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THE RELATIONSHIP BETWEEN DISTANCE FORMULAE AND COMPACT PERTURBATIONS FOR REFLEXIVE ALGEBRAS

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ABSTRACT. For completely distributive CSL algebras, hyper-reflexivity is equivalent to a description of the compact perturbation of the algebra analogous to the Fall-Arveson-Muhly Theorem for nest algebras.

The Arveson distance formula for nest algebras [2] is fundamental to the penetrating analysis that has been possible for these algebras. Power and I [8] showed that such a formula does not exist in general for the larger class of commutative subspace lattice (CSL) algebras. One consequence of the distance formula for nests is a pretty characterization [10] of the algebra of compact perturbations of a nest algebra. Conversely, Andersen's analysis [1] of the compact perturbations of nest algebras lead to Larson's solution [14] of the 'Ringrose problem', and ultimately to our complete similarity invariants for arbitrary nests [5].

The purpose of this note is to show that for the class of completely distributive CSL algebras, the existence of a distance formula is equivalent to the appropriate analogue of the Fall-Arveson-Muhly characterization of the compact perturbation of a nest algebra. A lattice is *completely distributive* if the most general possible distributive law for lattices holds (c.f. [6, § 23]). This class has proven to be *the* tractable subclass of all CSL's from a number of perspectives. For example, it is precisely this class for which the finite rank operators in the algebra are weak* dense [15]. The properties that we will use will be elaborated upon later.

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All Hilbert spaces, usually denoted $\mathcal{H}$, will be separable. The algebras of all bounded and compact operators are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$, respectively. Let $\mathcal{L}$ be a commutative subspace lattice endowed with its strong operator topology. (This is an intrinsic topology to the lattice, as it is equivalent to the metric topology obtained from any faithful, normal valuation on $\mathcal{L}$.) Let $C^*_{s*}(\mathcal{L}, \mathcal{B}(\mathcal{H}))$ denote the C*-algebra of continuous functions from $\mathcal{L}$ into $\mathcal{B}(\mathcal{H})$ endowed with the strong* topology. The ideal of norm continuous, compact valued functions is denoted by $C^*_n(\mathcal{L}, \mathcal{K})$. For T in $\mathcal{B}(\mathcal{H})$, let Φ_T be the element of $C^*_{s*}(\mathcal{L}, \mathcal{B}(\mathcal{H}))$ given by

$$\Phi_T(L) = P(L)^{\perp} T P(L).$$

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Here, $P(L)$ denotes the orthogonal projection onto L. The desired distance estimate asks about the relationship between dist(T, Alg($\mathcal{L}$)) and

$$\|\Phi_T\| = \sup_{L\in\mathcal{L}} \|P(L)^{\perp}TP(L)\|.$$

Clearly, $\|\Phi_T\| \leq \text{dist}(T, \text{Alg}(\mathcal{L}))$. The algebra is called *hyperreflexive* if there is a constant C such that $\text{dist}(T, \text{Alg}(\mathcal{L})) \leq C\|\Phi_T\|$. When $\mathcal{L}$ is a nest, one may take $C = 1$ [2]; for complemented CSL's, one may take $C = 2$ [4, 16]; but in general, there is no such constant [8]. From the Open Mapping Theorem, $\mathcal{L}$ is hyperreflexive if and only if the map $\Phi(T) := \Phi_T$ has closed range.

When K is compact, it is elementary that Φ_K is norm continuous and compact valued. Since the kernel of Φ is precisely Alg($\mathcal{L}$), Φ maps Alg($\mathcal{L}$) + $\mathcal{K}$ into $C_n^*(\mathcal{L}, \mathcal{K})$. The Fall-Arveson-Muhly Theorem is the converse for nests: *If* $\mathcal{N}$ *is a nest, and* T *is an operator such that* Φ_T *belongs to* $C_n^*(\mathcal{N}, \mathcal{K})$*, then* T *belongs to* $\mathcal{QT}(\mathcal{N}) = \text{Alg}(\mathcal{N}) + \mathcal{K}$. We will say that $\mathcal{L}$ has the FAM property provided that when Φ_T belongs to $C_n^*(\mathcal{L}, \mathcal{K})$, then T belongs to Alg($\mathcal{L}$) + $\mathcal{K}$.

In particular, FAM implies that Alg($\mathcal{L}$) + $\mathcal{K}$ is norm closed. Froelich [11] has shown that there are CSL's for which Alg($\mathcal{L}$) + $\mathcal{K}$ is not closed. It is always closed for completely distributive lattices [13]. The Laurie-Longstaff Theorem shows that Alg($\mathcal{L}$) $\cap$ $\mathcal{K}$ is weak* dense in Alg($\mathcal{L}$). The M-ideal arguments of [9] also show that Alg($\mathcal{L}$) + $\mathcal{K}$ is norm closed, and that the natural injection

$$\text{Alg}(\mathcal{L})/\,\text{Alg}(\mathcal{L})\cap\mathcal{K} \hookrightarrow \text{Alg}(\mathcal{L})+\mathcal{K}/\,\mathcal{K}$$

is isometric. Moreover, it shows that the unit ball of Alg($\mathcal{L}$) $\cap$ $\mathcal{K}$ is weak* dense in the unit ball of Alg($\mathcal{L}$). Thus, there is a norm one approximate identity for $\mathcal{K}$ in Alg($\mathcal{L}$) $\cap$ $\mathcal{K}$. For related results, see [12].

Wagner [18] has shown that completely distributive CSL's are compact (c.f.[7]). Arveson [3] generalized Voiculescu's Theorem [17] and Andersen's Theorem [1] to a common version valid for many non-separable C*-algebras. One consequence is that for compact CSL's, there is a norm one approximate identity for $\mathcal{K}$ consisting of positive finite rank operators R_n such that

$$\lim_{n\to\infty} \|R_nP(L) - P(L)R_n\| = 0$$

uniformly on $\mathcal{L}$. Of course, these operators do not lie in Alg($\mathcal{L}$). We will refer to such a sequence as a *quasi-central approximate unit for* $\mathcal{L}$. The commutator $XY - YX$ will be denoted by $[X, Y]$. Most of the results just described are systematically developed in [6].

THEOREM 1. *Let* $\mathcal{L}$ *be a completely distributive commutative subspace lattice. Then* $\mathcal{L}$ *is hyperreflexive if and only if* $\mathcal{L}$ *satisfies FAM.*

PROOF. Assume that $\mathcal{L}$ is hyperreflexive with distance constant C. Let T be an operator such that Φ_T belongs to $C_n^*(\mathcal{L}, \mathcal{K})$. Let R_n be a quasi-central approximate unit for $\mathcal{L}$; and set $K_n = R_nTR_n$. First, it will be shown that Φ_{K_n} converge uniformly to Φ_T. Indeed,

$$\begin{aligned}\Phi_T - \Phi_{K_n} &= P^{\perp}TP - P^{\perp}R_nP^{\perp}TR_nP - P^{\perp}R_nPTR_nP \\ &= P^{\perp}\left(P^{\perp}TP - R_nP^{\perp}TPR_n\right) + \\ &\qquad P^{\perp}R_nP^{\perp}T[P,R_n] + P^{\perp}[P,R_n]TPR_n.\end{aligned}$$

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Clearly, the last two terms of this sum converge uniformly to 0. Since Φ_T is continuous on the compact set $\mathcal{L}$, the range of Φ_T is a compact set. Thus, the first term of the sum also converges to 0 uniformly. For use in proving the converse, note that if T is arbitrary, then the sequence Φ_{K_n} converges to Φ_T in the strong* topology.

Now since $\mathrm{Alg}(\mathcal{L})$ has distance constant C, we obtain

$$\begin{aligned}\mathrm{dist}(T, \mathrm{Alg}(\mathcal{L}) + \mathcal{K}) &\leq \inf \mathrm{dist}(T - K_n, \mathrm{Alg}(\mathcal{L}))\\ &\leq C \inf \|\Phi_{T-K_n}\|\\ &= 0.\end{aligned}$$

Since $\mathrm{Alg}(\mathcal{L}) + \mathcal{K}$ is closed, property FAM holds.

Next, note that the distance constant can be defined merely by examining finite rank operators. Indeed, by weak* compactness, every coset of $\mathrm{Alg}(\mathcal{L})$ contains an operator T such that

$$\|T\| = \mathrm{dist}(T, \mathrm{Alg}(\mathcal{L})).$$

Without loss of generality, suppose that $\|T\| = 1$. Then $K_n := R_n T R_n$ are finite rank, and

$$\lim_{n\to\infty} \|K_n\| = \lim_{n\to\infty} \mathrm{dist}(K_n, \mathrm{Alg}(\mathcal{L})) = 1.$$

From the computation in the previous paragraph, it follows that the sequence Φ_{K_n} converges to Φ_T in the strong* topology, and

$$\lim_{n\to\infty} \|\Phi_{K_n}\| = \|\Phi_T\|.$$

Hence,

$$\lim_{n\to\infty} \frac{\mathrm{dist}(K_n, \mathrm{Alg}(\mathcal{L}))}{\|\Phi_{K_n}\|} = \frac{\mathrm{dist}(T, \mathrm{Alg}(\mathcal{L}))}{\|\Phi_T\|}.$$

Assume that $\mathcal{L}$ is not hyperreflexive. We will construct orthogonal families of projections $\{E_n\}$ and $\{F_n\}$ in $\mathcal{L}''$ so that the distance constant for the distance of elements of $E_n\mathcal{B}(\mathcal{H})F_n$ to $\mathrm{Alg}(\mathcal{L})$ tends to infinity. Once this is accomplished, choose finite rank operators $K_n = E_n K_n F_n$ such that $\|K_n\| = 1$, $\lim_{n\to\infty} \|\Phi_{K_n}\| = 0$, and

$$\lim_{n\to\infty} \mathrm{dist}(K_n, \mathrm{Alg}(\mathcal{L})) = 1.$$

Set $T = \sum_{n\geq 1} K_n$. Clearly, $\Phi_T = \sum_{n\geq 1} \Phi_{K_n}$ is a norm convergent sum, and hence belongs to $C_n^*(\mathcal{L}, \mathcal{K})$. But if K is any compact operator, then $\|E_n K F_n\| < 1/2$ for n sufficiently large. Thus, for n sufficiently large,

$$\begin{aligned}\mathrm{dist}(T - K, \mathrm{Alg}(\mathcal{L})) &\geq \mathrm{dist}(E_n(T-K)F_n, \mathrm{Alg}(\mathcal{L}))\\ &\geq \mathrm{dist}(K_n, \mathrm{Alg}(\mathcal{L})) - \|E_n K F_n\| \geq \frac{1}{2}.\end{aligned}$$

Consequently, T is not in $\overline{\mathrm{Alg}(\mathcal{L}) + \mathcal{K}}$, and FAM fails.

It remains to construct the families of projections. One can build a binary tree of projections in $\mathcal{L}''$ which, at each step, splits the subspaces into two, neither of which is a finite sum of atoms, so that the intersection along each infinite decreasing chain is 0 or a minimal projection. (If $\mathcal{L}''$ is isomorphic to $\mathcal{L}^{\infty}(0, 1)$ or to $\ell^{\infty}(\mathbf{Q} \cap (0, 1))$, this is easily achieved by disecting the interval into diadic subintervals. The general case just combines these two plans.)

For projections in $\mathcal{L}'$, let the distance constant for $E\mathcal{B}(\mathcal{H})F$ relative to $\mathrm{Alg}(\mathcal{L})$ be

$$\beta(E, F) := \sup_{T \in E\mathcal{B}(\mathcal{H})F} \frac{\mathrm{dist}(T, \mathrm{Alg}(\mathcal{L}))}{\sup_{L \in \mathcal{L}} \|P(L)^{\perp} T P(L)\|}.$$

Note that one of the four subspaces $E\mathcal{B}(\mathcal{H})F$, $E\mathcal{B}(\mathcal{H})F^{\perp}$, $E^{\perp}\mathcal{B}(\mathcal{H})F$, and $E^{\perp}\mathcal{B}(\mathcal{H})F^{\perp}$ has infinite distance constant. For if they were all bounded by a constant C, the distance constant for $\mathcal{L}$ would be at most $2C$. Using the binary tree, construct decreasing sequences $\{E_n\}$ and $\{F_n\}$ so that $\beta(E_n, F_n)$ is infinite, and so that $\bigwedge E_n = E_0$ and $\bigwedge F_n = F_0$ are 0 or atoms of $\mathcal{L}$. Either way, $\beta(E_0, F_0) = 1$.

Next, notice that $\beta(E_n^{\perp}, F_n^{\perp})$ increases to $\beta(E_0^{\perp}, F_0^{\perp})$, namely infinity. Thus it is easy to pick a sequence $n_1 < n_2 < n_3 < \cdots$ so that

$$\lim_{k \to \infty} \beta(E_{n_k} - E_{n_{k+1}}, F_{n_k} - F_{n_{k+1}}) = \infty.$$

This is the desired sequence. ∎

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