Can. J. Math., Vol. XXXIX, No. 6, 1987, pp. 1405-1412

ON THE STRUCTURE OF CERTAIN NEST ALGEBRA MODULES

G. J. KNOWLES

0. Introduction. Let $\mathscr{A}$ be a nest algebra of operators on some Hilbert space H. Weakly closed $\mathscr{A}$-modules were first studied by J. Erdos and S. Power in [**4**]. It became apparent that many interesting classes of non self-adjoint operator algebras arise as just such a module. This paper undertakes a systematic investigation of the correspondence which arises between such modules and order homomorphisms from Lat $\mathscr{A}$ into itself. This perspective provides a basis to answer some open questions arising from [**4**]. In particular, the questions concerning unique "determination" and characterization of maximal and minimal elements under this correspondence, are resolved. This is then used to establish when the determining homomorphism is unique.

Throughout this paper $\mathscr{E}$ will denote a complete nest of projections on a Hilbert space H. The nest algebra Alg $\mathscr{E}$ corresponding to $\mathscr{E}$ will be denoted by $\mathscr{A}$. The terminology and notation concerning nest algebras used in this paper are standard and may be found in [**3**, **5**]. All Hilbert spaces are complex and the term "projection" will always mean orthogonal projection. The rank one operator $x \to \langle x, e\rangle f$ will be denoted by $e \otimes f$. Order preserving homomorphisms from $\mathscr{E}$ into itself will be denoted by lower case Greek letters ϕ, $\psi, \dots$. Finally all modules considered will be two sided $\mathscr{A}$-submodules of $\mathscr{L}(H)$ under multiplication. The author would like to take this opportunity to thank J. Erdos for his many valuable suggestions during the research. The author also would like to thank the referee for the many thoughtful comments which both clarified the results and made the paper more readable.

1. Determination of modules. Of crucial importance in the results below will be the collection Hom $\mathscr{E}$ given by:

$$\text{Hom } \mathscr{E} = \{\phi:\phi \text{ is an order homomorphism from } \mathscr{E} \text{ into } \mathscr{E}\}.$$

Given ϕ in Hom $\mathscr{E}$, there is associated a weakly closed $\mathscr{A}$-submodule of $\mathscr{L}(H)$ given by:

$$\mathscr{U}_\phi = \{X \in \mathscr{L}(H): \quad [I - \phi(E)]XE = 0\}.$$

Received March 17, 1982 and in revised form April 14, 1983. This research was supported in part by a grant from the Science Research Council of Great Britain.

To each ϕ in Hom $\mathscr{E}$ there is naturally associated a ϕ_* in Hom $\mathscr{E}$ given by

$$\phi_*(E) = \inf\{\phi(F):F > E\}; \quad E \in \mathscr{E}$$

(with the convention that $\phi_*(I) = I$). Observe that Hom $\mathscr{E}$ has a natural partial ordering given by $\phi \leqq \psi$ if and only if $\phi(E) \leqq \psi(E)$ for all E in $\mathscr{E}$. It follows that $\phi < \psi$ implies $\phi_* \leqq \psi_*$.

The weakly closed $\mathscr{A}$-module $\mathscr{U}$ is said to be *determined* by the order homomorphism ϕ, if $\mathscr{U} = \mathscr{U}_\phi$, that is

$$\mathscr{U} = \{X \in \mathscr{L}(H): \quad [I - \phi(E)]XE = 0\}.$$

By $H_{\mathscr{U}}$ will be meant the subset of Hom $\mathscr{E}$ given by:

$$H_{\mathscr{U}} = \{\phi \in \text{Hom}\,\mathscr{E}: \quad \phi \text{ determines } \mathscr{U}\}.$$

Thus Hom $\mathscr{E}/_{H_{\mathscr{U}}}$ consists of all equivalence classes $[\phi]$ where $\phi \sim \psi$ if and only if $\mathscr{U}_\phi = \mathscr{U}_\psi$. Following [**4**] a ϕ in Hom $\mathscr{E}$ is said to be *left continuous* when, for each $F \neq 0$ in $\mathscr{E}$

$$\sup_{E<F} \phi(E) = \phi\left(\sup_{E<F} E\right) \quad [= \phi(F_-)].$$

It will be necessary to use the following result concerning rank 1 elements of $\mathscr{U}$ which first appeared for nest algebras in [**5**].

LEMMA 1.1 [**4**]. *A non-zero operator of rank l, $e \otimes f$ is in $\mathscr{U}_\phi$ if and only if, for some E in $\mathscr{E}$, $\phi_*(E)f = f$ and $E^\perp e = e$.*

Remark. It will be shown that each $[\phi]$ in Hom $\mathscr{E}/_{H_{\mathscr{U}}}$ contains a maximal and minimal element. Indeed $\psi \in [\phi]$ if and only if it is in the order interval in Hom $\mathscr{E}$ determined by those two elements. Furthermore $[\phi]$ contains a unique left continuous representative.

THEOREM 1.2. *Let ϕ, ψ be in* Hom $\mathscr{E}$, *then with the notation above*

$$\phi_* = \psi_* \textit{ if and only if } \mathscr{U}_\phi = \mathscr{U}_\psi.$$

Proof. Suppose that $\phi_*(E) \neq \psi_*(E)$ for some E in $\mathscr{E}$. It may as well be assumed that $\phi_*(E) > \psi_*(E)$. Thus an F can be found with the property that $F > E$ and $\psi(F) < \phi_*(E)$. Choose e a non-zero vector in the range of $F \ominus E$ and an f non-zero vector in the range of $\phi_*(E) \ominus \psi(F)$. An application of Lemma 1.1 above shows that $e \otimes f$ is in $\mathscr{U}_\phi$. However

$$[I - \psi(F)](e \otimes f)F = e \otimes f = 0.$$

Thus it is not the case that $e \otimes f$ is in $\mathscr{U}_\psi$, showing $\mathscr{U}_\psi$ and $\mathscr{U}_\phi$ are different modules.

For the reverse implication assume that $\phi_*(E) = \psi_*(E)$ for every E in $\mathscr{E}$. Given $e \otimes f$, a non-zero operator of rank 1, by Lemma 1.1 $e \otimes f$ is in $\mathscr{U}_\psi$ if and only if, for some F in $\mathscr{E}$

$$\psi_*(F)f = f \quad \text{and} \quad F^\perp e = e.$$

By assumption this holds if and only if

$$\phi_*(F)f = f \quad \text{and} \quad F^\perp e = e.$$

That is to say, if and only if $e \otimes f$ is in $\mathscr{U}_\phi$.

It is shown in ([4], Lemmas 1.2 and 1.6) that any weakly closed $\mathscr{A}$-submodule $\mathscr{V}$ of $\mathscr{L}(H)$ under multiplication is the weak closure of its finite rank elements. In addition, each finite rank operator in $\mathscr{V}$ is the sum of rank one members of $\mathscr{V}$.

By linearity, $\mathscr{U}_\phi$ and $\mathscr{U}_\psi$ contain the same finite rank operators. Hence the weak closures of these finite rank elements are also the same. By the above remarks we have $\mathscr{U}_\phi = \mathscr{U}_\psi$ as required.

Remark. The above theorem shows that, when dealing with a fixed module $\mathscr{U}$, there is no ambiguity in writing E_* for $\phi_*(E)$ whenever ϕ is in $H_\mathscr{U}$. It is also worth noting, for future use, that, if $\phi, \psi \in H_\mathscr{U}$ and $E \neq E_-$ then $\phi(E) = \psi(E)$. This is shown by observing that

$$\phi(E) = \psi_*(E_-) = \psi_*(E_-) = \psi(E).$$

With $\mathscr{U}$ as above, suppose, in addition, that the homomorphism ϕ in Hom $\mathscr{E}$ is left continuous. The following is a method of recovering the values of $\phi(E)$ for E in $\mathscr{E}$ non-zero.

LEMMA 1.3. *If ϕ in* Hom $\mathscr{E}$ *is left continuous then for each non-zero E in $\mathscr{E}$*

$$\phi(E) = \sup_{F<E} \phi_*(F) \quad \left(= \sup_{F<E} F_* \right).$$

Proof. When E has an immediate predecessor E_-

$$\sup_{F<E} \phi_*(F) = \phi_*(E_-) = \phi(E)$$

as required. Suppose then, that E has no immediate predecessor. It follows that

$$\sup_{F<E} \phi_*(F) \geqq \sup_{F<E} \phi(F) = \phi(E_-) = \phi(E)$$

by the left continuity of ϕ.

THEOREM 1.4. *Given a weakly closed $\mathscr{A}$-module $\mathscr{U}$ as above, there exists a minimal order homomorphism $\eta = \eta_\mathscr{U}^{\min}$ in $H_\mathscr{U}$. This homomorphism is the unique left continuous member of $H_\mathscr{U}$ satisfying $\eta(0) = 0$. Letting* c.l.s.

https://doi.org/10.4153/CJM-1987-065-7 Published online by Cambridge University Press

$\{\mathcal{R}(X): X \in \mathcal{W}\}$ denote the closed linear span of the ranges of all X's in $\mathcal{W}$, $\eta(E)$ is given by

$$\eta(E) = \textit{projection onto } \text{c.l.s.}\{\mathcal{R}(UE): \quad U \in \mathcal{U}\}.$$

Proof. It is shown in [**4**] that if η is given as above, then η is a left continuous element of $H_{\mathcal{U}}$. Clearly $\eta(0) = 0$.

Suppose that ϕ is another left continuous element of $H_{\mathcal{U}}$. By Theorem 1.2 and Lemma 1.3 it follows that

$$\phi(F) = \sup_{E<F} \phi_*(E) = \sup_{E<F} \eta_*(E) = \eta(F).$$

To show that η is minimal, choose ψ arbitrary in $H_{\mathcal{U}}$

$$\begin{aligned}\eta(E) &= \text{projection onto c.l.s.}\{\mathcal{R}(UE): \quad U \in \mathcal{H}\} \\ &= \text{projection onto c.l.s.}\{\mathcal{R}(\psi(E)UE): \quad U \in \mathcal{H}\} \\ &\leqq \psi(E).\end{aligned}$$

To achieve a sharper result of this theorem (i.e., one which dispenses with the left continuity condition on the entire domain of the homomorphism) we consider the structure of the order homomorphisms in $H_{\mathcal{U}}$. The following suggestive notation will now be introduced: Given $\mathcal{U}$ as above define $\eta_{\mathcal{U}}^{\max}$ by:

$$\eta_{\mathcal{U}}^{\max}(E) = \begin{cases} I & \text{if } E = I \\ (E_-)_* & \text{if } 0 < E < I \\ 0_* & \text{if } E = 0. \end{cases}$$

That this exists and is well defined follows from Theorems 1.2 and 1.4 above. In order to simplify the below, $\mathcal{U}$ a weakly closed $\mathcal{A}$-submodule of $\mathcal{L}(H)$ will be fixed and the notation $\eta^{\max}$ and $\eta^{\min}$ will be used for $\eta_{\mathcal{U}}^{\max}$ and $\eta_{\mathcal{U}}^{\min}$ respectively.

LEMMA 1.5. *With notation as above $\eta_*^{\max} = \eta_*^{\min}$. Consequently (Theorem 1.2) $\eta^{\max} \in H_{\mathcal{U}}$.*

Proof. First, suppose that E has an immediate successor E_+, then

$$\eta_{\mathcal{U}}^{\max}(E) = \inf_{F>E} \eta^{\max}(E) = \eta^{\max}(E_+) = \eta_*^{\min}(E).$$

Suppose then that E has no immediate successor. It is observed that, for $0 < F < I$

$$\eta^{\max}(F) = \begin{cases} \eta^{\min}(F) \text{ if } F \neq F_- \\ \eta_*^{\min}(F) \text{ if } F = F_-. \end{cases}$$

So it follows that $\eta_*^{\max} \geqq \eta_*^{\min}$.

https://doi.org/10.4153/CJM-1987-065-7 Published online by Cambridge University Press

Given $F > E$, by assumption there will exist a G with $E < G_- \leqq G < F$; hence

$$\eta^{\max}(G) \leqq \eta^{\min}(F),$$

showing

$$\inf_{G>E} \eta^{\max}(G) \leqq \eta^{\min}(F) \quad \text{for every } F > E.$$

Therefore

$$\eta_*^{\max}(E) = \inf_{F>E} \eta^{\max}(F) \leqq \inf_{F>E} \eta^{\min}(F) = \eta_*^{\min}(E).$$

The only case left to be dealt with is when $E = 0$ and 0 has no immediate successor. In this circumstance it follows that:

$$\eta_*^{\max}(0) = \inf_{E>0} \eta^{\max}(E) = \inf_{E>0} \eta_*^{\min}(E_-) = \inf_{E>0} \eta^{\min}(E) = \eta_*^{\min}(0)$$

which completes the proof.

2. On the structure of $H_{\mathscr{U}}$. We begin by determining a Key property of ϕ_* for $\phi \in H_{\mathscr{U}}$. To state this, it is necessary to introduce the following. An order homomorphism ϕ in Hom $\mathscr{E}$ is said to be *right continuous* if, for each $F \neq I$ in $\mathscr{E}$

$$\inf_{E>F} \phi(E) = \phi\left(\inf_{E>F} E\right) \quad [= \phi(F_+)].$$

LEMMA 2.1. *If $\phi \in$ Hom $\mathscr{E}$ then ϕ_* is right continuous.*

Proof. In the case that F has an immediate successor F_+ this result holds by virtue of ϕ_* being order preserving. Suppose then that F has no immediate successor. Given $F > E$ there will exist a G with $F < G < E$, then $\phi_*(G) \leqq \phi(E)$. This shows that

$$\inf_{G>E} \phi_*(G) \leqq \phi(E) \quad \forall E > F.$$

Therefore

$$\inf_{G>F} \phi_*(G) \leqq \inf_{E>F} \phi(E) = \phi_*(F),$$

from which it follows that

$$\phi_*(F) = \inf_{G>F} \phi_*(G)$$

completing the proof.

THEOREM 2.2. *For a given ϕ in* Hom $\mathscr{E}$ $\phi \in H_{\mathscr{U}}$ *if and only if*

$$\eta^{\min} \leqq \phi \leqq \eta^{\max}.$$

Proof. It was shown in Theorem 1.4 that if ϕ is in $H_{\mathscr{U}}$ then

$$\phi \geqq \eta^{\min}.$$

Now, for $E \neq I$, $\eta^{\max}(E)$ equals either $\phi(E)$ or $\phi_*(E)$; either way

$$\eta^{\max}(E) \geqq \phi(E).$$

Suppose then that $\eta^{\min} \leqq \phi \leqq \eta^{\max}$. Then

$$\eta^{\min}_* \leqq \phi_* \leqq \eta^{\max}_*.$$

An application of Lemma 1.2 shows that $\phi \in H_{\mathscr{U}}$.

Remark. It now follows that, if E has no immediate predecessor, then any left continuous homomorphism in $H_{\mathscr{U}}$ is uniquely defined at E.

Although $\eta^{\min}$ is left continuous it may very well be that $\eta^{\max}$ has no continuity properties whatsoever. For left continuity consider the case of an E in $\mathscr{E}$ with no immediate predecessor. By the left continuity of $\eta^{\min}$ and the observation that

$$\eta^{\max}(E_-) = \eta^{\max}(E) = E_*$$

it would follow that

$$\sup_{F<E} \eta^{\max}(F) = \sup_{F<E} \eta^{\min}(F) = \eta^{\min}(E).$$

It is now easy to arrange that $\eta^{\min}(E) \neq E_*$ (for example, let E have an immediate successor E_+, and have

$$\eta^{\min}(E_+) \geqq E_+ > E = \eta^{\min}(E)\,)$$

and consequently $\eta^{\max}$ is not left continuous at E.

For right continuity consider the case of a projection E in $\mathscr{E}$ with $E_+ = E \neq E_-$. It follows that

$$\inf_{F>E} \eta^{\max}(F) = \inf_{F>E} \eta^{\min}(F) = E_*$$

and

$$\eta^{\max}(E_+) = \eta^{\max}(E) = (E_-)_* = \eta^{\min}(E).$$

Again, it is easy to arrange that E_* and $\eta^{\min}(E)$ be different. (Note: $\eta^{\min}$ is not right continuous.) This shows that, in general $\eta^{\max}$ is also not right continuous.

LEMMA 2.3. *If $\mathscr{E}$ is a continuous nest then $\eta^{\max}$ is right continuous.*

Proof. Assuming that $\mathscr{E}$ is a continuous nest

$$\eta^{\max}(E) = E_* = \phi_*(E) \quad \text{for any } \phi \in H_{\mathscr{U}}, \text{ for all } E \neq I.$$

An application of Lemma 2.1 completes the argument.

3. Uniqueness. Let η in $H_{\mathscr{U}}$ and γ in Hom $\mathscr{E}$ satisfy $\eta = \gamma$ on $(0, I)$. It will now follow from Lemma 1.1 together with **[4]** Lemmas 1.2 and 1.6 that γ is also in $H_{\mathscr{U}}$. This consideration suggests introducing the following notation:

$$\langle \eta \rangle = \{\gamma \in \text{Hom}\,\mathscr{E}: \quad \gamma = \eta \text{ on } (0, I)\}.$$

Thus, if $\langle \eta \rangle = \langle \gamma \rangle$ then $\mathscr{U}_\eta = \mathscr{U}_\gamma$. The question posed by the referee and addressed in this section is, under which circumstances does the converse hold? That is to say, when does $H_{\mathscr{U}} = \langle \eta \rangle$? A nest $\mathscr{E}$ will be said to be *r-atomic* if every right atom is an atom. That is $E \neq E_+$ implies $E \neq E_-$. Obvious examples of r-atomic nests are any continuous or discrete nest.

LEMMA 3.1. *There exist two distinct right continuous homomorphisms in $H_{\mathscr{U}}$ for some* Alg *$\mathscr{E}$-submodule $\mathscr{U}$ if and only if $\mathscr{E}$ is not r-atomic.*

Proof. First suppose that $\mathscr{E}$ is r-atomic and $\phi, \psi \in H_{\mathscr{U}}$ are right continuous. Assume, contrary to above, that $\phi(E) \neq \psi(E)$ for some $E \in \mathscr{E}$. If $E \neq E_-$ then

$$\phi_*(E_-) = \phi(E) \neq \psi(E) = \psi_*(E_-).$$

An application of Theorem 1.2 will provide the required contradiction. Suppose then that $E = E_-$ by assumption of r-atomicity $E = E_+$ giving

$$\phi_*(E) = \phi(E) \neq \psi(E) = \psi_*(E).$$

Another application of Theorem 1.2 completes the necessity condition. To complete the proof let $\mathscr{E}$ with $E_- = E \neq E_+$. Define

$$\iota(F) = \phi(F) = F, \quad F \neq E \quad \text{and} \quad \iota(E) = E, \phi(E) = E_+.$$

It easily follows (Theorem 1.2) that $\mathscr{U}_\iota = \mathscr{U}_\phi = \text{Alg}\,\mathscr{E}$ and that both ι and ϕ are right continuous. Indeed, ι is continuous.

The above shows that for non r-atomic nests continuity will not imply uniqueness. The remainder of this section will be devoted to showing that, for r-atomic nests, continuity will imply uniqueness.

LEMMA 3.2. *If $\eta^{\min}$ is right continuous at E, then $\eta^{\max}$ is also right continuous at E.*

Proof. Only two cases need be considered (i) $E_+ = E = E_-$ and (ii) $E_+ = E \neq E_-$.

Case (i). As $E = E_+$ it will follow, as in the proof of Lemma 2.1 that:

$$\inf\{\eta^{\max}(F): \quad F > E\} = E_* = (E_-)_* = \eta^{\max}(E_+).$$

(Note that this argument will also deal with $E = 0$.)

Case (ii). As in case (i)

$$\inf\{\eta^{\max}(F):\quad F > E\} = \inf\{\eta^{\min}(F):\quad F > E\}$$
$$= \eta^{\min}(E_+) = \eta^{\min}(E)$$

by right continuity. Also

$$\eta^{\min}(E) = (E_-)_* = \eta^{\max}(E_+)$$

completing the proof.

For the remainder of this section $\mathcal{E}$ will be a fixed r-atomic nest of projections.

COROLLARY 3.3. *If there exists an* η *in* $H_{\mathcal{U}}$ *continuous on* $(0, I)$ *then* $H_{\mathcal{U}} = \langle\eta\rangle$.

Proof. An application of Theorem 1.4 will show that $\eta = \eta^{\min}$ on $(0, I)$. From Lemma 3.1 and Lemma 3.2 it also holds that $\eta = \eta^{\max}$ on $(0, I)$. The result will now immediately follow from Theorem 2.2.

COROLLARY 3.4. *If there is a* η *in* $H_{\mathcal{U}}$ *surjective then* $H_{\mathcal{U}} = \{\eta\}$.

Proof. Considerations of order topology will show that η is continuous on $[0, I]$, $\eta(0) = 0$, $\eta(I) = I$. Applications of Theorem 1.4 and Lemma 1.1 complete the proof.

The next result was first observed by the referee based upon Lemma 2.3

COROLLARY 3.5. *If* $\mathcal{E}$ *is a continuous nest then* $\eta^{\max} = \eta^{\min}$ ($\langle\eta^{\max}\rangle = \langle\eta^{\min}\rangle$) *if and only if* $H_{\mathcal{U}}$ *has a surjective (continuous) representative.*

The following example will show that the converse of Corollary 3.3 is false even when $\langle\eta\rangle = \{\eta\}$. Consider the nest $\{E_t]_{t\in[0,I]}$ of projections on $L^2[0, 1]$ corresponding to the subspaces $L^2[0, t]$ and consider the subnest $\{0\} \cup \{E_t\}_{t\geqq 1/2}$. Define ϕ in Hom $\mathcal{E}$ by $\phi(0) = 0 = \phi(E_{1/2})$ and $\phi(E_t) = I$ for $t > 1/2$. It is easily seen that ϕ uniquely determines $\mathcal{U}_\phi$. However ϕ is not right continuous at $E_{1/2}$.

REFERENCES

1. G. J. Knowles, *Dissertation*, Kings College (University of London, 1981).
2. ——— *Nest algebra-modules; commutants and cohomology*, (Submitted).
3. J. A. Erdos, *Operators of finite rank in nest algebras*, J. London Math. Soc. *43* (1968), 391-397.
4. J. A. Erdos and S. C. Power, *Weakly closed ideals of nest algebras*, J. Op. Theory *7* (1982), 219-235.
5. J. R. Ringrose, *On some algebras of operators*, Proc. London Math. Soc. (3) *15* (1965), 61-83.

Texas Tech University,
Lubbock, Texas