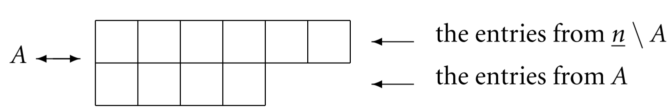
The $\mathcal{S}_n$-module $L\mathcal{P}_k(n)$ is isomorphic with a classical object, the permutation module $U_{(n-k,k)}$, the span of tabloids of type $(n-k,k)$ (see [J]). We will give a new proof for its decomposition into irreducible pieces.

Proposition 2.9 *For $0 \leq k \leq \frac{n}{2}$, the $\mathcal{S}_n$-modules $L\mathcal{P}_k(n)$ and $U_{(n-k,k)}$ are isomorphic.*

Proof At the level of sets we have the equivariant bijective map:

$$\mathcal{P}_k(n) \to \{\text{tabloids of type } (n-k,k)\}$$

given by



■

To describe the structure of the $\mathcal{S}_n$-modules $L\mathcal{P}_k(n)$ and $F_iL\mathcal{P}_k(n)$ we will use only the fact that the $\mathcal{S}_n$-module $U_{(n-k,k)}$ contains $V_{(n-k,k)}$ (with some multiplicity).

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Proposition 2.10 ([J]) *The irreducible decompositions of the $\mathcal{S}_n$-modules $L\mathcal{P}_k(n)$ and $F_iL\mathcal{P}_k(n)$ $(0 \le i \le k \le \frac{n}{2})$ are given by*

$$L\mathcal{P}_k(n) = V_{(n)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(n-k,k)},$$
$$F_iL\mathcal{P}_k(n) = V_{(n)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(n-i,i)}.$$

Proof The proof is by induction on k. We have $L\mathcal{P}_0(n) = V_{(n)}$ and, using the imbedding $V_{(n-k,k)} < U_{(n-k,k)} \cong L\mathcal{P}_k(n)$ and Lemma 2.8, we obtain

$$V_{(n)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(n-k+1,k-1)} \cong L\mathcal{P}_{k-1}(n) \cong F_{k-1}L\mathcal{P}_k(n) < L\mathcal{P}_k(n).$$

Using the hook formula [FH], we have $\dim V_{(n-k,k)} = \binom{n}{k-1}\frac{n-2k+1}{k}$, and counting the dimensions we find

$$\dim F_{k-1}L\mathcal{P}_k(n) + \dim V_{(n-k,k)} = \binom{n}{k-1} + \binom{n}{k-1}\frac{n-2k+1}{k} = \binom{n}{k},$$

and this gives the direct sum

$$L\mathcal{P}_{k-1}(n) \oplus V_{(n-k,k)} \cong F_{k-1}L\mathcal{P}_{k-1}(n) \oplus V_{(n-k,k)} \cong L\mathcal{P}_k(n). \qquad \blacksquare$$

Corollary 2.11 *The $\mathcal{S}_n$-decomposition of the module $L\mathcal{P}(n)$ is given by*

$$L\mathcal{P}(n) = (n+1)V_{(n)} \oplus (n-1)V_{(n-1,1)} \oplus \cdots \oplus (n-2k+1)V_{(n-k,k)} \oplus \cdots \oplus rV_{(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)},$$

where $r = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 1.$

A *natural operation* on the power set $\mathcal{P}(n)$ satisfies $\pi(A * B) = \pi(A) * \pi(B)$ for any permutation $\pi \in \mathcal{S}_n$. Given a natural operation ψ on $\mathcal{P}(n)$ (such as $\cup, \cap, \Delta, \dots$), we can linearize the map $\psi\colon \mathcal{P}(n) \times \mathcal{P}(n) \to \mathcal{P}(n)$ and obtain an $\mathcal{S}_n$-map $L\psi : L\mathcal{P}(n) \otimes L\mathcal{P}(n) \to L\mathcal{P}(n)$. Irreducible decomposition of the tensor product $L\mathcal{P}(n)^{\otimes 2}$ will add more irreducible representations of $\mathcal{S}_n$: $V_{(n-2,1,1)}, V_{(n-3,2,1)}, V_{(n-3,1,1,1)}, \dots$; each of them is contained in the kernel of $L\psi$.

3 Representation Stability

Using the stable notation, Proposition 2.10 gives the stable decompositions

$$L\mathcal{P}_k(n) = V(0)_n \oplus V(1)_n \oplus \cdots \oplus V(k)_n \qquad (\text{for } n \ge 2k),$$
$$F_iL\mathcal{P}_k(n) = V(0)_n \oplus V(1)_n \oplus \cdots \oplus V(i)_n \qquad (\text{for } n \ge 2k \ge 2i).$$

The natural maps

$$\mathcal{P}(n) \overset{\varphi_n}{\to} \mathcal{P}(n+1) \quad \text{and} \quad \mathcal{P}_k(n) \overset{\varphi_{k,n}}{\to} \mathcal{P}_k(n+1)$$

and their linearizations

$$L\mathcal{P}(n) \xrightarrow{L\varphi_n} L\mathcal{P}(n+1) \quad \text{and} \quad L\mathcal{P}_k(n) \xrightarrow{L\varphi_{k,n}} L\mathcal{P}_k(n+1)$$

are induced by the inclusion map $\underline{n} \hookrightarrow \underline{n+1}$. The sequences $(L\mathcal{P}_k(n))_{n\ge 0}$, $(F_iL\mathcal{P}_k(n))_{n\ge 0}$ are consistent, uniformly representation stable, and monotone in the sense of [C] and [CF]. Identifying the $\mathcal{S}_n$-representations $L\mathcal{P}_k(n) \cong Ind_{\mathcal{S}_k\times\mathcal{S}_{n-k}}^{\mathcal{S}_n} V_{(k)}$, the uniform representation stability is a special case of [H, Theorem 2.4] and monotonicity is a consequence of [C, Theorem 2.8]. We will give new proofs for these results, including also similar results for the sequences

$$F_i L\mathcal{P}_k(n) \to F_{i+1}L\mathcal{P}_k(n+1) \to \cdots \to F_{k-1}L\mathcal{P}_k(n+k-i-1)$$
$$\to L\mathcal{P}_k(n+k-i) \to L\mathcal{P}_k(n+k-i+1) \to \cdots$$

First we prove some "polynomial" identities.

Lemma 3.1 (i) *For an element A in $\mathcal{P}_i(n)$, $0 \le i \le k \le n$, we have*

$$\sigma_k^n(A) = \sigma_k^{n+1}(A) - \sigma_k^{n+1}\big(A \sqcup \{n+1\}\big).$$

(ii) *For $0 \le i \le k-1$, $k \le n$, we have*

$$\sigma_k^{n+1}(\underline{i+1}) = \frac{1}{(n-i)!(n-k+1)} \sum_{\pi \in \mathcal{S}_{n+1}^{\ge i+1}} \pi \cdot \sigma_k^n(\underline{i}) - (i+1, n+1) \cdot \sigma_k^n(\underline{i}),$$

where $\mathcal{S}_{n+1}^{\ge i+1}$ is the subgroup of permutations fixing the elements $1, 2, \ldots, i$.

Proof (i) The first equality is obvious. A term $A \sqcup B$, $B \subset \underline{n}$ is contained in $\sigma_k^n(A)$ and $\sigma_k^{n+1}(A)$ but not in the last sum, and a term $A \sqcup C \sqcup \{n+1\}$ is contained in the last two sums but not in the first one.

(ii) All the sets in this formula contain $\{1, \ldots, i\}$. In the left-hand side all the terms contain $\{i+1\}$, the sum $\sum_{\pi \in \mathcal{S}_{n+1}^{\ge i+1}} \pi \cdot \sigma_k^n(\underline{i})$ is symmetric in the elements $i+1, i+2, \ldots, n, n+1$, and therefore all its terms have the same multiplicity. Multiplicity of the term $\underline{k} = \underline{i} \sqcup \{i+1, \ldots, k\}$, $k \le n$, equals the number of permutations π in $\mathcal{S}_{n+1}^{\ge i+1}$ sending a $k-i$ subset of $\{i+1, i+2, \ldots, n\}$ into $\{i+1, \ldots, k\}$ (because any sum $\pi \cdot \sigma_k^n(\underline{i})$ contains $\underline{k}$ at most once), and this number is given by

$$\binom{n-i}{k-i}(k-i)!(n+1-k)! = (n-i)!(n+1-k).$$

Using the symmetry of the left-hand side and the different symmetry of the right-hand side, it is sufficient to show that the coefficients of the set $\underline{k} = \underline{i+1} \sqcup \{i+2, \ldots, k\} = \underline{i} \sqcup \{i+1, \ldots, k\}$ on the left hand side and the right hand side coincide, and the same for the coefficients of the set $\underline{i} \sqcup \{i+2, i+3, \ldots, k, n+1\}$. Now the term

$$\underline{k} = \underline{i} \sqcup \{i+1, \ldots, k\} = \underline{i+1} \sqcup \{i+2, \ldots, k\}$$

appears in the average of the sum $\sum_{\pi \in \mathcal{S}_{n+1}^{\ge i+1}} \pi \cdot \sigma_k^n(\underline{i})$ with coefficient 1, as in $\sigma_k^{n+1}(\underline{i+1})$, and does not appear in the sum $\sigma_k^n(\underline{i})$ modified by the transposition $(i+1, n+1)$. The term $\underline{i} \sqcup \{i+2, i+3, \ldots, k, n+1\}$ appears in the average of the sum $\sum_{\pi \in \mathcal{S}_{n+1}^{\ge i+1}} \pi \cdot \sigma_k^n(\underline{i})$ with the same coefficient 1 and has also the coefficient 1 in $(i+1, n+1) \cdot \sigma_k^n(\underline{i})$. ■

Lemma 3.2 (i) *For $0 \le i \le k-1$, $k \le n$ we have*

$$L\varphi_{k,n}(F_i L\mathcal{P}_k(n)) < F_{i+1}L\mathcal{P}_k(n+1),$$
$$\mathcal{S}_{n+1} \cdot L\varphi_{k,n}(F_i L\mathcal{P}_k(n)) = F_{i+1}L\mathcal{P}_k(n+1).$$

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(ii) *For $i = k \le n$ we have*

$$L\varphi_{k,n}(F_k L\mathcal{P}_k(n)) < F_k L\mathcal{P}_k(n+1),$$
$$\mathcal{S}_{n+1} \cdot L\varphi_{k,n}(F_k L\mathcal{P}_k(n)) = F_k L\mathcal{P}_k(n+1).$$

Proof Part (ii) is obvious at the set level: $\varphi_{k,n}(\mathcal{P}_k(n))$ is part of $\mathcal{P}_k(n+1)$, and $\mathcal{S}_{n+1}$ acts transitively on $\mathcal{P}_k(n+1)$. For part (i), the first inclusion is a consequence of Lemma 3.1(i), and this inclusion implies

$$\mathcal{S}_{n+1} \cdot L\varphi_{k,n}(F_i L\mathcal{P}_k(n)) < F_{i+1} L\mathcal{P}_k(n+1).$$

For the reverse inclusion it is enough to show that $\sigma_k^{n+1}(\underline{i+1})$ belongs to the $\mathcal{S}_{n+1}$ span of the image of $F_i L\mathcal{P}_k(n)$, and this is a consequence of Lemma 3.1(ii). ■

Remark 3.3 From Lemma 3.1(i), it is also clear that the image of $F_i L\mathcal{P}_k(n)$ is not contained in $F_i L\mathcal{P}_k(n+1)$ for $i \le k-1$.

Proposition 3.4 ([C, H])

(i) *The sequences $(L\mathcal{P}_k(n), L\varphi_{k,n})_{n\ge 0}$ are consistent, uniformly representation stable (with stable range $2k$), and monotone.*

(ii) *For $0 \le 2i \le 2k \le m$, the sequence $F_{\min(i+n-m,k)} L\mathcal{P}_k(n)_{n\ge m}$ is consistent, uniformly representation stable, and monotone.*

Proof (i) It is obvious that the maps $L\varphi_{k,n}\colon L\mathcal{P}_k(n) \to Res^{\mathcal{S}_{n+1}}_{\mathcal{S}_n} L\mathcal{P}_k(n+1)$ are injective, $\mathcal{S}_n$-equivariant, and also that $\mathcal{S}_{n+1} \cdot \mathrm{Im}(\varphi_{k,n}) = \mathcal{P}_k(n+1)$. The sequence of multiplicities of $V(\mu)_n$ in $L\mathcal{P}_k(n)$ is constant 1 for $\mu = (i)$, $0 \le i \le n/2$, and 0 for the other irreducible modules, by Proposition 2.10.

(ii) By Lemma 3.2, the injective map $L\varphi_{k,n}$ has restrictions $F_{\min(i+n-m,k)} L\mathcal{P}_k(n) \to F_{\min(i+n+1-m,k)} L\mathcal{P}_k(n+1)$, which are $\mathcal{S}_*$-surjective. The multiplicities are eventually stable, as in part (i).

The proof of monotonicity will be given at the end of Section 5. ■

Remark 3.5 The sequence $(L\mathcal{P}(n), L\varphi_n)_{n\ge 0}$ is consistent but not representation stable.

4 Stability of the Symmetric Group Actions

We give a set-theoretical analogue of the representation stability for a (direct) sequence of finite $\mathcal{S}_n$-sets X_n and maps $X_0 \to^{\varphi_0} X_1 \to^{\varphi_1} X_2 \to^{\varphi_2} \cdots$. The following definitions are obvious.

Definition 4.1 (i) The sequence $(X_n, \varphi_n)_{n\ge 0}$ of $\mathcal{S}_n$-sets is *consistent* if and only if the map $X_n \overset{\varphi_n}{\to} Res^{\mathcal{S}_{n+1}}_{\mathcal{S}_n}(X_{n+1})$ is $\mathcal{S}_n$-equivariant.

(ii) The sequence is *injective* if φ_n is (eventually) injective.

(iii) The sequence is $\mathcal{S}_*$-*surjective* if $\mathcal{S}_{n+1} \cdot \varphi_n(X_n) = X_{n+1}$ for large n.

To define "stability" we need a "stable notation" for transitive $\mathcal{S}_n$-sets. These are of the form $\mathcal{S}_n / H$, where H is a subgroup of $\mathcal{S}_n$ defined up to conjugation.

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Definition 4.2 A transitive $\mathcal{S}_n$-set X_n *has type* $(\lambda_1, \dots, \lambda_t)$ if it is equivalent to $\mathcal{S}_n / H$ as $\mathcal{S}_n$-sets and the action of H on $\underline{n}$ has t orbits of cardinalities $\lambda_1, \dots, \lambda_t$.

Remarks 4.3 (a) If H and K are conjugate in $\mathcal{S}_n$, then $\mathcal{S}_n / H$ and $\mathcal{S}_n / K$ have the same type.

(b) If $\mathcal{S}_n / H$ is of the type $(\lambda_1, \lambda_2, \dots, \lambda_t)$, then (up to conjugation) H is a subgroup of $\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_t}$; if pr_i is the projection of $\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_t}$ onto $\mathcal{S}_{\lambda_i}$, then $pr_i(H) < \mathcal{S}_{\lambda_i}$ acts transitively on the set $\underline{\lambda_i}$ of cardinality λ_i.

(c) In general there are many non-equivalent transitive $\mathcal{S}_n$-sets of the same type $(\lambda_1, \lambda_2, \dots, \lambda_t)$. There is a minimal one corresponding to the largest subgroup, $\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_t}$. Its linearization, $L(\mathcal{S}_n / \mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_t})$, is the permutation module $U_{(\lambda_1, \lambda_2, \dots, \lambda_t)}$ containing the irreducible representation $V_{(\lambda_1, \lambda_2, \dots, \lambda_t)}$ (with multiplicity one).

Example 4.4 The sequence $(\mathcal{S}_n / \mathcal{A}_n = \mathbb{Z}_2, \mathrm{Id})$ is uniformly and strongly action stable. More generally, any consistent, injective and $\mathcal{S}_*$-surjective sequence of transitive actions whose isotopy groups act transitively on $\{1, 2, \dots, n\}$ is strongly action stable.

Example 4.5 The sequence of $\mathcal{S}_n$-sets, $\underline{n} = \{1, 2, \dots, n\}$ (with natural action of $\mathcal{S}_n$ and canonical inclusion $i_n \colon \underline{n} \hookrightarrow \underline{n+1}$) is uniformly and strongly action stable.

Theorem B generalizes this last example.

Proof of Theorem B (i) As in Section 2, we take $n \geq 2k$. The group $\mathcal{S}_n$ acts on $\mathcal{P}_k(n)$ transitively and its corresponding subgroup is $\mathcal{S}_{n-k} \times S_k$; $\mathcal{P}_k(n)$ is of a unique stable type $(k)_n$, with multiplicity 1. Obviously the canonical inclusions $\mathcal{P}_k(n) \hookrightarrow \mathcal{P}_k(n+1)$ are consistent, injective, and $\mathcal{S}_*$-surjective.

(ii) The orbits of $\mathcal{P}(n)$ are $\{\mathcal{P}_k(n)\}_{0 \leq k \leq n}$, with corresponding subgroups $\mathcal{S}_{n-k} \times \mathcal{S}_k$; the stable types are $(k)_n$ with multiplicity 2 for $n \geq 2k+1$, hence the sequence is not uniformly stable. Moreover, condition (iii) of Definition 1.2 is not satisfied: for $\mu_* = (k), n \geq 2k+1$,

$$X_n(\mu_*) = \mathcal{P}_k(n) \sqcup \mathcal{P}_{n-k}(n), \quad X_{n+1}(\mu_*) = \mathcal{P}_k(n+1) \sqcup \mathcal{P}_{n+1-k}(n+1),$$

and $\mathcal{S}_{n+1} \cdot \varphi_n(X_n(\mu_*)) = \mathcal{P}_k(n+1) \sqcup \mathcal{P}_{n-k}(n+1)$. ∎

5 Canonical Polynomial Basis

Now we translate the power set representations into a quotient representation of the polynomial algebra $\mathbb{Q}[x_1, \dots, x_n]$; we compute canonical basis for the irreducible components in this isomorphic algebraic model. The set of squares $\mathrm{sq}(n) = \{x_1^2, \dots, x_n^2\}$ is $\mathcal{S}_n$-invariant, hence the ideal generated by $\mathrm{sq}(n)$ is $\mathcal{S}_n$-invariant and we obtain a quotient representation of $\mathcal{S}_n$ on the space of "square free" polynomials (*i.e.*, the $\mathbb{Q}$-span of monomials in which the exponents of $x_1, x_2, \dots, x_n$ are ≤ 1.)

$$\mathcal{S}f(n) = \mathbb{Q}[x_1, \dots, x_n] / \langle \mathrm{sq}(n) \rangle \cong \mathbb{Q}\langle \underline{\mathbf{x}}_A \mid A \in \mathcal{P}(n) \rangle$$

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where $\underline{\mathbf{x}}_A = x_{a_1}x_{a_2}\cdots x_{a_k}$ is the square free monomial corresponding to the subset $A = \{a_1, \ldots, a_k\} \in \mathcal{P}_k(n)$.

Lemma 5.1 *The power set $L\mathcal{P}(n)$ and the space of square free polynomials $\mathcal{S}f(n)$ are isomorphic $\mathcal{S}_n$-modules.*

Proof The power set $\mathcal{P}(n)$ and the canonical basis $\{\underline{\mathbf{x}}_A \mid A \in \mathcal{P}(n)\}$ are isomorphic as $\mathcal{S}_n$-sets. ∎

In the new setting we have a $\mathcal{S}_n$-decomposition by grading $\mathcal{S}f(n) = \bigoplus_{k=0}^{n} \mathcal{S}f_k(n)$, the $\mathcal{S}_n$-filtration $(0 \le k \le \lfloor \frac{n}{2} \rfloor)$

$$F_*\mathcal{S}f_k(n) : 0 < F_0\mathcal{S}f_k(n) < F_1\mathcal{S}f_k(n) < \cdots < F_k\mathcal{S}f_k(n) = \mathcal{S}f_k(n),$$

and also the irreducible components.

Corollary 5.2 *For any i, k, n satisfying $0 \le i \le k \le \lfloor \frac{n}{2} \rfloor$ we have*

$$\begin{aligned}
\mathcal{S}f_k(n) &\cong \mathcal{S}f_{n-k}(n),\\
F_i\mathcal{S}f_k(n) &= V_{(n)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(n-i,i)},\\
\mathcal{S}f_k(n) &= V_{(n)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(n-k,k)},\\
\mathcal{S}f(n) &= (n+1)V_{(n)} \oplus \cdots \oplus (n-2k+1)V_{(n-k,k)} \oplus \cdots \oplus rV_{(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)},
\end{aligned}$$

where $r = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 1$.

Remark 5.3 For $k \ge \lfloor \frac{n}{2} \rfloor$ we have

$$F_i\mathcal{S}f_k(n) = \begin{cases} V_{(n)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(n-i,i)} & \text{for } 0 \le i \le n-k,\\ V_{(n)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(k,n-k)} & \text{for } n-k \le i \le k. \end{cases}$$

Corollary 5.4 *The sequences $(\mathcal{S}f_k(n))_{n\ge 0}$ and $(F_{\min(i+n-m,k)}\mathcal{S}f_k(n))_{n\ge m}$ (for $m \ge 2k$) are consistent, uniformly representation stable, and monotone.*

Using the isomorphism of Lemma 5.1, we will use the same notation for elements in $L\mathcal{P}(n)$ introduced in Section 2 and the corresponding polynomials in $\mathcal{S}f(n)$ (the first ones are elementary symmetric polynomials in n variables):

$$\begin{aligned}
\sigma_k^n &= \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k},\\
\sigma_k^B &= \sum_{C\in\mathcal{P}_k(B)} \underline{\mathbf{x}}_C = \sum_{\substack{b_i\in B\\ b_1<b_2<\cdots<b_k}} x_{b_1}x_{b_2}\cdots x_{b_k},\\
\sigma_k^n(A) &= \underline{\mathbf{x}}_A\sigma_{k-|A|}^{A'} = x_{a_1}\cdots x_{a_i}\Big(\sum_{b_j\notin A} x_{b_{i+1}}\cdots x_{b_k}\Big).
\end{aligned}$$

(In the last two formulae, $\operatorname{card}(B) \ge k \ge \operatorname{card}(A)$, $A = \{a_1, \ldots, a_i\}$, and $A' = \underline{n}\setminus A$). If $|A| = k$, then $\sigma_k^A = \underline{\mathbf{x}}_A = \sigma_k^n(A)$. We will use new polynomials

$$\begin{aligned}
\delta_{hj} &= x_h - x_j \quad (1 \le h < j \le n)\\
\delta_{H_*J_*} &= \delta_{h_1j_1}\delta_{h_2j_2}\ldots\delta_{h_sj_s},
\end{aligned}$$

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where $H_* = (h_1, h_2, \ldots, h_s)$, $J_* = (j_1, j_2, \ldots, j_s)$, $h_\alpha < j_\alpha$ and $H_* \cup J_*$ contains $2s$ elements. Using this notation, $\{\sigma_k^n(A) \mid \operatorname{card}(A) = i\}$ is a basis of $F_i \mathcal{S} f_k(n)$ for $0 \le i \le k \le \frac{n}{2}$. Now we will describe bases for the irreducible $\mathcal{S}_n$-submodules $V_{(n-i,i)}$ contained in $\mathcal{S} f_k(n)$. For this, the following facts are important:

Remark 5.5 (i) The space $\mathcal{S} f(n)$ (like $L\mathcal{P}(n)$) has a canonical inner product, *i.e.,*

$$\langle \underline{\mathbf{x}}_A, \underline{\mathbf{x}}_B \rangle = \delta_{A,B}$$

and the natural representation of $\mathcal{S}_n$ is an orthogonal representation.

(ii) The homogenous components $\mathcal{S} f_k(n)$ are pairwise orthogonal.

(iii) The isotypic components are pairwise orthogonal, *i.e.*, if $W(\lambda)$ and $W(\mu)$ are two isotypic components of an $\mathcal{S}_n$-module W corresponding to the irreducible modules V_λ and V_μ respectively ($\lambda \ne \mu$), P_λ and P_μ are the corresponding projections $P_\lambda \colon W \to W(\lambda)$, $P_\mu \colon W \to W(\mu)$, and $x \in W(\lambda)$, $y \in W(\mu)$, then

$$\begin{aligned}\langle x, y \rangle = \langle x, P_\mu y \rangle &= \frac{1}{n!} \dim V_\mu \sum_{\pi \in \mathcal{S}_n} \overline{\chi_{V_\mu}(\pi)} \langle x, \pi y \rangle \\ &= \frac{1}{n!} \dim V_\mu \sum_{\pi \in \mathcal{S}_n} \chi_{V_\mu}(\pi) \langle \pi^{-1} x, y \rangle \\ &= \frac{1}{n!} \dim V_\mu \sum_{\pi \in \mathcal{S}_n} \chi_{V_\mu}(\pi^{-1}) \langle \pi^{-1} x, y \rangle = \langle P_\mu x, y \rangle = 0\end{aligned}$$

(We used the projection formula, the fact that W is a real or rational representation, the equality $\pi^t = \pi^{-1}$ because π is an orthogonal transformation, and also the equality $\chi(\pi) = \chi(\pi^{-1})$ because π and π^{-1} are conjugate in $\mathcal{S}_n$).

(iv) In the case of $\mathcal{S} f(n)$, in any isotypic component $\mathcal{S} f(n)(\lambda)$ we can find irreducible $\mathcal{S}_n$-modules given by homogenous polynomials and for a given degree k, there is at most one irreducible $\mathcal{S}_n$-module V_λ. As a consequence we have a canonical orthogonal decomposition of $\mathcal{S} f(n)$ into irreducible $\mathcal{S}_n$-modules.

Our method is to find vectors in $F_{i-1} \mathcal{S} f_k(n)^\perp$, the orthogonal complement of $F_{i-1} \mathcal{S} f_k(n)$ in $F_i \mathcal{S} f_k(n)$, because this complement corresponds to the irreducible component $V_{(n-i,i)}$ of $\mathcal{S} f_k(n)$. We then describe an independent subset of these vectors, and finally the computation of cardinality and dimension will give the basis.

Lemma 5.6 *For $k \le \frac{n}{2}$, the following vectors from $\mathcal{S} f_k(n)$ are orthogonal to $F_{k-1} \mathcal{S} f_k(n)$:*

$$\{\delta_{H_* J_*} \mid H_* = (h_1, h_2, \ldots, h_k), J_* = (j_1, \ldots, j_k), h_\alpha < j_\alpha, \operatorname{card}(H_* \cup J_*) = 2k\}.$$

Proof Obviously $\delta_{H_* J_*} \in \mathcal{S} f_k(n)$. The canonical basis of $F_{k-1} \mathcal{S} f_k(n)$ is given by $\{\sigma_k^n(A) \mid \operatorname{card}(A) = k-1\}$. Computing $\langle \underline{\mathbf{x}}_A \cdot \sigma_1^{A'}, \delta_{h_1 j_1} \delta_{h_2 j_2} \cdots \delta_{h_k j_k} \rangle$ we obtain the following.

(i) If $A \not\subseteq H_* \cup J_*$, there is no match between the monomials of these two polynomials, hence the inner product is zero.

In the next cases $A \subset H_* \cup J_*$:

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(ii) if there is an index $s \in \{1, \dots, k\}$ such that $\{h_s, j_s\} \subset A$, there is no match between the monomials of the two polynomials;

(iii) If A contains α elements $H_\alpha \subset H_*$ and β elements $J_\beta \subset J_*$ (hence $\alpha + \beta = k - 1$ and the indices of these elements are disjoint), there are precisely two common monomials of the given polynomials, $\underline{\mathbf{x}}_{\mathbf{A}} x_{h_s}$ and $\underline{\mathbf{x}}_{\mathbf{A}} x_{j_s}$, where the index s is the unique index from 1 to k that does not appear as an index in $H_\alpha \cup J_\beta$. The two monomials have coefficients (1,1) in the first polynomial, $\underline{\mathbf{x}}_A \cdot \sigma_1^{A'}$, and $(\pm 1, \mp 1)$ in the second one, $\delta_{H_* J_*}$.

Therefore, in all three cases the inner product is zero. ■

In the following two lemmas we generalize the last result.

Lemma 5.7 *For $0 \le i \le k \le \frac{n}{2}$, the following vectors are in $F_i \mathcal{S} f_k(n)$:*

$$\{\delta_{H_* J_*} \sigma_{k-i}^L \mid H_* = (h_1, \dots, h_i), J_* = (j_1, \dots, j_i), H_* \sqcup J_* \sqcup L = \underline{n}\}.$$

Proof Translating Lemma 2.8 into polynomial notation we obtain a linear isomorphism

$$\psi \colon \mathcal{S} f_i(n) \xrightarrow{\cong} F_i \mathcal{S} f_k(n), \quad \underline{\mathbf{x}}_A \mapsto \underline{\mathbf{x}}_A \sigma_{k-i}^{A'}.$$

A direct computation shows that

$$\psi(\delta_{H_* J_*}) = \psi\Big(\sum_{\alpha \sqcup \beta = \underline{i}} (-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \Big) = \sum_{\alpha \sqcup \beta = \underline{i}} (-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \sigma_{k-i}^{(H_\alpha \sqcup J_\beta)'}.$$

Using the decomposition formula

$$\sigma_p^{X \sqcup Y} = \sigma_p^X + \sigma_p^Y + \sum_{\substack{q+q'=p \\ q,q' \ge 1}} \sigma_q^X \sigma_{q'}^Y,$$

the symmetric sum $\sigma_{k-i}^{(H_\alpha \sqcup J_\beta)'}$ splits into

$$\sigma_{k-i}^{(H_* \sqcup J_*)'} + \sigma_{k-i}^{H_{\alpha'} \sqcup J_{\beta'}} + \sum_{\substack{q+q'=k-i \\ q,q' \ge 1}} \sigma_q^{H_{\alpha'} \sqcup J_{\beta'}} \cdot \sigma_{q'}^{(H_* \sqcup J_*)'},$$

where $(H_* \sqcup J_*)' = \underline{n} \setminus (H_* \sqcup J_*) = L$, $H_{\alpha'} = H_* \setminus H_\alpha$, and $J_{\beta'} = J_* \setminus J_\beta$. The first sum in this splitting gives the desired result:

$$\sum_{\alpha \sqcup \beta = \underline{i}} (-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \sigma_{k-i}^{(H_* \sqcup J_*)'} = \delta_{H_* J_*} \sigma_{k-i}^L.$$

To show that the second sum

$$\sum_{\alpha \sqcup \beta = \underline{i}} (-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \sigma_{k-i}^{H_{\alpha'} \sqcup J_{\beta'}}$$

and the third sum

$$\sum_{\alpha \sqcup \beta = \underline{i}} \Big[(-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \sum_{q+q'=k-i} \sigma_{q'}^{H_{\alpha'} \sqcup J_{\beta'}} \cdot \sigma_{q'}^{(H_* \sqcup J_*)'} \Big] =$$

$$\sum_{q+q'=k-i} \Big[\sum_{\alpha \sqcup \beta = \underline{i}} (-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \sigma_q^{H_{\alpha'} \sqcup J_{\beta'}} \Big] \cdot \sigma_{q'}^{(H_* \sqcup J_*)'}$$

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are zero, it is enough to prove that for any q in the interval $[1, k-i-1]$ the following sum is zero

$$S = \sum_{\alpha\sqcup\beta=\underline{i}} (-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \sigma_q^{H_{\alpha'}\sqcup J_{\beta'}}.$$

This sum contains monomials from two disjoint sets of variables, $\{x_{h_1}, \dots, x_{h_i}\}$ and $\{x_{j_1}, \dots, x_{j_i}\}$ ($H_* = (h_1, \dots, h_i)$, $J_* = (j_1, \dots, j_i)$). Therefore, in such a monomial $\mathbf{m} = \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \underline{\mathbf{x}}_M$ (M is a q-subset of $H_{\alpha'} \sqcup J_{\beta'}$), there are indices p such that x_{h_p} and x_{j_p} are both contained in $\mathbf{m}$. On the other hand, precisely one of them is in the "first part" and the other is in the "second part": either $h_p \in H_\alpha$, $j_p \in M$, or $h_p \in M$, $j_p \in J_\beta$. We define an involution (without fixed points) on the set of monomials $\mathbf{m}$ in S, choosing the maximal common index p and changing the places of x_{h_p} and x_{j_p} ($h_p \in H_\alpha$):

$$\mathbf{m} = \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \underline{\mathbf{x}}_M \leftrightarrow \mathbf{m}' = \underline{\mathbf{x}}_{H_\alpha \setminus \{h_p\}} \underline{\mathbf{x}}_{J_\beta \sqcup \{j_p\}} \underline{\mathbf{x}}_{M \sqcup \{h_p\} \setminus \{j_p\}}.$$

In S, these two monomials have coefficients $(-1)^{|\beta|}\mathbf{m}$ and $(-1)^{|\beta|+1}\mathbf{m}'$, hence the total sum is zero. ∎

Lemma 5.8 *For $0 \le i \le k \le \frac{n}{2}$, the following vectors from $F_i \mathcal{S} f_k(n)$ are orthogonal to $F_{i-1}\mathcal{S} f_k(n)$:*

$$\{\delta_{H_* J_*} \sigma_{k-i}^L \mid H_* = (h_1, \dots, h_i), J_* = (j_1, \dots, j_i), H_* \sqcup J_* \sqcup L = \underline{n}\}.$$

Proof Choose two elements

$$V = \sigma_k^n(A) = \sum_{B \in \mathcal{P}_{k-i+1}(A')} \underline{\mathbf{x}}_A \underline{\mathbf{x}}_B \in F_{i-1}\mathcal{S} f_k(n),$$

$$W = \delta_{H_* J_*} \sigma_{k-i}^L = \sum_{\alpha\sqcup\beta=\underline{i}, L_\gamma \in \mathcal{P}_{k-i}(L)} (-1)^{|\beta|} \underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \underline{\mathbf{x}}_{L_\gamma},$$

where $|A| = i-1, H_* \sqcup J_* \sqcup L = \underline{n}, |H_*| = |J_*| = i, H_\alpha \subseteq H_*, J_\beta \subseteq J_*$ and H_α is determined by J_β. If $\rho \colon H_* \to J_*$ is the bijection given by $\delta_{H_* J_*} = \prod_{h \in H_*}(x_h - x_{\rho(h)})$, then $H_\alpha = H_* \setminus \rho^{-1}(J_\beta)$. Modifying W by a permutation, one can suppose that $H_* = \{1, \dots, i\}, J_* = \{i+1, \dots, 2i\}$.

We have to show that $\langle V, W \rangle = 0$, which means to count the number of their common monomials and to identify their signs. We will use the following notation:

$$H_{\alpha_1} = A \cap H_*, \quad J_{\beta_1} = A \cap J_*, \quad L_{\gamma_1} = A \cap L.$$

A monomial $\underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \underline{\mathbf{x}}_{L_\gamma}$, which is also contained in V, should satisfy the relations

$$H_{\alpha_1} \subseteq H_\alpha, \quad J_{\beta_1} \subseteq J_\beta, \quad L_{\gamma_1} \subseteq L_\gamma.$$

Therefore, we have the decompositions

$$H_\alpha = H_{\alpha_1} \sqcup H_{\alpha_2}, \quad J_\beta = J_{\beta_1} \sqcup J_{\beta_2}, \quad L_\gamma = L_{\gamma_1} \sqcup L_{\gamma_2}.$$

Let us denote the cardinalities of $H_\alpha, H_{\alpha_1}, \dots, L_{\gamma_2}$ by $a, a_1, \dots, c_2$ respectively; for a common monomial $\underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \underline{\mathbf{x}}_{L_\gamma}$, the numbers a_1, b_1, c_1 are uniquely defined by V and W. For a fixed b, $b_1 \le b \le i - a_1$, a common monomial $\underline{\mathbf{x}}_{H_\alpha} \underline{\mathbf{x}}_{J_\beta} \underline{\mathbf{x}}_{L_\gamma}$ ($|J_\beta| = b$) is given by an arbitrary subset $J_{\beta_2} \subseteq J_* \setminus (J_{\beta_1} \sqcup \rho(H_{\alpha_1}))$ of cardinality $b_2 = b - b_1$ and an arbitrary subset $L_{\gamma_2} \subseteq \underline{n} \setminus (A \cup H_* \cup J_*)$ of cardinality $k - 2i + a_1 + b_1 + 1$ (of course,

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the set H_{α_2} is equal to $H_* \setminus (H_{\alpha_1} \sqcup \rho^{-1}(J_{\beta_1} \sqcup J_{\beta_2}))$). Now we can compute the inner product

$$\begin{aligned}\langle V, W\rangle &= \sum_{b=b_1}^{i-a_1} (-1)^b \binom{i-a_1-b_1}{b-b_1}\binom{n-3i+a_1+b_1+1}{k-2i+a_1+b_1+1}\\ &= (-1)^{b_1}\binom{n-3i+a_1+b_1+1}{k-2i+a_1+b_1+1}\sum_{b_2=0}^{i-a_1-b_1}\binom{i-a_1-b_1}{b_2} =\\ &= (-1)^{b_1}\binom{n-3i+a_1+b_1+1}{k-2i+a_1+b_1+1}(1-1)^{i-a_1-b_1} = 0,\end{aligned}$$

where in the last equality we use the hypothesis $a_1 + b_1 \leq |A| = i - 1 < i$. ■

Lemma 5.9 *For $2 \leq 2k \leq n$, the set*

$$\{\delta_{H_*J_*} \in \mathcal{S}f_k(n) \mid H_* = (h_1, \dots, h_k),\\ J_* = (j_1, \dots, j_k), h_\alpha < j_\alpha, \operatorname{card}(H_* \cup J_*) = 2k\}$$

contains a linearly independent set of cardinality $\binom{n}{k} - \binom{n}{k-1}$.

Proof By induction on n, starting with $n = 2$, $\Delta_{12} \in \mathcal{S}f_1(2)$. Suppose we have a linearly independent subset of polynomials in $\mathcal{S}f_k(n-1)$, Δ_k^{n-1}, having cardinality $\delta_k^{n-1} = \binom{n-1}{k} - \binom{n-1}{k-1}$ and a second set, Δ_{k-1}^{n-1}, of linearly independent polynomials in $\mathcal{S}f_{k-1}(n-1)$ with cardinality $\delta_{k-1}^{n-1} = \binom{n-1}{k-1} - \binom{n-1}{k-2}$. Then we can define a subset in $\mathcal{S}f_k(n)$ taking δ_k^{n-1} and all polynomials $\delta_{r,n} \cdot \delta_{L_*M_*}$ with $\delta_{L_*M_*}$ in δ_{k-1}^{n-1}, where the index r is the smallest element in the complement of $L_* \sqcup M_* \sqcup \{n\}$. These polynomials are linearly independent: $\sum c_{r,L_*M_*}\delta_{r,n} \cdot \delta_{L_*M_*} + \sum c_{H_*J_*}\delta_{H_*J_*} = 0$ implies $c_{r,L_*,M_*} = 0$ (look at the coefficient of x_n) and next, by induction, $c_{H_*J_*} = 0$. Their total number is

$$\delta_k^{n-1} + \delta_{k-1}^{n-1} = \binom{n-1}{k} + \binom{n-1}{k-1} - \binom{n-1}{k-1} - \binom{n-1}{k-2} = \binom{n}{k} - \binom{n}{k-1}. \ ■$$

Proposition 5.10 *For $0 \leq i \leq k \leq \frac{n}{2}$, there is a set of pairs*

$$\mathcal{B} = \{(H_*, J_*) \mid H_* = (h_1, h_2, \dots, h_i), J_* = (j_1, j_2, \dots, j_i), H_* \sqcup J_* \subset \underline{n}\}$$

such that the following set is a basis of the irreducible component $V_{(n-i,i)}$ of $\mathcal{S}f_k(n)$

$$\{\delta_{H_*J_*}\sigma_{k-i}^L \mid (H_*, J_*) \in \mathcal{B}, H_* \sqcup J_* \sqcup L = \underline{n}\}.$$

Proof The irreducible component $V_{(n-i,i)}$ of $\mathcal{S}f_k(n)$ is the orthogonal complement of $F_{i-1}\mathcal{S}f_k(n)$ in $F_i\mathcal{S}f_k(n)$. Its dimension is $\binom{n}{i} - \binom{n}{i-1}$; by Lemma 5.6, any polynomial $\Delta_{H_*J_*}\sigma_{k-i}^L$ belongs to $V_{(n-i,i)}$. From Lemma 5.9 there is a set of linearly independent polynomials $\{\delta_{H_*J_*} \in \mathcal{S}f_i(n)\}$ of cardinality $\binom{n}{i} - \binom{n}{i-1}$. The image of this set through the isomorphism $\psi : \mathcal{S}f_i(n) \to F_i\mathcal{S}f_k(n)$ gives the required basis. ■

An Algorithm Using the proofs of Lemma 5.9 and Proposition 5.10 we can describe an algorithm to compute bases of the irreducible modules $V_{(n-i,i)}$ of $\mathcal{S}f_k(n)$, $0 \leq i \leq k \leq \frac{n}{2}$.

(a) If $i = 0$, the elementary symmetric polynomial σ_k^n gives a basis of $V_{(n)}$.
For $i \geq 1$, the component $V_{(n-i,i)}$ in $\mathcal{S}f_k(n)$ is given by the orthogonal complement of $F_{i-1}\mathcal{S}f_k(n)$ in $F_i\mathcal{S}f_k(n) < \mathcal{S}f_k(n)$.

(b) First part of the algorithm: we construct a basis Δ_i^n of $F_{i-1}\mathcal{S}f_k(n)^{\perp}$ in $F_i\mathcal{S}f_i(n) = \mathcal{S}f_i(n)$, by induction on n. We start with $x_1 - x_2 \in F_0\mathcal{S}f_1(2)^{\perp}$; after the construction of the bases Δ_{i-1}^{n-1}, Δ_i^{n-1} of $F_{i-2}\mathcal{S}f_{i-1}(n-1)^{\perp}$ and $F_{i-1}\mathcal{S}f_i(n-1)^{\perp}$ respectively, take the basis $\Delta_i^n = \Delta_i^{n-1} \sqcup (x_* - x_n)\Delta_{i-1}^{n-1}$, where the index r in $(x_r - x_n)\Delta_{L_* M_*}$ is the smallest element in $\underline{n-1} \setminus (L_* \sqcup M_*)$.

(c) Second part of the algorithm: let $i = k$, Δ_i^n be a the basis of the $V_{(n-k,k)}$ component. If $1 \leq i \leq k-1$, multiply each polynomial $\Delta_{H_* J_*} \in \Delta_i^n$ with $\sigma_{k-i}^{\underline{n}\setminus(H_* \sqcup J_*)}$.

Example 5.11 Using the previous algorithm, we find the following basis of the component $V_{(5,2)}$ of $\mathcal{S}f_3(7)$:

$$
\begin{array}{ll}
(x_3 - x_4)(x_1 - x_2)(x_5 + x_6 + x_7), & (x_2 - x_6)(x_1 - x_4)(x_3 + x_5 + x_7),\\
(x_2 - x_4)(x_1 - x_3)(x_5 + x_6 + x_7), & (x_2 - x_6)(x_1 - x_5)(x_3 + x_4 + x_7),\\
(x_3 - x_5)(x_1 - x_2)(x_4 + x_6 + x_7), & (x_3 - x_7)(x_1 - x_2)(x_4 + x_5 + x_6),\\
(x_2 - x_5)(x_1 - x_3)(x_4 + x_6 + x_7), & (x_2 - x_7)(x_1 - x_3)(x_4 + x_5 + x_6),\\
(x_2 - x_5)(x_1 - x_4)(x_3 + x_6 + x_7), & (x_2 - x_7)(x_1 - x_4)(x_3 + x_5 + x_6),\\
(x_3 - x_6)(x_1 - x_2)(x_4 + x_5 + x_7), & (x_2 - x_7)(x_1 - x_5)(x_3 + x_4 + x_6),\\
(x_2 - x_6)(x_1 - x_3)(x_4 + x_5 + x_7), & (x_2 - x_7)(x_1 - x_6)(x_3 + x_4 + x_5).
\end{array}
$$

Proof of Proposition 3.4: monotonicity In order to show that $\mathcal{S}_{n+1} \cdot L\varphi_{k,n}(V(i)_n) \supseteq V(i)_{n+1}$, it is enough to prove that for $P_* \sqcup Q_* \sqcup R = \underline{n+1}$, $|P_*| = |Q_*| = i$ we have

$$\delta_{P_* Q_*}\sigma_{k-i}^R \in \mathcal{S}_{n+1} \cdot \{\delta_{H_* J_*}\sigma_{k-i}^L \mid H_* \sqcup J_* \sqcup L = \underline{n}, |H_*| = |J_*| = i\}.$$

One can suppose that $n+1 \in R$; otherwise, choose an index $j \in R$ ($|R| = n+1-2i > 0$) and take $(j, n+1) \cdot \delta_{P_* Q_*}\sigma_{k-i}^R$. If we multiply the equality ($n+1 > 2k \geq k+i$)

$$\sigma_{k-i}^R = \frac{1}{n+1-k-i}\sum_{t \in R}(t, n+1) \cdot \sigma_{k-i}^{R\setminus\{n+1\}}$$

by $\delta_{P_* Q_*}$, we obtain

$$\delta_{P_* Q_*}\sigma_{k-i}^R = \frac{1}{n+1-k-i}\sum_{t \in R}(t, n+1) \cdot \delta_{P_* Q_*}\sigma_{k-i}^{R\setminus\{n+1\}}$$

(the "transposition" $(n+1, n+1)$ is the identity permutation). ■

6 An Application to the Arnold Algebra

V. I. Arnold [A] computed the cohomology algebra of the pure braid group P_n, describing the first nontrivial cohomology algebra of a complex hyperplane arrangement, later generalized by Orlik-Solomon [OS] to arbitrary hyperplane arrangements. We denote this algebra by $\mathcal{A}(n)$.

Definition 6.1 (Arnold) The Arnold algebra $\mathcal{A}(n)$ is the graded commutative algebra (over $\mathbb{Q}$) generated in degree one by $\binom{n}{2}$ generators $\{w_{ij}\}$ having the following defining relations of degree two (the Yang-Baxter or the infinitesimal braid relations) YB_{ijk}:

$$\mathcal{A}(n) = \langle w_{ij},\ 1 \le i < j \le n \mid YB_{ijk} : w_{ij}w_{ik} - w_{ij}w_{jk} + w_{ik}w_{jk},\ 1 \le i < j < k \le n \rangle.$$

With the convention $w_{ij} = w_{ji}$ $(i \neq j)$, we define the natural action of the symmetric group $\mathcal{S}_n$ on the exterior algebra $\Lambda^*(w_{ij})$ by $\pi \cdot w_{ij} = w_{\pi(i)\pi(j)}$. The set of infinitesimal braid relations $\{YB_{ijk}\}$ is invariant (up to a sign) so we have a natural action of $\mathcal{S}_n$ on the Arnold algebra $\mathcal{A}(n)$. Church and Farb [CF] proved the representation stability of $\mathcal{A}(n)$ (see also [H]). We will use some results of the previous sections to describe the irreducible $\mathcal{S}_n$ submodules of $\mathcal{A}^1(n)$, $\mathcal{A}^2(n)$, and $\mathcal{A}^3(n)$. We also use the results of Section 5 to describe bases of the irreducible representations appearing in $\mathcal{A}^1(n)$ and $\mathcal{A}^2(n)$.

Proof of Theorem C (degree 1) This is a consequence of the isomorphism of $\mathcal{S}_n$-modules $\mathcal{A}^1(n) \cong L\mathcal{P}_2(n)$, $w_{ij} \mapsto \{i, j\}$, and of Proposition 2.10. ∎

In the same way we obtain the unstable decomposition.

Proposition 6.2 *In the unstable cases the decompositions are*

$$\mathcal{A}^1(2) = V_{(2)}, \quad \mathcal{A}^1(3) = V_{(3)} \oplus V_{(2,1)}.$$

Proof of Theorem D This is a consequence of the inductive method for constructing bases of the different pieces of $\mathcal{S}f_2(n) \cong L\mathcal{P}_2(n) \cong \mathcal{A}^1(n)$. For instance, the polynomial

$$\delta_{12}\sigma_n^{(12)'} = (x_1 - x_2)(x_3 + x_4 + \cdots + x_n)$$

corresponds to the linear combination of sets

$$\big(\{1,3\} + \{1,4\} + \cdots + \{1,n\}\big) - \big(\{2,3\} + \{2,4\} + \cdots + \{2,n\}\big),$$

and this corresponds to Ω^n_{12}. Similarly, the polynomial $\delta_{ij}\delta_{lk} = (x_i - x_j)(x_l - x_k)$ corresponds to Ω_{ijkl}. ∎

The vector space $L\mathcal{P}_3(n)$ is isomorphic to $I^2(n)$, the degree two component of the ideal of the infinitesimal braid relations

$$\{i, j, k\} \leftrightarrow YB_{ijk} : w_{ij}w_{ik} - w_{ij}w_{jk} + w_{ik}w_{jk},$$

but they are not isomorphic as $\mathcal{S}_n$-modules, since the symmetric group action on $I^2(n)$ involves signs. For instance, $(12)\cdot YB_{123} = w_{12}w_{23} - w_{12}w_{13} + w_{23}w_{13} = -YB_{123}$.

Proposition 6.3 *For $n \ge 4$ the degree two component of the ideal of relations decomposes as*

$$I^2(n) = V(1,1)_n \oplus V(1,1,1)_n.$$

For $n = 2$ we have $I^2(2) = 0$ and for $n = 3$ we have $I^2(3) = V_{(1,1,1)}$.

Proof The characters of the irreducible modules $V_{(n-2,1,1)}$ and $V_{(n-3,1,1,1)}$ can be computed using the Frobenius formula and are given in the character table in the proof of the next lemma. We obtain the character of $I^2(n)$ by direct computation. The symmetric group acts on the canonical basis $\{YB_{ijk}\}$ of $I^2(n)$ by permuting the

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elements of this basis and adding a $\pm$ sign due to graded commutativity. The relation YB_{ijk} is invariant (up to sign) by a permutation π if and only if $\{i, j, k\}$ is a union of cycles of π. If the permutation π has type $(i_1; i_2; \dots; i_n)$ (i_q is the number of cycles of length q), then π leaves invariant the elements YB_{ijk} corresponding to three fixed points i, j, k (the number of relations of this first type is $\binom{i_1}{3}$) and also elements YB_{pqr} corresponding to a three cycle (p, q, r) (and the number of relations of this second type is i_3). In the last case, if $\{i\}$ is a fixed point of π and (u, v) is a two-cycle, we have $\pi \cdot Y_{iuv} = -Y_{iuv}$ (and the total number of such relations is $i_1 i_2$). Therefore the value of the character on π is

$$\chi_{I^2(n)}(i_1; i_2; \dots; i_n) = \binom{i_1}{3} + i_3 - i_1 i_2,$$

and this is equal to $\chi_{V(n-2,1,1)}(i_1; i_2; \dots; i_n) + \chi_{V(n-3,1,1,1)}(i_1; i_2; \dots; i_n)$. ∎

Lemma 6.4 *For $n \geq 7$ the degree two component of the exterior algebra $\Lambda^2(n) = \Lambda^*(w_{ij})_{1\leq i<j\leq n}$ decomposes as*

$$\Lambda^2(n) = 2V(1)_n \oplus 2V(2)_n \oplus 3V(1,1)_n \oplus V(3)_n \oplus 2V(2,1)_n \oplus V(1,1,1)_n \oplus V(3,1)_n.$$

The unstable cases have the following decompositions:

$$\begin{aligned}
&\Lambda^2(2) = 0,\\
&\Lambda^2(3) = V_{(2,1)} \oplus V_{(1,1,1)},\\
&\Lambda^2(4) = 2V_{(3,1)} \oplus V_{(2,2)} \oplus 2V_{(2,1,1)} \oplus V_{(1,1,1,1)},\\
&\Lambda^2(5) = 2V_{(4,1)} \oplus 2V_{(3,2)} \oplus 3V_{(3,1,1)} \oplus V_{(2,2,1)} \oplus V_{(2,1,1,1)},\\
&\Lambda^2(6) = 2V_{(5,1)} \oplus 2V_{(4,2)} \oplus 3V_{(4,1,1)} \oplus V_{(3,3)} \oplus 2V_{(3,2,1)} \oplus V_{(3,1,1,1)}.
\end{aligned}$$

Proof These decompositions are obtained from the expansion

$$\begin{aligned}
\Lambda^2(\mathcal{A}^1) &= \Lambda^2(V(0)_n \oplus V(1)_n \oplus V(2)_n)\\
&= \Lambda^2 V(1)_n \oplus \Lambda^2 V(2)_n \oplus V(1)_n \oplus V(2)_n \oplus (V(1)_n \otimes V(2)_n),
\end{aligned}$$

where

$$\begin{aligned}
V(1)_n \otimes V(2)_n &= V(1)_n \oplus V(2)_n \oplus V(1,1)_n \oplus V(3)_n \oplus V(2,1)_n,\\
\Lambda^2 V(1)_n &= V(1,1)_n,\\
\Lambda^2 V(2)_n &= V(1,1)_n \oplus V(2,1)_n \oplus V(1,1,1)_n \oplus V(3,1)_n.
\end{aligned}$$

The decomposition of the tensor product is from [M] (and can be checked using Littlewood–Richardson rule or using the characters from the following table). For the degree two exterior algebra one can use the following character table ($(i_1; i_2; \dots; i_n)$ stands for the conjugacy class with i_q cycles of length q):

	$\chi_V(i_1;\ldots;i_n)$	$\chi_V((i_1;\ldots;i_n)^2)$	$\chi_{\Lambda^2 V}(i_1;\ldots;i_n)$
$V(1)_n$	i_1-1	i_1+2i_2-1	$\binom{i_2-1}{2}-i_2$
$V(1,1)_n$	$\binom{i_1-1}{2}-i_2$		
$V(2)_n$	$\frac{i_1(i_1-3)}{2}+i_2$	$\frac{(i_1+2i_2)(i_1+2i_2-3)}{2}+2i_4$	$\frac{i_1(i_1-3)(i_1^2-3i_1-2)}{8}$ $+\frac{(i_1^2-5i_1+3)i_2-i_2^2}{2}-i_4$
$V(3)_n$	$\frac{i_1(i_1-1)(i_1-5)}{6}$ $+i_2(i_1-1)+i_3$		
$V(2,1)_n$	$\frac{i_1(i_1-2)(i_1-4)}{3}-i_3$		
$V(1,1,1)_n$	$\binom{i_1-1}{3}+i_2(1-i_1)+i_3$		
$V(3,1)_n$	$\frac{i_1(i_1-1)(i_1-3)(i_1-6)}{8}$ $+i_2\binom{i_1-1}{2}-\binom{i_2}{2}-i_4$		

The entries in the second column are computed using the Frobenius formula; in the third column we used

$$(i_1;i_2;i_3;i_4;\ldots)^2=(i_1+2i_2;2i_4;i_3;\ldots),$$

and in the last column we used the formula

$$\chi_{\Lambda^2(V)}(\pi)=\frac{1}{2}\big[(\chi_V(\pi))^2-\chi_V(\pi^2)\big]$$

(see [K]). ■

Proof of Theorem C (degree 2) This is a consequence of Proposition 6.3 and Lemma 6.4. ■

Similarly we have the following proposition.

Proposition 6.5 *In the unstable cases we have*

$$\mathcal{A}^2(2)=0,$$
$$\mathcal{A}^2(3)=V_{(2,1)},$$
$$\mathcal{A}^2(4)=2V_{(3,1)}\oplus V_{(2,2)}\oplus V_{(2,1,1)},$$
$$\mathcal{A}^2(5)=2V_{(4,1)}\oplus 2V_{(3,2)}\oplus 2V_{(3,1,1)}\oplus V_{(2,2,1)},$$
$$\mathcal{A}^2(6)=2V_{(5,1)}\oplus 2V_{(4,2)}\oplus 2V_{(4,1,1)}\oplus V_{(3,3)}\oplus 2V_{(3,2,1)}.$$

These decompositions coincide with the formulae from [CF]. The last proposition is refined in [AAB], using the "type" decomposition of the Križ model for the configuration space of a complex projective manifold. The results of this section are necessary for the cohomological computations of [AAB].

Proof of Theorem E For the degree three part of the Arnold algebra, we compute the character polynomial directly. For an arbitrary permutation $\sigma\in\mathcal{S}_n$ of type $(i_1;i_2;\ldots;i_n)$, any 6-tuple of 1-cycles $(i)(j)(k)(l)(m)(p)$, $1\le i<j<k<l<m<p\le n$, fixes the monomials having six distinct indices from $\{i,j,k,l,m,p\}$ and there are $15\binom{i_1}{6}$ such monomials. The permutations $(i,j)\,(k)\,(l)\,(m)\,(p)$ have non-zero

contribution to the character for the monomials $w_{ij}w_{kl}w_{mp}, w_{ij}w_{km}w_{lp}, w_{ij}w_{lm}w_{kp}$. For a permutation involving $(i, j)(k, l, m)$ for $i < j$ and $k < l < m$ the monomial in the Arnold basis

$$w_{ij}w_{kl}w_{lm} \to w_{ij}w_{lm}w_{km} = -w_{ij}w_{km}w_{lm} = -w_{ij}w_{kl}w_{lm} + w_{ij}w_{kl}w_{km}$$

contributes -1 to the character giving in total $-i_2 i_3$; similar computations for other permutations give the character of $\mathcal{A}^3(n)$:

$$\begin{aligned}\chi_{\mathcal{A}^3(n)}&(i_1; i_2; \dots; i_n)\\ &= 15\binom{i_1}{6} + 3\binom{i_1}{4}i_2 - \binom{i_1}{2}\binom{i_2}{2} - 5\binom{i_2}{3} + 3\binom{i_3}{2} - \binom{i_1}{2}i_4 - i_2 i_4\\ &\quad + i_6 + 20\binom{i_1}{5} + 2\binom{i_1}{3}i_2 - i_2 i_3 - \binom{i_1}{2}i_3 + 6\binom{i_1}{4} - 2\binom{i_2}{2}.\end{aligned}$$

Using the characters of the corresponding irreducible modules given explicitly in [S] we get the decomposition. ∎

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