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CAN A SEMI-PRIME RING BE A FINITE UNION OF RIGHT ANNIHILATORS?

BY

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ABSTRACT. The interesting question of the title was posed by J. Bergen and this note answers it in the negative. The main result characterizes rings which can be a finite union of proper right annihilators, and shows that any such commutative ring must have a total annihilator.

The interesting and appealing question of the title was raised by J. Bergen. The term "union" means set theoretic union. The answer to the question is no, as one might expect on the grounds that an algebraic structure is rarely a finite union of proper substructures. Indeed, a first thought might be that a semi-prime ring cannot be a finite union of proper right ideals. Somewhat surprisingly, this can occur and shows that Bergen's question is not trivial. Our main result, which answers Bergen's question is elementary, but not transparent, and gives a characterization of those rings which can be a finite union of proper right annihilators. It also shows that any such commutative ring must have a nonzero total annihilator. Some examples are given to show that the main result is quite sharp.

Throughout this note let R be an associative ring, and for any nonempty subset $A \subset R$, let $r(A) = \{r \in R | ar = 0 \text{ for all } a \in A\}$ and $\ell(A) = \{r \in R | ra = 0 \text{ for all } a \in A\}$. Of course we do not assume that $1 \in R$, since then it is trivial that R could not be a union of proper right ideals. Essential to our argument is a result of B. H. Neumann [2; Lemma, p. 239] which shows that if a group is a union of proper subgroups then one may assume that each subgroup is of finite index. For convenience, we state his result.

LEMMA. *Let $H_1, \ldots, H_n$ be distinct subgroups of a group G, and let $\{C_{ij} = H_i g_{ij}\}$ be a finite collection of cosets. If $G = \bigcup C_{ij}$, then some H_i has finite index in G. Furthermore if H_j has infinite index for $j \leqq m$ and finite index for $j > m$, then $G = \bigcup\{C_{ij} | i > m\}$.*

It follows from the Lemma that if a ring is a union of proper ideals then one can assume that the intersection of the ideals must be of finite index. This gives rise to an easy example of a domain which is a finite union of proper ideals.

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EXAMPLE 1. Let F be a finite field and $F\{x, y\}$ the free algebra over F in indeterminates x and y. Set $R = (x, y)$, the ideal of $F\{x, y\}$ of all elements with zero constant term, and $I = (x^2, xy, yx, y^2)$, the ideal of R of all elements with no monomial of degree one. Since $Fx + Fy$ is a two dimensional vector space over F, there is $\{v_i | 1 \leqq i \leqq \text{card}(F) + 1\} \subset Fx + Fy$ so that $\bigcup Fv_i = Fx + Fy$. It follows that R is the union of its proper ideals $J_i = Fv_i + I$.

Before stating our main result, we recall that a semi-prime ring R is a ring which contains no nonzero nilpotent ideal. Note that a semi-prime ring R cannot contain a nonzero nilpotent right ideal B since then $RB + B$ would be a nilpotent ideal of R. We can now give the main result of this note.

THEOREM. *Let R be a ring with nonempty subsets $A_1, \ldots, A_n$ so that $R = \bigcup r(A_i)$. Then either $A_i = \{0\}$ for some i, or $\ell(R) \neq 0$, or for some $a \in \bigcup A_i$, aR is a nonzero nilpotent right ideal of R. If R is a commutative ring, then either some $A_i = \{0\}$, or $\ell(R) \neq 0$.*

PROOF. There is nothing to prove unless $A_i \neq \{0\}$ for all i, so let $a_i \in A_i - \{0\}$ and observe that $r(A_i) \subset r(a_i)$. Hence, without loss of generality we may assume that $A_i = \{a_i\}$. By applying the Lemma, we may assume also that each $(r(a_i), +)$ has finite index in $(R, +)$, and consequently, $T = \bigcap r(a_i)$ has finite index in $(R, +)$. Thus $(R/T, +)$ is a finite group and End (R/T), its ring of group endomorphisms, must also be finite. Consider $F : R \longrightarrow$ End $(R/T, +)$ given by $(s + T)((r)F) = sr + T$. It is easy to see that F is a function and ring homomorphism, so if Ker $F = I$, then R/I is a finite ring. Furthermore, since $I =$ Ker F, we have that $RI \subset T$.

Since R/I is a finite ring, its radical N/I is nilpotent by standard structure theory [1], and so satisfies $N^k \subset I$ for some k. Suppose first that $N = R$, so $R^{k+1} \subset RI \subset T$, and $a_1 R^{k+1} = 0$ results. If $a_1 R = 0$ then $a_1 \in \ell(R)$, and if $a_1 R \neq 0$ but m is minimal with $a_1 R^m = 0$, then $a_1 R^{m-1} \subset \ell(R)$. In either case, $\ell(R) \neq 0$ and the theorem is proved. Therefore, we may assume that $N \neq R$, and so R/N is a finite semi-prime ring with identity element $1_{R/N}$.

Since $R/N = \bigcup (r(a_i) + N)/N$ and each $(r(a_i) + N)/N$ is a right ideal in R/N, one of these contains $1_{R/N}$, and so $R = r(a_i) + N$ for some i. Therefore, $a_i R = a_i N$, and since $N^k \subset I, a_i N^{k+1} \subset a_i R N^k \subset a_i RI \subset a_i T = 0$, it follows that $(a_i R)^{k+1} = (a_i N)^{k+1} \subset a_i N^{k+1} = 0$. If $a_i R = 0$, then $a_i \in \ell(R)$, and otherwise $a_i R$ is a nonzero nilpotent right ideal of R, finishing the proof when R is not commutative. When R is commutative, it remains to show that $\ell(R) \neq 0$. Let m be minimal so that $a_i N^m = 0$ and note that when $m = 1, 0 = a_i N = a_i R$, so $a_i \in \ell(R)$. When $m > 1$ there is $y \in N^{m-1}$ so that $a_i y \neq 0$, but of course, $a_i y N \subset a_i N^m = 0$. Now $a_i y R = a_i y r(a_i) + a_i y N = a_i y r(a_i) = 0$ since $a_i y = y a_i$. Therefore $a_i y \in \ell(R)$, completing the proof of the theorem. □

To answer the question of the title, consider the Theorem when R is a semi-prime ring. The possibility that $\ell(R) \neq 0$ cannot occur, and by our earlier observation, R contains no nilpotent right ideal. Thus, the following corollary is immediate.

COROLLARY. *If R is a semi-prime ring and $R = \bigcup r(A_i)$ for nonempty subsets*

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$A_1, \ldots, A_n$ *of* R, *then some* $A_i = \{0\}$.

We end the paper with a few examples which tie up some loose ends in the Theorem. First, in the commutative case, one might wonder whether some $a \in \cup A_i$ must be in $\ell(R)$. Our next example shows that this need not happen.

EXAMPLE 2. Let F be any finite field, and let R be the ideal in $F[x,y]/(x^2,y^2)$ generated by $x+(x^2,y^2)$ and $y+(x^2,y^2)$. Equivalently $R = \{ax+by+cxy | a,b,c \in F\}$ where $x^2 = y^2 = 0$. Now, $\ell(R) = Fxy$. For $ax+by \in R - \{0\}$ there is $cx+dy \neq 0$ so that $(cx+dy)(ax+by) = (ad+bc)xy = 0$. Clearly $F(ax+by)+Fxy \subset r(cx+dy) \neq R$, and so for appropriate $t_i = c_ix+d_iy, R = \bigcup r(t_i)$ and no $r(t_i) = R$.

Another question concerning the Theorem is whether, in fact, one has the same dichotomy in general as for commutative rings. That is, if no $A_i = \{0\}$, must $\ell(R) \neq 0$? Our next example shows that neither of these two choices must occur.

EXAMPLE 3. Let F be a finite field, denote the number of elements in F by card(F), and set

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix} \in M_3(F) \right\}.$$

In terms of the usual matrix units $\{e_{ij}\}, R = \{ae_{12}+be_{13}+ce_{23}+de_{33} | a,b,c,d, \in F\}$. It is easy to see that R is a ring. For $y = ae_{12}+be_{13}+ce_{23}+de_{33} \in R, y \in r(e_{13})$ if $d = 0$ and $y \in r(e_{12} - cd^{-1}e_{13})$ if $d \neq 0$, so R is the union of card(F) + 1 right annihilators. Furthermore, $\ell(R) = 0$ since $y \in \ell(R)$ yields $0 = ye_{33}$, forcing $y = ae_{12}$, and then $0 = ye_{23}$ shows $y = 0$.

One can extend Example 3 to get a similar example of an infinite ring. Specifically, let $\{t_i\}$ be the card(F) + 1 elements of R described above and satisfying $R = \bigcup r(t_i)$. For any ring $S, R \oplus S = \bigcup r((t_i, 0))$. Furthermore, if $\ell(S) = 0$ then $\ell(R \oplus S) = 0$. When $\ell(S) \neq 0$, one obtains an example, like that in the commutative case; that is, $\ell(R \oplus S) \neq 0, R \oplus S = \bigcup r((t_i, 0))$ and $(t_i, 0) \notin \ell(R \oplus S)$ for any i. In fact, $(t_i, 0)(R \oplus S)^k \not\subset \ell(R \oplus S)$ for any i and any k.

As a final comment, we point out the well-known fact that the question of the title has a positive answer if the union is not required to be finite.

EXAMPLE 4. For each positive integer i, let R_i be a semi-prime ring, and set $R = \oplus R_i$ (direct sum). If $f_i : R_i \longrightarrow R$ is the usual injection of R_i into the "ith co-ordinate" of R, then $R = \bigcup r(f_i(a_i))$ for any choice of $a_i \in R_i - \{0\}$.

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