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Word complexity of (measure-theoretically) weakly mixing rank-one subshifts

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Abstract. We exhibit, for arbitrary $\epsilon > 0$, subshifts admitting weakly mixing (probability) measures with word complexity p satisfying $\limsup p(q)/q < 1.5 + \epsilon$. For arbitrary $f(q) \to \infty$, said subshifts can be made to satisfy $p(q) < q + f(q)$ infinitely often. We establish that every subshift associated to a rank-one transformation (on a probability space) which is not an odometer satisfies $\limsup p(q) - 1.5q = \infty$ and that this is optimal for rank-ones.

Key words: symbolic dynamics, word complexity, weak mixing, rank-one transformations
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1. *Introduction*

Morse and Hedlund [**MH38**] initiated the study of word complexity of symbolic systems: given an infinite word $x \in \mathcal{A}^{\mathbb{Z}}$, on some finite set $\mathcal{A}$—the alphabet—the word complexity $p(q)$ is the number of distinct subwords of x of length q; more generally, for a closed, shift-invariant $X \subseteq \mathcal{A}^{\mathbb{Z}}$, i.e. a subshift, the complexity $p(q)$ is the number of distinct subwords of length q appearing in any of the $x \in X$.

The same authors [**MH40**] established the first lower bound on the word complexity in terms of the structure of the subshift: if x is aperiodic, then $p(q) \geq q + 1$ for all q. A natural question, considering aperiodicity to be a weak form of mixing-like behavior, is to what extent mixing-type properties impose lower bounds on complexity, especially in light of recent results (e.g. [**CFPZ19**, **CK19**, **CK20a**, **CK20b**, **DDMP16**, **DOP21**, **OP19**, **PS22**]) regarding subshifts with low word complexity being highly structured.

Morse and Hedlund [**MH40**] also exhibited words with $p(q) = q + 1$, called Sturmian words, which can be encoded by irrational rotations [**CH73**]. As irrational rotations are totally ergodic, the natural question is whether weak mixing imposes any sort of stronger lower bound on word complexity. Topological mixing properties were considered by Gao and Ziegler [**GZ19**] (see also Gao and Hill [**GH16a**, **GH16b**]); here we address the measure-theoretic question.

The lowest previously known complexity for a subshift admitting a weakly mixing (probability) measure, due to Ferenczi [**Fer95**], is a subshift with complexity satisfying $\limsup p(q)/q = 5/3$ and $\liminf p(q)/q = 1.5$. We exhibit subshifts, admitting weakly mixing (probability) measures, with lower complexity.

THEOREM A. (Theorem 6.9) *For every* $\epsilon > 0$, *there exists a weakly mixing rank-one transformation (on a probability space) such that the associated subshift has complexity* $\limsup p(q)/q < 1.5 + \epsilon$.

THEOREM B. (Theorem 6.9) *For any* $f(q) \to \infty$, *the subshifts can be made to satisfy* $p(q) < q + f(q)$ *infinitely often.*

Naturally, one wonders whether these bounds are sharp. Cassaigne [**Cas98**] showed that if $p(q) = q + c$ for some constant c, then it is the image of a Sturmian word (so cannot admit a weakly mixing measure); this implies $p(q) < q + f(q)$ infinitely often is the best possible (see Proposition 2.6 for specifics).

The analogous question for strong mixing was first explored by Ferenczi [**Fer96**] who showed that the classical staircase transformation (proved mixing by Adams [**Ada98**]) has quadratic complexity and conjectured that was the minimal possible. The author, Pavlov and Rodock [**CPR22**] disproved this conjecture; recently, the author [**Cre22**] showed that strong mixing manifests exactly at superlinear complexity: every strongly mixing subshift satisfies $\lim p(q)/q = \infty$ and for any $f(q) \to \infty$, there exist strongly mixing subshifts with $\lim p(q)/(qf(q)) = 0$.

We establish that $\limsup p(q)/q = 1.5$ is optimal for rank-one transformations.

THEOREM C. (Theorem 4.3) *Let T be a rank-one transformation (on a probability space) which is not an odometer. Then the associated subshift has complexity satisfying* $\limsup p(q) - 1.5q = \infty$ *(and* $\liminf p(q) - q = \infty$*).*

While Sturmian words are encoded by irrational rotations (which are totally ergodic and rank-one), Rote [**Rot94**] showed that the general word encoded by an irrational rotation has complexity $p(q) = 2q$, so if one treats an irrational rotation as a rank-one subshift, then the complexity satisfies $p(q) \geq 2q$.

There appears to be a complexity distinction between totally ergodic and weakly mixing rank-one subshifts, namely that we can exhibit examples of totally ergodic rank-one subshifts with strictly lower complexity than any of our weakly mixing examples. Specifically, Theorem C is optimal.

THEOREM D. (Theorem 6.12) *For every* $f(q) \to \infty$, *there exists a totally ergodic rank-one transformation (on a probability space) such that the associated subshift satisfies* $p(q) < 1.5q + f(q)$ *for all sufficiently large q and* $p(q) < q + f(q)$ *infinitely often.*

It is worth remarking that $\limsup p(q) - 1.5q = \infty$ distinguishing behavior in subshifts also appears in the work of Ormes and Pavlov [**OP19**] who showed that if $\limsup p(q) - 1.5q < \infty$, then the words in question are necessarily uniformly recurrent or bidirectionally eventually periodic. For rank-one transformations, having bounded

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spacers implies uniform recurrence, so their result and ours do not meaningfully overlap. However, it is interesting that $\limsup p(q) - 1.5q < \infty$ is exactly the bound that rules out total ergodicity for rank-one subshifts as it is well known that the lack of total ergodicity for rank-ones is equivalent to factoring onto a finite cyclic permutation, which is similar in spirit to their conclusion.

In connection with other properties often discussed with rank-one transformations, if we replace $p(q) < q + f(q)$ infinitely often with a slightly weaker condition, then the work of Ryzhikov [**Ryz13**] gives the following theorem.

THEOREM E. (Theorem 6.10) *For every* $\epsilon > 0$*, there exists a subshift with complexity satisfying* $\limsup p(q)/q < 1.5 + \epsilon$ *and* $\liminf p(q)/q < 1 + \epsilon$ *such that the associated rank-one transformation is weakly mixing (on a probability space) and has minimal self-joinings (hence also has trivial centralizer and is mildly mixing).*

The proof of Theorem C is worth outlining briefly. First we establish that for a rank-one subshift with $\limsup p(q)/q < 2$, there is a rank-one subshift which generates the same language such that the spacer sequence eventually takes on at most two values. Not being an odometer implies that both values must occur infinitely often and one can arrange for both to occur at every level (this arranging can lead to the cut sequence growing very rapidly).

The proof then proceeds by an analysis of all possible rank-one subshifts with exactly two spacer values. We remark that finding our low complexity examples was a direct result of this examination, which both indicated 1.5 ought to be the optimal bound and led to which subshifts were the correct candidates.

There remain questions regarding the precise nature of the complexity of subshifts admitting weakly mixing measures; we discuss these in §7. The main question left open is whether there exists a subshift, necessarily not rank-one, admitting a weakly mixing (probability) measure such that $\limsup p(q)/q < 1.5$. We tentatively conjecture that this is not the case and a bit more: for every subshift admitting a weakly mixing (probability) measure, we tentatively conjecture that $\limsup p(q)/q > 1.5$.

Section 5 where the examples are constructed (and §6 where weak mixing is proved) may be read independently; the reader primarily interested in the examples may opt to skip §§3 and 4 which are aimed at proving Theorem C.

2. *Definitions and preliminaries*

2.1. *Symbolic dynamics*

Definition 2.1. A *subshift* on the finite set $\mathcal{A}$ is any subset $X \subset \mathcal{A}^{\mathbb{Z}}$ which is closed in the product topology and shift-invariant: for $x = (x_n)_{n\in\mathbb{Z}} \in X$ and $k \in \mathbb{Z}$, the translate $(x_{n+k})_{n\in\mathbb{Z}}$ is also in X.

Definition 2.2. A *word* is any element of $\mathcal{A}^{\ell}$ for some ℓ, the *length* of w, written $\ell en(w)$. A word w is a *subword* of a word or biinfinite sequence x if there exists k so that $w_i = x_{i+k}$ for $1 \leq i \leq \ell en(w)$. A word u is a *prefix* of w if $u_i = w_i$ for $1 \leq i \leq \ell en(u)$ and a word v

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is a *suffix* of w if $v_i = w_{i+\ell en(w)-\ell en(v)}$ for $1 \leq i \leq \ell en(v)$. A subword (or prefix or suffix) is *proper* when it is not the entire word.

For words v, w, we denote by vw their concatenation—the word obtained by following v immediately by w. We also write such concatenations with product or exponential notation, e.g. $\prod_i w_i$ or 0^n.

Definition 2.3. The *language* of a subshift X is $\mathcal{L}(X) = \{w : w \text{ is a subword of some } x \in X\}$.

Definition 2.4. The *word complexity function* of a subshift X over $\mathcal{A}$ is the function $p_X : \mathbb{N} \to \mathbb{N}$ defined by $p_X(q) = |\mathcal{L}(X) \cap \mathcal{A}^q|$, the number of words of length q in the language of X.

When X is clear from context, we suppress the subscript and just write $p(q)$.

2.1.1. *Right-special words.* All subshifts we consider are on the alphabet {0, 1} so it is natural to consider the following definition.

Definition 2.5. The set of *right-special* words is $\mathcal{L}^{RS}(X) = \{w \in X : w0, w1 \in \mathcal{L}(X)\}$.

Cassaigne [**Cas97**] showed the following well-known relationship: $p(q) = p(m) + \sum_{\ell=m}^{q-1} |\{w \in \mathcal{L}^{RS} : \ell en(w) = \ell\}|$ for $m < q$.

2.1.2. *Quasi-Sturmian words.* An infinite $x \in \mathcal{A}^{\mathbb{N}}$ is *Sturmian* when $p_x(q) = q + p_x(1)$. Morse and Hedlund [**MH40**] exhibited examples of such words and showed that if $p_x(q) \leq q$ or $p_x(q+1) = p_x(q)$ for any q, then x is periodic.

Cassaigne [**Cas98**] termed infinite words x such that $p_x(q) = q + c$ for some constant c and all sufficiently large q *quasi-Sturmian*, and showed such a word must be the image of a Sturmian word under a morphism $f : \mathcal{A}^* \to \mathcal{A}^*$ which is non-periodic.

Indeed, his result quickly gives a bit more to obtain the following proposition.

PROPOSITION 2.6. *Let X be an aperiodic subshift such that $p_X(q) \leq q + d$ for some constant d and infinitely many q. Then X is quasi-Sturmian (in the sense that all $x \in X$ are quasi-Sturmian), and hence cannot admit a weakly mixing measure.*

Proof. By the Hedlund–Morse theorem, we may assume $p(\ell + 1) - p(\ell) \geq 1$ for all ℓ since otherwise, the subshift is periodic. For infinitely many q,

$$\begin{aligned} q + d \geq p(q) &= p(1) + \sum_{\ell=1}^{q-1} (p(\ell+1) - p(\ell)) \\ &\geq p(1) + q - 1 + |\{\ell < q : p(\ell+1) - p(\ell) \geq 2\}| \end{aligned}$$

so for infinitely many q, we have $|\{\ell < q : p(\ell+1) - p(\ell) \geq 2\}| < d$, meaning $|\{\ell : p(\ell+1) - p(\ell) \geq 2\}| < d$.

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Set $c = p(1) - 1 + \sum_{\ell=1}^{\infty}(p(\ell+1) - p(\ell) - 1)$, which must be finite as there are only finitely many ℓ with $p(\ell+1) - p(\ell) > 1$. Then for all $q > \max\{\ell : p(\ell+1) - p(\ell) \geq 2\}$,

$$p(q) = p(1) + q - 1 + \sum_{\ell=1}^{q-1}(p(\ell+1) - p(\ell) - 1) = q + c.$$

Since Sturmian words can be encoded by irrational rotations, Sturmian (and therefore quasi-Sturmian) subshifts cannot admit weakly mixing measures. □

2.2. *Ergodic theory*

Definition 2.7. A *transformation* T is a measurable map on a standard Borel or Lebesgue measure space $(Y, \mathcal{B}, \mu)$ that is measure-preserving: $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$.

Definition 2.8. Two transformations T on $(Y, \mathcal{B}, \mu)$ and T' on $(Y', \mathcal{B}', \mu')$ are *measure-theoretically isomorphic* if there exists a bijective map ϕ between full measure subsets $Y_0 \subset Y$ and $Y'_0 \subset Y'$, where $\mu(\phi^{-1}A) = \mu'(A)$ for all measurable $A \subset Y'_0$ and $(\phi \circ T)(y) = (T' \circ \phi)(y)$ for all $y \in Y_0$.

Definition 2.9. A transformation T is *ergodic* when $A = T^{-1}A$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$.

Definition 2.10. A transformation T is *totally ergodic* when T^k is ergodic for all $k \in \mathbb{N}$.

Definition 2.11. A transformation T on a probability space is *weakly mixing* when any of the following equivalent conditions hold:

- for all measurable sets A, B, there exists $\{t_n\}$ such that $\mu(T^{t_n}A \cap B) \to \mu(A)\mu(B)$;
- there exists a density one $\{t_n\}$ such that $\mu(T^{t_n}A \cap B) \to \mu(A)\mu(B)$ for all measurable sets A, B;
- $T \times T$ is ergodic;
- for all measurable A, B, there exists n such that $\mu(T^nA \cap A)\mu(T^nA \cap B) > 0$.

2.3. *Rank-one transformations.* A *rank-one transformation* is a transformation T constructed by 'cutting and stacking'. Here, Y represents a (possibly infinite) interval, $\mathcal{B}$ is the induced σ-algebra from $\mathbb{R}$ and μ is Lebesgue measure. We give a brief description, referring the reader to [FGH+21] or [Sil08] for more details and to [Fer97] for equivalent definitions.

The transformation is defined inductively on increasingly larger portions of the space through Rohlin towers or *columns*, denoted C_n. Each column C_n consists of *levels* $I_{n,j}$, where $0 \leq j < h_n$ is the height of the level within the column. All levels $I_{n,j}$ in C_n are intervals with the same length, $\mu(I_n)$, and the total number of levels in a column is the *height* of the column, denoted by h_n. The transformation T is defined on all levels $I_{n,j}$ except the top one I_{n,h_n-1} by sending each $I_{n,j}$ to $I_{n,j+1}$ using the unique order-preserving affine map.

Start with $C_1 = [0, 1)$ with height $h_1 = 1$. To obtain C_{n+1} from C_n, we require a *cut sequence*, $\{r_n\}$, such that $r_n \geq 1$ for all n. Make r_n vertical cuts of C_n to create $r_n + 1$

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subcolumns of equal width. Denote a *sublevel* of C_n by $I^{[i]}_{n,j}$, where $0 \leq a < h_n$ is the height of the level within that column and i represents the position of the subcolumn, where $i = 0$ represents the leftmost subcolumn and $i = r_n$ is the rightmost subcolumn. After cutting C_n into subcolumns, add extra intervals called *spacers* on top of each subcolumn to function as levels of the next column. The *spacer sequence*, $\{s_{n,i}\}$, such that $0 \leq i \leq r_n$ and $s_{n,i} \geq 0$, specifies how many sublevels to add above each subcolumn. Spacers are the same width as the sublevels, act as new levels in the column C_{n+1} and are taken to be the leftmost intervals in $[1, \infty)$ not in C_n. After the spacers are added, stack the subcolumns with their spacers right on top of left, i.e. so that $I^{[i+1]}_{n,0}$ is directly above $I^{[i]}_{n,h_n-1}$. This gives the next column, C_{n+1}.

Each column C_n defines T on $\bigcup_{j=0}^{h_n-2} I_{n,j}$ and the partially defined map T on C_{n+1} agrees with that of C_n, extending the definition of T to a portion of the top level of C_n where it was previously undefined. Continuing this process gives the sequence of columns $\{C_1, \ldots, C_n, C_{n+1}, \ldots\}$ and T is then the limit of the partially defined maps.

Though this construction could result in Y being an infinite interval with infinite Lebesgue measure, Y has finite measure if and only if $\sum_n (1/r_n h_n) \sum_{i=0}^{r_n} s_{n,i} < \infty$, see [**CS10**]. All rank-one transformations we define satisfy this condition and for convenience, we renormalize so that $Y = [0, 1)$. Every rank-one transformation is ergodic and invertible.

The reader should be aware that we are making r_n cuts and obtaining $r_n + 1$ subcolumns (following Ferenczi [**Fer96**]), while other papers (e.g. [**Cre21**]) use r_n as the number of subcolumns.

2.4. *Odometers*

Definition 2.12. A rank-one transformation which can be constructed using a spacer sequence such that there exists N so that $s_{n,i} = 0$ for all $n \geq N$ and $0 \leq i < r_n$ is an *odometer*.

Odometers have discrete spectrum and all their eigenvalues are rational in the sense that they are of the form $\exp(2\pi i q)$ for $q \in \mathbb{Q}$.

2.5. *Symbolic models of rank-one transformations.*

For a rank-one transformation defined as above, we define a subshift $X(T)$ on the alphabet $\{0, 1\}$ which is measure-theoretically isomorphic to T.

Definition 2.13. The *symbolic model* $X(T)$ of, or *subshift associated* to, a rank-one transformation T is given by the sequence of words: $B_1 = 0$ and

$$B_{n+1} = B_n 1^{s_{n,0}} B_n 1^{s_{n,1}} \cdots B_n 1^{s_{n,r_n}} = \prod_{i=0}^{r_n} B_n 1^{s_{n,i}}$$

and $X(T)$ is the set of all biinfinite sequences such that every subword is a subword of some B_n.

The words B_n are a symbolic coding of the column C_n: 0 represents C_1 and 1 represents the spacers, and $h_n = \ell en(B_n)$. There is a natural measure associated to $X(T)$.

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Definition 2.14. The *empirical measure* for a symbolic model $X(T)$ of a rank-one transformation T is the measure ν defined by, for each word w,

$$\nu([w]) = \lim_{n\to\infty} \frac{|\{1 \leq j \leq \ell en(B_n) - \ell en(w) \ : \ B_n[j; \ell en(w)] = w\}|}{\ell en(B_n) - \ell en(w)},$$

where $B_n[j; \ell]$ denotes the subword of B_n starting at position j with length ℓ.

Danilenko [**Dan16**] (combined with [**dJ77**]) proved that the symbolic model $X(T)$ of a rank-one subshift, equipped with its empirical measure, is measure-theoretically isomorphic to the cut-and-stack construction (see [**AFP17**]; see [FGH+21] for the full generality including odometers).

Due to this isomorphism, we move back and forth between rank-one and symbolic model terminology as needed and write $\mathcal{L}(T)$ for the language of $X(T)$, or simply $\mathcal{L}$ if $X(T)$ is clear from context, and make the following definition.

Definition 2.15. A *rank-one subshift* is the symbolic model of a rank-one transformation.

Likewise, when the measure is clear from text, such as the empirical measure for a rank-one subshift, we make the following definition.

Definition 2.16. A *(measure-theoretically) weakly mixing subshift* is a subshift for which the measure is weakly mixing.

3. *Properties of rank-one subshifts*

LEMMA 3.1. *For $n < m$, B_m has B_n as a prefix and $B_n 1^{s_{n,r_n}}$ as a suffix.*

Proof. This is immediate from the construction. □

LEMMA 3.2. *B_n has 0 as a prefix for all n.*

Proof. By Lemma 3.1, B_n has $B_1 = 0$ as a prefix. □

We next need a result of Danilenko.

PROPOSITION 3.3. [**Dan19**, Lemma 1.10] *Every rank-one subshift is measure-theoretically isomorphic to a rank-one subshift with $s_{n,r_n} = 0$ and the two subshifts generate the same language.*

PROPOSITION 3.4. *For a rank-one subshift on a finite measure space, $(1/h_n)\inf\{s_{n,i} : 0 \leq i < r_n\} \to 0$.*

Proof. Suppose $\inf\{s_{n,i} : 0 \leq i < r_n\} \geq \delta h_n$ infinitely often for some $\delta > 0$. Then for such n, we have $\mu(C_{n+1}) \geq \mu(C_n) + \inf\{s_{n,i} : 0 \leq i < r_n\}\mu(I_n) \geq (1+\delta)\mu(C_n)$. So for any k, if we choose N such that at least k values of $n < N$ have $\inf\{s_{n,i} : 0 \leq i < r_n\} \geq \delta h_n$, then $\mu(C_N) \geq (1+\delta)^k \mu(C_0)$. Taking $k \to \infty$ shows the measure would then be infinite. □

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3.1. *Rank-one subshifts with at least three distinct spacer values*

PROPOSITION 3.5. *For a rank-one subshift on a finite measure space with $s_{n,r_n} = 0$ for all sufficiently large n, if the set $\{s_{m,i} : m \geq n, 0 \leq i < r_m\}$ contains at least three distinct values for infinitely many n, then* $\limsup p(q)/q \geq 2$.

Proof. Choose n such that $t_n = \inf\{s_{n,i} : 0 \leq i < r_n\}$ has the property that $t_n = \inf\{s_{m,i} : m \geq n, 0 \leq i < r_m\}$ (such an n must exist since otherwise, there is a sequence $\{m_t\}$ along which $\inf\{s_{m_t,i} : 0 \leq i < r_{m_t}\}$ is strictly decreasing, which would contradict that $s_{m,i} \geq 0$).

Let $u_n, v_n \in \{s_{m,i} : m \geq n, 0 \leq i < r_m\}$ such that $t_n < u_n < v_n$. Such must exist since otherwise, $|\{s_{m,i} : m \geq n, 0 \leq i < r_{m'}\}| = 2$, so the same holds for all $n' \geq n$.

The word $B_n 1^{t_n} B_n$ is a subword of B_{n+1}. As B_n has 0 as a prefix, $B_n 1^{t_n} 0 \in \mathcal{L}$. As $u_n > t_n$ and $B_n 1^{u_n}$ is a subword of $B_m 1^{u_n}$ which is a subword of B_{m+1}, this shows $B_n 1^{t_n} \in \mathcal{L}^{RS}$. Likewise, $B_n 1^{u_n} \in \mathcal{L}^{RS}$.

Let N such that $s_{n,r_n} = 0$ for $n \geq N$. Let $c \geq 1$ such that B_N has 01^{c-1} as a suffix (such $c \leq h_N$ must exist as B_N has 0 as a prefix). Since $s_{n,r_n} = 0$ for $n \geq N$, the word B_n, for all $n \geq N$, has B_N as a suffix and hence has 01^{c-1} as a suffix.

Therefore, $B_n 1^{t_n}$ has 01^{c-1+t_n} as a suffix and $B_n 1^{u_n}$ has 01^{c-1+u_n} as a suffix meaning that for every $t_n + c \leq \ell < h_n + t_n$, the suffixes of $B_n 1^{t_n}$ and $B_n 1^{u_n}$ of length ℓ are distinct (as $u_n > t_n$).

Then $p(\ell + 1) - p(\ell) = |\{w \in \mathcal{L}^{RS} : \ell en(w) = \ell\}| \geq 2$ for $t_n + c \leq \ell < h_n + t_n$, meaning that $p(h_n) \geq 2(h_n - t_n - c)$ so, as Proposition 3.4 implies $t_n/h_n \to 0$, $p(h_n)/h_n \geq 2(1 - (t_n + c)/h_n) \to 2$. □

3.2. *Rank-one subshifts with the same language*

LEMMA 3.6. *Let T be a rank-one subshift with cut sequence $\{r_n\}$ and spacer sequence $\{s_{n,i}\}$.*

Let $N \in \mathbb{N}$. For $n < N$, set $\tilde{r}_n = r_n$ and $\tilde{s}_{n,i} = s_{n,i}$.

Set $\tilde{r}_N = (r_N + 1)(r_{N+1} + 1) - 1$ and for $0 \leq a \leq r_{N+1}$, set $\tilde{s}_{N,a(r_N+1)+b} = s_{N,b}$ for $0 \leq b < r_N$ and set $\tilde{s}_{N,a(r_N+1)+r_N} = s_{N,r_N} + s_{N+1,a}$.

For $n > N$, set $\tilde{r}_n = r_{n+1}$ and $\tilde{s}_{n,i} = s_{n+1,i}$.

Then the rank-one subshift $\tilde{T}$ generates the same language as T.

Proof. Clearly, $\tilde{B}_n = B_n$ for $n \leq N$. By design,

$$\tilde{B}_{N+1} = \prod_{a=0}^{r_{N+1}} \left(\prod_{b=0}^{r_N} \tilde{B}_N 1^{\tilde{s}_{N,a(r_N+1)+b}} \right)$$
$$= \prod_{a=0}^{r_{N+1}} \left(\left(\prod_{b=0}^{r_N-1} B_N 1^{s_{N,b}} \right) B_N 1^{s_{N,r_N}+s_{N+1,a}} \right) = \prod_{a=0}^{r_{N+1}} B_{N+1} 1^{s_{N+1,a}} = B_{N+2},$$

so $\tilde{B}_n = B_{n+1}$ for all $n > N$. □

PROPOSITION 3.7. *Let T be a rank-one transformation such that $s_{n,r_n} = 0$ and $s_{n,i} = c_n$ for $0 \leq i < r_n$ for all sufficiently large n. If c_n is not eventually constant, then there exists* *a rank-one subshift $\tilde{T}$ which generates the same language as T, with the property that $\tilde{s}_{n,\tilde{r}_n} = 0$ and $\tilde{s}_{n,i}$ is not constant over $0 \le i < \tilde{r}_n$ for infinitely many n.*

Proof. If c_n is not eventually constant, then there exist infinitely many $n < m$ such that $c_n \neq c_m$, so there exist infinitely many n such that $c_n \neq c_{n+1}$.

If we apply Lemma 3.6 at such an n, then $\tilde{s}_{n,i}$ is not constant over $0 \le i < \tilde{r}_n$ since $\tilde{s}_{n,r_n} = s_{n,r_n} + s_{n+1,0} = 0 + c_{n+1} \neq c_n = \tilde{s}_{n,0}$ and $\tilde{s}_{n,\tilde{r}_n} = s_{n,r_n} + s_{n+1,r_{n+1}} = 0$.

Let $\mathcal{N}$ be a set of n such that $c_n \neq c_{n+1}$ such that $\mathcal{N}$ does not contain any pairs of consecutive integers. Applying Lemma 3.6 for each $n \in \mathcal{N}$ gives the claim. □

In fact, one can do a similar modification across multiple stages simultaneously.

LEMMA 3.8. *Let T be a rank-one subshift with cut sequence $\{r_n\}$ and spacer sequence $\{s_{n,i}\}$, and let $\{n_t\}$ be a strictly increasing sequence with $n_1 = 1$. For $t \ge 1$, set*

$$\tilde{r}_t = \left(\prod_{n=n_t}^{n_{t+1}-1} (r_n + 1) \right) - 1$$

and for $0 \le j < n_{t+1} - n_t$ and $0 \le i_j \le r_{n_t+j}$,

$$\tilde{s}_{t,i_0+i_1(r_{n_t}+1)+i_2(r_{n_t+1}+1)(r_{n_t}+1)+\cdots+i_{n_{t+1}-n_t-1}(r_{n_{t+1}-1}+1)\cdots(r_{n_t}+1)} = s_{n_t,i_0} + \sum_{j=1}^{n_{t+1}-n_t-1} \begin{cases} s_{n_t+j,i_j} & \text{if } i_k = r_{n_t+k} \text{ for all } 0 \le k < j, \\ 0 & \text{otherwise.} \end{cases}$$

Then T and $\tilde{T}$ generate the same language: $\tilde{B}_t = B_{n_t}$ for all $t \ge 1$.

Proof. We have $\tilde{B}_1 = 0 = B_1 = B_{n_1}$, so we may assume $\tilde{B}_t = B_{n_t}$ and then

$$\begin{aligned}
\tilde{B}_{t+1} &= \prod_{a=0}^{\tilde{r}_t} \tilde{B}_t 1^{\tilde{s}_{t,a}} \\
&= \prod_{i_0,\ldots,i_{n_{t+1}-n_t-1}} B_{n_t} 1^{s_{n_t,i_0}} 1^{\sum_{j=1}^{n_{t+1}-n_t-1} s_{n_t+j,i_j} \mathbb{1}_{i_k=r_{n_t+k} \text{ for all } k<j}} \\
&= \prod_{i_1,\ldots,i_{n_{t+1}-n_t-1}} \left(\prod_{i_0=0}^{r_{n_t}} B_{n_t} 1^{s_{n_t,i_0}} \right) 1^{s_{n_t+1,i_1}} 1^{\sum_{j=2}^{n_{t+1}-n_t-1} s_{n_t+j,i_j} \mathbb{1}_{i_k=r_{n_t+k} \text{ for all } k<j}} \\
&= \prod_{i_1,\ldots,i_{n_{t+1}-n_t-1}} B_{n_t+1} 1^{s_{n_t+1,i_1}} 1^{\sum_{j=2}^{n_{t+1}-n_t-1} s_{n_t+j,i_j} \mathbb{1}_{i_k=r_{n_t+k} \text{ for all } k<j}} \\
&= \prod_{i_2,\ldots,i_{n_{t+1}-n_t-1}} \left(\prod_{i_1=0}^{r_{n_t+1}} B_{n_t+1} 1^{s_{n_t+1,i_1}} \right) 1^{s_{n_t+2,i_2}} 1^{\sum_{j=3}^{n_{t+1}-n_t-1} s_{n_t+j,i_j} \mathbb{1}_{i_k=r_{n_t+k} \text{ for all } k<j}} \\
&\ \ \vdots \\
&= \prod_{i_{n_{t+1}-n_t-1}=0}^{r_{n_{t+1}-n_t-1}} B_{n_{t+1}-1} 1^{s_{n_{t+1}-1,i_{n_{t+1}-n_t-1}}} = B_{n_{t+1}}.
\end{aligned}$$

□

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PROPOSITION 3.9. *Let T be a rank-one subshift such that $s_{n,r_n} = 0$ for all sufficiently large n and that there exists $0 \le i, i' < r_n$ such that $s_{n,i} \ne s_{n,i'}$ for infinitely many n. Then there exists a rank-one subshift $\tilde{T}$, which generates the same language, such that for all sufficiently large n, $\tilde{s}_{n,\tilde{r}_n} = 0$ and there exists $0 \le i, i' < \tilde{r}_n$ with $\tilde{s}_{n,i} \ne \tilde{s}_{n,i'}$.*

Proof. Let $n_1 = 1$ and $\{n_t\}_{t\ge 2}$ be the sequence of n for which $s_{n,i} \ne s_{n,i'}$. Lemma 3.8 then gives the claim since $s_{n_t,i}$ being non-constant over $0 \le i < r_{n_t}$ implies $\tilde{s}_{t,a}$ is non-constant over $0 \le a < r_{n_t}$ and hence over $0 \le a < \tilde{r}_t$. Clearly, $\tilde{s}_{t,\tilde{r}_t} = 0$ for sufficiently large t as $s_{n,r_n} = 0$ for all sufficiently large n. □

PROPOSITION 3.10. *Let T be a rank-one subshift such that $s_{n,r_n} = 0$ for all sufficiently large n and that for infinitely many n, $s_{n,0} = s_{n,r_n-1} = 0$. Then there exists a rank-one subshift, which generates the same language, such that $\tilde{s}_{n,r_n} = 0$ and $\tilde{s}_{n,0} = \tilde{s}_{n,\tilde{r}_n-1} = \tilde{s}_{n+1,\tilde{r}_{n+1}-1} = 0$ for all sufficiently large n.*

Proof. Let $n_1 = 1$ and $\{n_t\}_{t\ge 2}$ be the sequence of n for which $s_{n,0} = s_{n,r_n-1} = 0$. Lemma 3.8 then gives the subshift since $\tilde{s}_{t,0} = s_{n_t,0} = 0$ and $\tilde{s}_{t,\tilde{r}_t-1}$ has $i_0 = r_{n_t} - 1$ so $\tilde{s}_{t,\tilde{r}_t-1} = s_{n_t,r_{n_t}-1} = 0$ and, likewise, $\tilde{s}_{t+1,\tilde{r}_{t+1}-1} = s_{n_{t+1},r_{n_{t+1}}-1} = 0$. □

PROPOSITION 3.11. *If a rank-one subshift has the property that $s_{n,r_n} = 0$ for all sufficiently large n and there exist constant non-negative integers $c < d$ such that $s_{n,i} \in \{c, d\}$ for all $0 \le i < r_n$ (with both occurring) for sufficiently large n, then there exists a rank-one subshift which generates the same language such that $s_{n,r_n} = 0$ and $s_{n,i} \in \{0, d-c\}$ for all $0 \le i < r_n$ (with both occurring) for all sufficiently large n.*

Proof. For all n, set $\tilde{r}_n = r_n$. Let N such that for all $n \ge N$, we have $s_{n,i} \in \{c, d\}$ for all $0 \le i < r_n$ and $s_{n,r_n} = 0$. For $n < N$, set $\tilde{s}_{n,i} = s_{n,i}$.

Set $\tilde{s}_{N,i} = s_{N,i}$ for $0 \le i < r_N$ and $\tilde{s}_{N,r_N} = c$. For $n > N$, set $\tilde{s}_{n,i} = s_{n,i} - c$ for $0 \le i < r_n$ and $\tilde{s}_{n,r_n} = 0$.

Clearly, $\tilde{B}_n = B_n$ for $n \le N$. Observe that

$$\tilde{B}_{N+1} = \left(\prod_{i=0}^{r_N-1} B_N 1^{s_{N,i}} \right) B_N 1^c = B_{N+1} 1^c.$$

If $\tilde{B}_n = B_n 1^c$, then

$$\tilde{B}_{n+1} = \left(\prod_{i=0}^{r_n-1} \tilde{B}_n 1^{\tilde{s}_{n,i}} \right) \tilde{B}_n = \left(\prod_{i=0}^{r_n-1} B_n 1^c 1^{s_{n,i}-c} \right) B_n 1^c = B_{n+1} 1^c,$$

so $\tilde{B}_n = B_n 1^c$ for all $n > N$, meaning they generate the same language. □

3.3. *Totally ergodic rank-one subshifts*

PROPOSITION 3.12. *Let T be a rank-one transformation such that there exists c so that for all sufficiently large n, it holds that $s_{n,i} = c$ for all $0 \le i < r_n$ and $s_{n,r_n} = 0$. Then T is an odometer.*

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Proof. Let $N > 1$ such that for all $n \geq N$, $s_{n,i} = c$ for all $0 \leq i < r_n$ and $s_{n,r_n} = 0$. Let $S_{n,j}^{[i]}$ for $1 \leq j \leq c$ be the spacer levels added above $C_n^{[i]}$ for $0 \leq i < r_n$ (we do not add spacers above $C_n^{[r_n]}$ as $s_{n,r_n} = 0$). Since $T(S_{n,c}^{[i]}) = I_{n,0}^{[i+1]}$ for $0 \leq i < r_n$, and since $I_{n,0} \subseteq I_{N,0}^{[0]}$, we have that $T(S_{n,c}^{[i]}) \subseteq I_{N,0}^{[0]}$ for all $n \geq N$ and all $0 \leq i < r_n$. Since $I_{N,h_N-1}^{[r_N]} = \bigsqcup_{n>N} \bigsqcup_{i=0}^{r_n-1} I_{n,h_n-1}^{[i]}$, this means $T^{h_N+c}(I_{N,0}) = I_{N,0}$.

Define $I_{N,h_N} = \bigsqcup_{n \geq N} \bigsqcup_{0 \leq i < r_n} S_{n,1}^{[i]}$. Then, $T(I_{N,h_N-1}) = I_{N,h_N}$ and $T^c(I_{N,h_N}) = I_{N,0}$. Define the column $C'_N = \bigsqcup_{j=0}^{h_N-1} I_{N,j} \sqcup \bigsqcup_{j=0}^{c-1} T^j(I_{N,h_N})$ and the columns C'_{N+n} via cutting and stacking starting from C'_N using cut sequence $r'_{N+n} = r_{N+n}$ and spacer sequence $s'_{n,i} = 0$. The resulting odometer is the same map as X, so X is an odometer. □

PROPOSITION 3.13. *Let T be a rank-one transformation on a finite measure space which is not an odometer. If* $\limsup p(q)/q < 2$, *then there exists a rank-one subshift, which generates the same language as T, such that there exists a constant positive integer d so that for all sufficiently large n, it holds that* $s_{n,r_n} = 0$ *and* $s_{n,i} \in \{0, d\}$ *for all* $0 \leq i < r_n$ *and there exists* $0 \leq i, i' < r_n$ *so that* $s_{n,i} = 0$ *and* $s_{n,i'} = d$.

Proof. By Proposition 3.3, T is measure-theoretically isomorphic to a transformation $\tilde{T}$ which generates the same language and has $\tilde{s}_{n,\tilde{r}_n} = 0$ for all n. By Proposition 3.12, $\tilde{T}$ has the property that for every n and $0 \leq i < r_n$, there exists $m \geq n$ and $0 \leq i' < r_m$ such that $s_{m,i} \neq s_{n,i'}$.

By Proposition 3.5, if $\limsup_N |\{s_{n,i} : n \geq N, 0 \leq i < r_n\}| \geq 3$, then $\limsup p(q)/q \geq 2$. So there exists N such that $|\{s_{m,i} : m \geq N, 0 \leq i < r_m\}| \leq 2$. Therefore, $|\{s_{m,i} : m \geq n, 0 \leq i < r_m\}| = 2$ for all sufficiently large n.

Proposition 3.7 gives a rank-one subshift generating the same language such that $s_{n,r_n} = 0$ for all sufficiently large n and $s_{n,i} \neq s_{n,i'}$ for infinitely many n. Proposition 3.9 then gives a rank-one subshift generating the same language with that property for all sufficiently large n. Finally, Proposition 3.11 gives a rank-one subshift, still generating the same language, such that $s_{n,i} \in \{0, d\}$ and $0 \leq i < r_n$, and $s_{n,r_n} = 0$ for all sufficiently large n. □

4. *Subshifts with exactly one non-zero spacer value*

THEOREM 4.1. *Let p be the complexity function for a rank-one subshift such that for all sufficiently large n, the spacer sequence satisfies* $s_{n,i} \in \{0, d\}$ *for some constant positive integer d and* $s_{n,r_n} = 0$, *and that* $s_{n,i}$ *is not constant over* $0 \leq i < r_n$. *Then,* $\limsup p(q) - 1.5q = \infty$.

This is a quick consequence of the following theorem.

THEOREM 4.2. *Let p be the complexity function for a rank-one subshift such that for all sufficiently large n, the spacer sequence satisfies* $s_{n,i} \in \{0, d\}$ *for some constant positive integer d and* $s_{n,r_n} = 0$, *and that* $s_{n,i}$ *is not constant over* $0 \leq i < r_n$.

Then there exists a constant C such that for all sufficiently large n, there exists $q_n \geq h_n$ *such that* $p(q_n) \geq 1.5q_n + (p(h_n) - h_n) - C$.

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Proof of Theorem 4.1 from Theorem 4.2. Let N such that for all $n \geq N$, there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + (p(h_n) - h_n) - C$. Let $m > n$ such that $h_m \geq q_n$. As $s_{n,i} > 0$ for $i < r_n$ implies aperiodicity, $p(\ell + 1) - p(\ell) \geq 1$ for all ℓ so $p(h_n) \geq h_n$ and $p(h_m) - p(q_n) \geq h_m - q_n$. Then,

$$\begin{aligned} p(h_m) - h_m &= (p(h_m) - p(q_n)) + p(q_n) - h_m \\ &\geq (h_m - q_n) + 1.5q_n - C - h_m = 0.5q_n - C \to \infty \end{aligned}$$

and therefore $p(q_m) - 1.5q_m \geq p(h_m) - h_m - C \to \infty$. □

Before proving Theorem 4.2, we show how Theorem 4.1 implies the following theorem.

THEOREM 4.3. *Let T be a rank-one transformation (on a probability space) which is not an odometer. Then the associated subshift has complexity satisfying* $\limsup p(q) - 1.5q = \infty$ *(and* $\liminf p(q) - q = \infty$*).*

Proof. By Proposition 3.13, either $\limsup p(q)/q \geq 2$ or there exists a rank-one subshift which generates the same language with the property that there exists a constant non-negative integer d such that for all sufficiently large n, $s_{n,i} \in \{0, d\}$ for all $0 \leq i < r_n$ and $s_{n,r_n} = 0$, and such that there exists $0 \leq i, i' < r_n$ with $s_{n,i} = 0$ and $s_{n,i'} = d$. Theorem 4.1 applied to that subshift then gives that $\limsup p(q) - 1.5q = \infty$. Proposition 2.6 ensures $\liminf p(q) - q = \infty$ as otherwise, $p(q) = q + c$ for a constant c for all sufficiently large q. □

The remainder of this section is the proof of Theorem 4.2.

4.1. *Some notation and basic facts.*

Write $\widehat{1}$ to represent 1^d.

We use repeatedly the facts that 0 is a prefix of every B_n (Lemma 3.2) and that B_n is a suffix of B_m for $m \geq n$ for sufficiently large n (due to $s_{n,r_n} = 0$).

We also use repeatedly the fact that $B_n B_n$ and $B_n \widehat{1} B_n$ are subwords of B_{n+1} due to $s_{n,i}$ not being constant over $0 \leq i < r_n$.

LEMMA 4.4. *There exists a constant $c \geq 1$ such that for all $n \geq N$, the words B_n^2 and $B_n \widehat{1} B_n$ differ on suffixes of length at least $h_n + c$.*

Proof. Choose c such that B_N has 01^{c-1} as a suffix (possible as B_N has 0 as a prefix). Since $B_n B_n$ has $B_N B_n$ as a suffix, $B_n B_n$ has $01^{c-1} B_n$ as a suffix. As $B_n \widehat{1} B_n$ has $1^{c-1} 1^d B_n$ as a suffix, this shows the words differ on the suffixes $01^{c-1} B_n$ and $1^c B_n$. □

4.2. *Counting via right-special words*

LEMMA 4.5. $B_n \in \mathcal{L}^{RS}$ *for all n.*

Proof. B_{n+1} contains $B_n B_n$ and $B_n \widehat{1}$ as subwords. B_n has 0 as a prefix, so $B_n \in \mathcal{L}^{RS}$. □

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LEMMA 4.6. *Write* $f_n = p(h_n) - h_n$.

If there are t_n *distinct right-special words, all of length at least* h_n *and less than* q_n*, which are not suffixes of* B_{n+m} *for any* $m \geq 1$*, then*

$$p(q_n) \geq q_n + f_n + t_n.$$

Proof. Since $p(q_n) - p(h_n) = |\{w \in \mathcal{L}^{RS} : h_n \leq \ell en(w) < q_n\}|$ and since, by Lemma 4.5, we have at least $q_n - h_n$ suffixes of some B_{n+m} of length at least h_n and less than q_n which are right-special and distinct from the t_n hypothesized, $p(q_n) \geq p(h_n) + q_n - h_n + t_n$. □

The proof of Theorem 4.2 will proceed by establishing the existence of right-special words which are not suffixes of any B_{n+m}. To this end, rewrite the defining words as

$$B_{n+1} = \left(\prod_{j=1}^{z_n-1} B_n^{a_{n,j}}\widehat{1}\right) B_n^{a_{n,z_n}},$$

where $a_{n,j} \geq 1$ and $z_n \geq 2$, and $a_{n,j} \geq 2$ for at least one j as 0 and d both occur in $\{s_{n,i} : 0 \leq i < r_n\}$.

4.3. *The (straightforwardly) 5/3 cases.* Throughout this section, let N such that $s_{n,i} \in \{0, d\}$ and $s_{n,r_n} = 0$, and $s_{n,i}$ is not constant over $0 \leq i < r_n$ for all $n \geq N$.

PROPOSITION 4.7. *If, for* $n \geq N$*, one of the following holds:*

- $a_{n,z_n} \geq 2$ *and* $a_{n,1} = 1$*, i.e.* $B_n\widehat{1}$ ——— B_n^2*;*
- $a_{n,z_n} = 1$ *and* $a_{n,1} \geq 2$*, i.e.* B_n^2 ——— $\widehat{1}B_n$*;*
- $a_{n,z_n} = 1$ *and* $a_{n,1} = 1$*, and* $a_{n,j} \geq 3$ *for some* j*, i.e.* $B_n\widehat{1}$ ——— B_n^3 ——— $\widehat{1}B_n$*,*

then there exists $q_n \geq h_n$ *such that* $p(q_n) \geq (5/3)q_n + f_n - c$.

LEMMA 4.8. *Words of the form* $B_n\widehat{1}B_n\widehat{1}$ ———

For $n \geq N$*, if* $a_{n,1} = a_{n,2} = 1$*, then* $B_n\widehat{1}B_n\widehat{1}B_n \in \mathcal{L}^{RS}$.

Proof. Let j minimal such that $a_{n,j} \geq 2$.

If $j > 3$, then B_{n+1} has the subword $B_n^{a_{n,j-3}}\widehat{1}B_n^{a_{n,j-2}}\widehat{1}B_n^{a_{n,j-1}}\widehat{1}B_n^{a_{n,j}} = B_n\widehat{1}B_n\widehat{1}B_n\widehat{1}B_n^{a_{n,j}}$, which has $B_n\widehat{1}B_n\widehat{1}B_n\widehat{1}B_n^2$ as a prefix.

If $j = 3$, then the word $B_{n+1}\widehat{1}B_{n+1}$ has the subword $B_n^{a_{n,z_n}}\widehat{1}B_n^{a_{n,1}}\widehat{1}B_n^{a_{n,2}}\widehat{1}B_n^{a_{n,3}}$ as a subword, which has $B_n\widehat{1}B_n\widehat{1}B_n\widehat{1}B_n^2$ as a subword.

Then, $B_n\widehat{1}B_n\widehat{1}B_n \in \mathcal{L}^{RS}$ as $B_n\widehat{1}B_n\widehat{1}B_n\widehat{1}$ and $B_n\widehat{1}B_n\widehat{1}B_nB_n$ are both subwords of $B_n\widehat{1}B_n\widehat{1}B_n\widehat{1}B_n^2$. □

LEMMA 4.9. *Words of the form* $B_n\widehat{1}B_n^2$ ——— B_n^2

For $n \geq N$*, if* $a_{n,z_n} \geq 2$ *and* $a_{n,1} = 1$ *and* $a_{n,2} \geq 2$*, then* $B_n^2\widehat{1}B_n \in \mathcal{L}^{RS}$.

Proof. $B_{n+1}\widehat{1}B_{n+1} \in \mathcal{L}$ implies $B_n^{a_{n,z_n}}\widehat{1}B_n^{a_{n,1}}\widehat{1}B_n^{a_{n,2}} \in \mathcal{L}$, so $B_n^2\widehat{1}B_n\widehat{1} \in \mathcal{L}$ as $a_{n,z_n} \geq 2$ and $a_{n,1} = 1$.

$B_{n+1}B_{n+1} \in \mathcal{L}$ implies $B_n^{a_{n,z_n}}B_n^{a_{n,1}}\widehat{1}B_n^{a_{n,2}} \in \mathcal{L}$. Since $a_{n,2} \geq 2$, this gives $B_n^2\widehat{1}B_n^2$, so $B_n^2\widehat{1}B_n0 \in \mathcal{L}$. □

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LEMMA 4.10. *Words of the form* $B_n^2\widehat{1}$ ———
For $n \geq N$, *if* $a_{n,1} = 2$, *then* $B_n\widehat{1}B_n^2 \in \mathcal{L}^{RS}$.

Proof. $B_n^{a_{n,z_n-1}}\widehat{1}B_n^{a_{n,z_n}}B_n^{a_{n,1}} \in \mathcal{L}$ as it is a subword of $B_{n+1}B_{n+1}$ so, as $a_{n,z_n-1} \geq 1$ and $a_{n,z_n} + a_{n,1} \geq 3$, also $B_n\widehat{1}B_n^3 \in \mathcal{L}$. $B_n^{a_{n,z_n}}\widehat{1}B_n^{a_{n,1}}\widehat{1}$ is a subword of $B_{n+1}\widehat{1}B_{n+1}$ so, as $a_{n,z_n} \geq 1$, also $B_n\widehat{1}B_n^2\widehat{1} \in \mathcal{L}$. □

LEMMA 4.11. *Words of the form* B_n^3 ——— *or* ——— B_n^4 ———
For $n \geq N$, *if* $a_{n,1} > 2$ *or* $a_{n,j} > 3$ *for some* j, *then* $B_n^3 \in \mathcal{L}^{RS}$.

Proof. If $a_{n,j} \geq 4$, since $B_n^{a_{n,j}}\widehat{1}$ is a subword of B_{n+1}, so is $B_n^4\widehat{1}$. If $a_{n,1} \geq 3$, then since $B_n^{a_{n,z_n}}B_n^{a_{n,1}}\widehat{1}$ is a subword of $B_{n+1}B_{n+1}$ and $a_{n,z_n} + a_{n,1} \geq 4$, also $B_n^4\widehat{1} \in \mathcal{L}$. □

LEMMA 4.12. *Words of the form* $B_n\widehat{1}$ ——— $\widehat{1}B_n^3\widehat{1}$ ——— $\widehat{1}B_n$
For $n \geq N$, *if* $a_{n,1} = a_{n,z_n} = 1$ *and* $a_{n,j} = 3$ *for some* $j > 1$, *then* $B_n\widehat{1}B_n^2 \in \mathcal{L}^{RS}$.

Proof. The word $B_{n+1}B_{n+1} \in \mathcal{L}$ so $B_n^{a_{n,z_n-1}}\widehat{1}B_n^{a_{n,z_n}}B_n^{a_{n,1}}\widehat{1} \in \mathcal{L}$ so $B_n\widehat{1}B_n^2\widehat{1} \in \mathcal{L}$. As $B_n^{a_{n,j-1}}\widehat{1}B_n^{a_{n,j}} \in \mathcal{L}$, also $B_n\widehat{1}B_n^3 \in \mathcal{L}$. □

Proof of Proposition 4.7. First consider when $a_{n,z_n} \geq 2$ and $a_{n,1} = 1$. If $a_{n,2} = 1$, then Lemma 4.8 gives $D_n = B_n\widehat{1}B_n\widehat{1}B_n \in \mathcal{L}^{RS}$. Since B_{n+1} has B_n^2 as a suffix, every suffix of D_n of length at least $h_n + c$ is not a suffix of B_{n+1} (Lemma 4.4) and is right-special. If $a_{n,2} \geq 2$, then Lemma 4.9 gives $D_n = B_n^2\widehat{1}B_n \in \mathcal{L}^{RS}$ which likewise has the property that every suffix of D_n of length at least $h_n + c$ is right-special and not a suffix of B_{n+1}.

Now consider when $a_{n,z_n} = 1$ and $a_{n,1} \geq 2$. If $a_{n,1} = 2$, then Lemma 4.10 gives $D_n = B_n\widehat{1}B_n^2 \in \mathcal{L}^{RS}$. As $\widehat{1}B_n$ is a suffix of B_{n+1} in this case, again every suffix of D_n of length at least $h_n + c$ is not a suffix of B_{n+1} and is right-special. If $a_{n,1} > 2$, then Lemma 4.11 gives $D_n = B_n^3$ which has the same property.

Last consider the case when $a_{n,z_n} = 1$ and $a_{n,1} = 1$, and $a_{n,j} \geq 3$ for some j. If $a_{n,j} = 3$, then Lemma 4.12 gives $D_n = B_n\widehat{1}B_n^2 \in \mathcal{L}^{RS}$ and as $\widehat{1}B_n$ is a suffix of B_{n+1} in this case, D_n has the same property as above. If $a_{n,j} > 3$, then Lemma 4.11 gives $D_n = B_n^3$ with the same property.

In all cases, we have a word D_n of length at least $3h_n$ with every suffix of length at least $h_n + c$ being right-special and not a suffix of B_{n+1}, so $2h_n - c$ right-special words which are not suffixes of B_{n+1} all of length less than $3h_n$. By Lemma 4.6, then $p(3h_n) \geq 3h_n + f_n + 2h_n - c = (5/3)(3h_n) + f_n - c$. □

4.4. *Words of the form* B_n^2 ——— B_n^2

PROPOSITION 4.13. *If* $a_{n,1} \geq 2$ *and* $a_{n,z_n} \geq 2$ *and* $a_{n+1,z_{n+1}} \geq 2$, *then there exists* $q_n \geq h_n$ *such that* $p(q_n) \geq 1.5q_n + f_n - 3c$.

These subshifts include the examples studied in [**Fer95**] defined by $B_{n+1} = B_n^p 1 B_n^q$ for $p, q > 1$.

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The proof of Proposition 4.13 is a series of lemmas. Write

$$B_{n+1} = B_n^{\alpha}\widehat{1}uB_n^{\beta}$$

for some word u which is either empty or ends in $\widehat{1}$. Then, $\alpha, \beta \geq 2$.

LEMMA 4.14. *If $\alpha \neq \beta$, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - c$.*

Proof. The word $B_{n+1}B_{n+1}$ has $\widehat{1}B_n^{\alpha+\beta}\widehat{1}$ as a subword, so $B_n^{\alpha+\beta-1} \in \mathcal{L}^{RS}$. The word B_{n+1} has $\widehat{1}B_n^{\beta}$ as a suffix, so our word differs from B_{n+1} on suffixes of length at least $\beta h_n + c$ (Lemma 4.4, which we henceforth use implicitly) and so gives at least $(\alpha - 1)h_n - c$ right-special words which are not suffixes of B_{n+1} with length less than $(\alpha + \beta - 1)h_n$. Then by Lemma 4.6,

$$\begin{aligned} p((\alpha+\beta-1)h_n) &\geq (\alpha+\beta-1)h_n + f_n + (\alpha-1)h_n - c \\ &= \tfrac{3}{2}(\alpha+\beta-1)h_n + \tfrac{1}{2}(\alpha-1-\beta)h_n + f_n - c. \end{aligned}$$

If $\alpha \geq \beta + 1$, then $\frac{3}{2}(\alpha+\beta-1)h_n + \frac{1}{2}(\alpha-1+\beta)h_n + f_n - c \geq \frac{3}{2}(\alpha+\beta-1)h_n + f_n - c$.

Now consider when $\alpha < \beta$. Let α' minimal such that $\widehat{1}B_n^{\alpha'}\widehat{1}$ is a subword of $\widehat{1}B_{n+1}$. Then, $\alpha' \leq \alpha < \beta$. If $\alpha' < \alpha$, then $B_n^{\alpha'}\widehat{1}B_n^{\alpha'}\widehat{1}$ is a subword of B_{n+1} as α' is minimal so $B_n^{\alpha'}$ must precede $\widehat{1}B_n^{\alpha'}\widehat{1}$ in B_{n+1}. If $\alpha' = \alpha$, then as $\alpha < \beta$, the word $B_n^{\alpha'}\widehat{1}B_n^{\alpha'}\widehat{1}$ is a subword of $B_{n+1}\widehat{1}B_{n+1}$ (with the first $\widehat{1}$ in our word being the middle $\widehat{1}$ in $B_{n+1}\widehat{1}B_{n+1}$). Since α' is minimal, B_{n+1} has $B_n^{\alpha'}\widehat{1}B_n^{\beta}$ as a suffix and, as $\alpha' < \beta$, that word has $B_n^{\alpha'}\widehat{1}B_n^{\alpha'}B_n$ as a subword. Then, $B_n^{\alpha'}\widehat{1}B_n^{\alpha'} \in \mathcal{L}^{RS}$.

Since B_{n+1} has $B_n^{\alpha'+1}$ as a suffix, our word gives at least $\alpha' h_n + d - c$ right-special words which are not suffixes of B_{n+1} with length less than $2\alpha' h_n + d$. Then by Lemma 4.6 (which we will henceforth use implicitly),

$$p(2\alpha' h_n + d) \geq 2\alpha' h_n + d + f_n + \alpha' h_n + d - c = \tfrac{3}{2}(2\alpha' h_n + d) + \tfrac{1}{2}d - c + f_n.$$

□

From here on, assume $\alpha = \beta$.

LEMMA 4.15. *If $\widehat{1}B_n^{t}\widehat{1}$ is a subword of B_{n+1} for some $t \neq \beta$ and $t \neq 2\beta$, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - c$.*

Proof. As B_{n+1} has $B_n^{\beta}\widehat{1}$ as a prefix, there is some $t' \neq \beta, 2\beta$ such that $B_n^{\beta}\widehat{1}B_n^{t'}\widehat{1}$ is a subword of B_{n+1}.

Suppose first that there is such a $t' < \beta$. As $B_n^{\beta}\widehat{1}B_n^{\beta}$ is a subword of $B_{n+1}\widehat{1}B_{n+1}$, then $B_n^{\beta}\widehat{1}B_n^{t'} \in \mathcal{L}^{RS}$. Since $B_n^{t'+1}$ is a suffix of B_{n+1} (as $t' < \beta$), this gives at least $\beta h_n + d - c$ right-special suffixes that are not suffixes of B_{n+1}, all of length less than $(\beta + t')h_n + d$. Then, as $t' < \beta$,

$$\begin{aligned} p((\beta+t')h_n + d) &\geq (\beta+t')h_n + d + f_n + \beta h_n + d - c \\ &= \tfrac{3}{2}((\beta+t')h_n + d) + \tfrac{1}{2}(\beta - t')h_n + \tfrac{1}{2}d + f_n - c \\ &> \tfrac{3}{2}((\beta+t')h_n + d) + f_n - c. \end{aligned}$$

So we may assume that for all t such that $\widehat{1}B_n^{t}\widehat{1}$ is a subword of B_{n+1}, we have $t \geq \beta$.

Suppose now that $\beta < t' < 2\beta$. As B_{n+1} has $B_n^\beta \hat{1} B_n^\beta$ as a suffix (since we have ruled out $t < \beta$), the word $B_{n+1}B_{n+1}$ has $B_n^\beta \hat{1} B_n^{2\beta}$ as a subword. Then, $B_n^\beta \hat{1} B_n^{t'} \in \mathcal{L}^{RS}$ as $t' < 2\beta$. This gives at least $(\beta + t' - \beta)h_n + d - c$ right-special suffixes which are not suffixes of B_{n+1} (which ends in $\hat{1}B_n^\beta$) all of length less than $(\beta + t')h_n + d$. Then as $t' > \beta$,

$$\begin{aligned} p((\beta + t')h_n + d) &\geq (\beta + t')h_n + d + f_n + t'h_n + d - c \\ &= \tfrac{3}{2}((\beta + t')h_n + d) + \tfrac{1}{2}(t' - \beta)h_n + \tfrac{1}{2}d - c + f_n \\ &> \tfrac{3}{2}((\beta + t')h_n + d) + \tfrac{1}{2}d - c + f_n. \end{aligned}$$

If $t' > 2\beta$, then $B_n^{2\beta+1}\hat{1} \in \mathcal{L}$, so $B_n^{2\beta} \in \mathcal{L}^{RS}$, which gives at least $\beta h_n - c$ right-special suffixes which are not suffixes of B_{n+1}, all of length less than $2\beta h_n$. Then,

$$p(2\beta h_n) \geq 2\beta h_n + f_n + \beta h_n - c = \tfrac{3}{2}(2\beta h_n) + f_n - c. \qquad \square$$

We are left with the case when every $\hat{1}B_n^t\hat{1}$ in B_{n+1} has $t = \beta$ or $t = 2\beta$.

LEMMA 4.16. *If $\hat{1}B_n^{2\beta}\hat{1}$ is a subword of B_{n+1}, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - 2c$.*

Proof. First observe that $B_n^{2\beta-1} \in \mathcal{L}^{RS}$, which gives at least $(\beta - 1)h_n - c$ right-special suffixes of length less than $(2\beta - 1)h_n$ which are not suffixes of B_{n+1}.

Choose $x, y \geq 1$ so that B_{n+1} has $(B_n^\beta\hat{1})^x B_n^{2\beta}$ as a prefix and $B_n^{2\beta}(\hat{1}B_n^\beta)^y$ as a suffix.

Then, $B_{n+1}\hat{1}B_{n+1}$ has the subword $(B_n^\beta\hat{1})^{x+y+1}B_n^{2\beta}$, which means that $(B_n^\beta\hat{1})^{x+y}B_n^\beta \in \mathcal{L}^{RS}$. This gives at least $x(\beta h_n + d) - c$ right-special suffixes which are not suffixes of B_{n+1} of length less than $(x + y + 1)\beta h_n + (x + y)d$. As there is no overlap between these and the suffixes of $B_n^{2\beta-1}$, this gives a total of at least $((x + 1)\beta - 1)h_n + xd - 2c$ right-special suffixes of length less than $(x + y + 1)\beta h_n + (x + y)d$ which are not suffixes of B_{n+1}. Then,

$$\begin{aligned} p((x + y + 1)\beta h_n + (x + y)d) &\geq (x + y + 1)\beta h_n + (x + y)d \\ &\quad + f_n + ((x + 1)\beta - 1)h_n + xd - 2c \\ &= \tfrac{3}{2}((x + y + 1)\beta h_n + (x + y)d) + \tfrac{1}{2}(x + 1 - y)\beta h_n - h_n + \tfrac{1}{2}(x - y)d - 2c + f_n \end{aligned}$$

and, as $\beta \geq 2$, this means that if $x \geq y$, then

$$\begin{aligned} p((x + y + 1)\beta h_n + (x + y)d) &\geq \tfrac{3}{2}((x + y + 1)\beta h_n + (x + y)d) + \tfrac{1}{2}\beta h_n - h_n - 2c + f_n \\ &\geq \tfrac{3}{2}((x + y + 1)\beta h_n + (x + y)d) - 2c + f_n. \end{aligned}$$

So we may assume from here on that $x < y$.

Write $B_{n+1} = \big(\prod_{i=1}^s (B_n^\beta\hat{1})^{x_i} B_n^{2\beta}\hat{1}\big)(B_n^\beta\hat{1})^{y-1}B_n^\beta$ for some $s \geq 1$ and $x_i \geq 1$ with $x_1 = x$. Choose i' such that $x_{i'}$ is minimal and i' is the minimal such i.

First we consider the case when $i' > 1$. Then, $x_{i'} < x$ since otherwise, we would have chosen $i' = 1$. Since $B_{n+1}\hat{1}$ has $B_n^{2\beta}\hat{1}(B_n^\beta\hat{1})^{x_s}B_n^{2\beta}\hat{1}(B_n^\beta\hat{1})^{y-1}B_n^\beta\hat{1}$ as a suffix, it also has $(B_n^\beta\hat{1})^{x_{i'}+1}B_n^{2\beta}(\hat{1}B_n^\beta)^y\hat{1}$ as a suffix since $x_s \geq x_{i'}$. As $y > x_{i'}$, then $(B_n^\beta\hat{1})^{x_{i'}+1}B_n^{2\beta}(\hat{1}B_n^\beta)^{x_{i'}+1}\hat{1} \in \mathcal{L}$.

Since $i' > 1$, B_{n+1} has $(B_n^{\beta}\widehat{1})^{x_{i'-1}} B_n^{2\beta} (\widehat{1}B_n^{\beta})^{x_{i'}} \widehat{1} B_n^{2\beta}$ as a subword. Then, $(B_n^{\beta}\widehat{1})^{x_{i'}+1} B_n^{2\beta} (\widehat{1}B_n^{\beta})^{x_{i'}} \widehat{1} B_n^{2\beta} \in \mathcal{L}$ as $x_{i'-1} \geq x_{i'}+1$ by the choice of i', so $(B_n^{\beta}\widehat{1})^{x_{i'}+1} B_n^{2\beta} (\widehat{1}B_n^{\beta})^{x_{i'}} \widehat{1} B_n^{\beta}$ is right-special.

Since $x_{i'} < x < y$ implies $x_{i'} < y-1$ and B_{n+1} has $\widehat{1}(B_n^{\beta}\widehat{1})^{y-1} B_n^{\beta}$ as a suffix, this gives at least $(x_{i'}+2)\beta h_n + (x_{i'}+1)d - c$ right-special words which are not suffixes of B_{n+1}, all of length less than $(2x_{i'}+4)\beta h_n + (2x_{i'}+2)d$. Therefore,

$$\begin{aligned}
&p((2x_{i'}+4)\beta h_n + (2x_{i'}+2)d) \\
&\qquad \geq (2x_{i'}+4)\beta h_n + (2x_{i'}+2)d + f_n + (x_{i'}+2)\beta h_n + (x_{i'}+1)d - c \\
&\qquad = \tfrac{3}{2}((2x_{i'}+4)\beta h_n + (2x_{i'}+2)d) - c + f_n.
\end{aligned}$$

Now consider when $i = 1$, i.e. $x_i \geq x$ for all i. Here, $B_{n+1}B_{n+1}$ has $B_n^{2\beta}\widehat{1}(B_n^{\beta}\widehat{1})^{y-1} B_n^{2\beta}\widehat{1}(B_n^{\beta}\widehat{1})^{x-1} B_n^{2\beta}$ as a subword and $B_{n+1}\widehat{1}$ has $(B_n^{\beta}\widehat{1})^{x_s} B_n^{2\beta}\widehat{1}(B_n^{\beta}\widehat{1})^{y-1} B_n^{\beta}\widehat{1}$ as a subword. As $x < y$ and $x \leq x_s$, this means $(B_n^{\beta}\widehat{1})^{x} B_n^{2\beta}\widehat{1}(B_n^{\beta}\widehat{1})^{x-1} B_n^{\beta} \in \mathcal{L}^{RS}$.

This gives at least $(x+1)\beta h_n + xd - c$ right-special words which are not suffixes of B_{n+1}, all of length less than $(2x+2)\beta h_n + 2xd$. Therefore,

$$\begin{aligned}
p((2x+2)\beta h_n + 2xd) &\geq (2x+2)\beta h_n + 2xd + f_n + (x+1)\beta h_n + xd - c \\
&= \tfrac{3}{2}((2x+2)\beta h_n + 2xd) + f_n - c.
\end{aligned}$$

□

4.4.1. *Proof of Proposition 4.13*

Proof of Proposition 4.13. By Lemmas 4.14, 4.15 and 4.16, we are left with the situation when $B_{n+1} = (B_n^{\beta}\widehat{1})^{L} B_n^{\beta}$ for some $L \geq 1$.

Since B_{n+2} has $B_{n+1}B_{n+1}$ as a suffix, and since Proposition 4.7 then covers the case when B_{n+2} has $B_{n+1}\widehat{1}$ as a prefix, we may assume that $B_{n+2} = B_{n+1}^{\alpha_{n+1}}\widehat{1}uB_{n+1}^{\beta_{n+1}}$ for some $\alpha_{n+1}, \beta_{n+1} \geq 2$, where u is either empty or ends with $\widehat{1}$. Lemmas 4.14 and 4.15 applied to $n+1$ mean we may assume $\alpha_{n+1} = \beta_{n+1}$.

As $B_{n+2}B_{n+2} \in \mathcal{L}$, the word $B_{n+1}^{2\beta_{n+1}}\widehat{1} \in \mathcal{L}$. Then, $B_{n+1}^{2\beta_{n+1}-1} \in \mathcal{L}^{RS}$. As B_{n+2} has $\widehat{1}B_n^{\beta_{n+1}}$ as a suffix, this gives at least $(\beta_{n+1}-1)h_{n+1} - c$ right-special words of length less than $(2\beta_{n+1}-1)h_{n+1}$ which are not suffixes of B_{n+2}.

As $B_{n+2}\widehat{1}B_{n+2} \in \mathcal{L}$ and B_{n+2} has $B_{n+1}B_{n+1}$ as a prefix, $B_{n+2}\widehat{1}B_{n+2}$ has $B_{n+1}\widehat{1}B_{n+1}B_n^{\beta}$ as a subword. Then, $B_{n+1}\widehat{1}B_{n+1}B_n^{\beta} = (B_n^{\beta}\widehat{1})^{2L+1}B_n^{2\beta} \in \mathcal{L}$. Therefore, $(B_n^{\beta}\widehat{1})^{2L}B_n^{\beta} \in \mathcal{L}^{RS}$. As B_{n+2} has $B_{n+1}B_{n+1}$ as a suffix and that word has $B_n^{2\beta}(\widehat{1}B_n^{\beta})^{L}$ as a suffix, this gives at least $(L\beta-1)h_n + Ld - c$ right-special words of length less than $(2L+1)(\beta h_n + d)$ which are not suffixes of B_{n+2}.

As $B_n^{2\beta}\widehat{1}$ is a subword of $B_{n+1}B_{n+1}$, this means $B_n^{2\beta-1} \in \mathcal{L}^{RS}$ which gives at least $(\beta-1)h_n - c$ right-special words of length less than $(2\beta-1)h_n$ which are not suffixes of B_{n+1}, and hence not of B_{n+2} as B_{n+1} has $\widehat{1}B_n^{\beta}$ as a suffix.

As none of these right-special words overlap with one another, the three cases above provide at least $(\beta_{n+1}-1)h_{n+1} + (L\beta-1)h_n + (\beta-1)h_n + Ld - 3c$ right-special words which are not suffixes of B_{n+2} all of length less than $(2\beta_{n+1}-1)h_{n+1}$.

Since $h_{n+1} = (L+1)\beta h_n + Ld$, we then have $(\beta_{n+1}-1)h_{n+1} + (L\beta-1+\beta-1)h_n + Ld - 3c = \beta_{n+1}h_{n+1} - 2h_n - 3c = \beta_{n+1}h_{n+1} - (2/((L+1)\beta))(h_{n+1} - Ld) - 3c$ extra

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right-special words of length at most $(2\beta_{n+1}-1)h_{n+1}$. Therefore, since $L \geq 1$ and $\beta \geq 2$ so $(2/((L+1)\beta)) \leq 2/4$, we have

$$\begin{aligned} p((2\beta_{n+1}-1)h_{n+1}) &\geq (2\beta_{n+1}-1)h_{n+1} + f_n + \beta_{n+1}h_{n+1} - \frac{2}{(L+1)\beta}h_{n+1} - 3c \\ &= \frac{3}{2}(2\beta_{n+1}-1)h_{n+1} + f_n + \left(\frac{1}{2} - \frac{2}{(L+1)\beta}\right)h_{n+1} - 3c \\ &\geq \frac{3}{2}(2\beta_{n+1}-1)h_{n+1} + f_n - 3c. \end{aligned}$$

□

4.5. *Words of the form* $B_n\widehat{1}$ —— $\widehat{1}B_n^2\widehat{1}$ —— $\widehat{1}B_n$ *with* B_n^3 *never appearing.* This section handles the most difficult case, when $a_{n,1} = a_{n,z_n} = 1$ and $a_{n,j} \leq 2$ for all j. This difficulty is likely unavoidable as this case contains the examples we exhibit which are near $1.5q$ in complexity.

PROPOSITION 4.17. *If, for infinitely many* $n \geq N$, *it holds that* $a_{n,1} = a_{n,z_n} = a_{n+1,z_{n+1}} = 1$ *and* $a_{n,j} \leq 2$ *for all* j *and* $a_{n,j} = 2$ *for at least one* j, *then for all sufficiently large* n, *there exists* $q_n \geq h_n$ *such that* $p(q_n) \geq 1.5q_n + f_n - 2c$.

Let $n \geq N$ such that $a_{n,1} = a_{n,z_n} = a_{n,z_{n+1}} = 1$ and $a_{n,j} \leq 2$ for all j, and $a_{n,j} = 2$ for at least one j. Then we may write

$$B_{n+1} = (B_n\widehat{1})^{\alpha}(B_n^2\widehat{1})^{\beta}u(B_n^2\widehat{1})^{\kappa}(B_n\widehat{1})^{\gamma-1}B_n$$

for some word u, which has prefix $B_n\widehat{1}$ and suffix $\widehat{1}B_n\widehat{1}$, and where $\alpha, \beta, \gamma, \kappa \geq 1$, or else u is empty and $\kappa = 0$ and $\alpha, \beta, \gamma \geq 1$. Then,

$$\begin{aligned} B_{n+1}B_{n+1} &= \text{———}\ B_n^2\widehat{1}(B_n\widehat{1})^{\gamma-1}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha-1}(B_n^2\widehat{1})^{\beta}B_n\widehat{1}\ \text{———}, \\ B_{n+1}\widehat{1}B_{n+1} &= \text{———}\ B_n^2\widehat{1}(B_n\widehat{1})^{\gamma+\alpha}(B_n^2\widehat{1})^{\beta}B_n\widehat{1}\ \text{———}. \end{aligned}$$

The proof of Proposition 4.17 is a series of lemmas.

LEMMA 4.18. *There are at least* $\alpha(h_n + d) - c$ *right-special words which are not suffixes of* B_{n+1} *and with length less than* $(\alpha + \gamma + 1)(h_n + d)$, *all of which do not contain* B_n^2 *as a subword.*

Proof. $B_n\widehat{1}(B_n\widehat{1})^{\gamma+\alpha}B_n^2$ is a subword of $B_{n+1}\widehat{1}B_{n+1}$ so $(B_n\widehat{1})^{\gamma+\alpha}B_n \in \mathcal{L}^{RS}$. Since B_{n+1} has suffix $B_n^2\widehat{1}(B_n\widehat{1})^{\gamma-1}B_n$, every suffix of $(B_n\widehat{1})^{\gamma+\alpha}B_n$ at least c longer than $(B_n\widehat{1})^{\gamma}B_n$ is not a suffix of B_{n+1}. □

LEMMA 4.19. *If* $\alpha = 1$ *and* $\gamma = 1$, *then there exists* $q_n \geq h_n$ *such that* $p(q_n) \geq 1.5q_n + f_n - 2c$.

Proof. First consider the case when $\kappa = 0$ and u is empty. Here, $B_{n+1} = (B_n\widehat{1})^{\alpha}(B_n^2\widehat{1})^{\beta}(B_n\widehat{1})^{\gamma-1}B_n = B_n\widehat{1}(B_n^2\widehat{1})^{\beta}B_n$. Since $a_{n+1,z_{n+1}} = 1$ and $a_{n+1,j} \geq 2$ for some j, we have $B_{n+1}B_{n+1}\widehat{1} \in \mathcal{L}$. Since $B_{n+1}B_{n+1}\widehat{1} = B_n\widehat{1}(B_n^2\widehat{1})^{2\beta+1}B_n\widehat{1}$, we then have $B_n\widehat{1}(B_n^2\widehat{1})^{2\beta}B_n \in \mathcal{L}^{RS}$.

Since B_{n+2} has $B_{n+1}\widehat{1}B_{n+1}$ as a suffix, it has $\widehat{1}B_n\widehat{1}B_n\widehat{1}(B_n^2\widehat{1})^{\beta}B_n$ as a suffix. This means our right-special word gives at least $(4\beta+2)h_n+(2\beta+1)d-(2\beta+2)h_n-(\beta+1)d-c=2\beta h_n+\beta d-c$ right-special words which are not suffixes of B_{n+2}, all of length less than $(2\beta+1)(2h_n+d)$. As Lemma 4.18 gives at least h_n+d-c additional right-special words which are not suffixes of B_{n+2} and do not contain B_n^2, we conclude that

$$\begin{aligned} p((2\beta+1)(2h_n+d)) &\geq (2\beta+1)(2h_n+d)+f_n+(2\beta+1)h_n+(\beta+1)d-2c \\ &= \tfrac{3}{2}(2\beta+1)(2h_n+d)+\tfrac{1}{2}d+f_n-2c. \end{aligned}$$

We now consider when $\kappa \geq 1$ and u is non-empty.

Here, $B_{n+1}B_{n+1} = $ ——— $B_n\widehat{1}(B_n^2\widehat{1})^{\kappa+1+\beta}B_n\widehat{1}$ ——— meaning that $B_n\widehat{1}(B_n^2\widehat{1})^{\beta+\kappa}B_n \in \mathcal{L}^{RS}$. As B_{n+1} has suffix $\widehat{1}B_n\widehat{1}(B_n^2\widehat{1})^{\kappa}B_n$, every suffix of our word of length at least $(2\kappa+2)h_n+(\kappa+1)d+c$ is not a suffix of B_{n+1}. So there are at least $2\beta h_n+\beta d-c$ right-special words of length less than $2(\kappa+\beta+1)h_n+(\kappa+\beta+1)d$ which are not suffixes of B_{n+1}.

Lemma 4.18 in this case also gives h_n+d-c right-special words of length less than $3(h_n+d)$ which are not suffixes of B_{n+1} and do not contain B_n^2. So,

$$\begin{aligned} &p(2(\kappa+\beta+1)h_n+(\beta+\kappa+1)d) \\ &\quad\geq 2(\kappa+\beta+1)h_n+(\beta+\kappa+1)d+f_n+(2\beta+1)h_n+(\beta+1)d-2c \\ &\quad= \tfrac{3}{2}(2(\kappa+\beta+1)h_n+(\beta+\kappa+1)d)+f_n+(\beta-\kappa)h_n+\tfrac{1}{2}(\beta-\kappa+1)d-2c, \end{aligned}$$

so if $\beta \geq \kappa$, then

$$p(2(\kappa+\beta+1)h_n+(\beta+\kappa+1)d) \geq \tfrac{3}{2}(2(\kappa+\beta+1)h_n+(\beta+\kappa+1)d)+f_n-2c.$$

So from here on, assume $\beta < \kappa$.

Observe that if $(B_n\widehat{1})^4B_n \in \mathcal{L}$, then necessarily $(B_n\widehat{1})^4B_n^2 \in \mathcal{L}$ as $\gamma = 1$, so $(B_n\widehat{1})^3B_n \in \mathcal{L}^{RS}$. As B_{n+1} has $B^2\widehat{1}B_n$ as a suffix, every suffix of our word of length at least $2h_n+d+c$ is not a suffix of B_{n+1}. This gives at least $2h_n+2d-c$ right-special words of length less than $4h_n+3d$ which are not suffixes of B_{n+1}. Then,

$$p(4h_n+3d) \geq 4h_n+3d+f_n+2h_n+2d-c = \tfrac{3}{2}(4h_n+3d)+f_n+\tfrac{1}{2}d-c.$$

So, from here on, we assume that $\widehat{1}B_n\widehat{1}B_n\widehat{1}B_n\widehat{1} \notin \mathcal{L}$.

Suppose that $B_n^2\widehat{1}B_n\widehat{1}B_n^2$ is a subword of B_{n+1}. Then, $B_n\widehat{1}B_n^2\widehat{1}B_n\widehat{1}B_n0 \in \mathcal{L}$ as the initial $B_n^2\widehat{1}$ is preceded by $B_n\widehat{1}$. Also, $B_{n+1}\widehat{1}B_{n+1}$ has the subword $B_n\widehat{1}B_n^2\widehat{1}B_n\widehat{1}B_n\widehat{1}$, where the next-to-last $\widehat{1}$ is the $\widehat{1}$ appearing between the B_{n+1} in $B_{n+1}\widehat{1}B_{n+1}$. Then, $B_n\widehat{1}B_n^2\widehat{1}B_n\widehat{1}B_n \in \mathcal{L}^{RS}$.

As B_{n+1} has $B_n^2\widehat{1}B_n$ as a suffix, our word gives at least $3h_n+2d-c$ right-special words which are not suffixes of B_{n+1}, all of length less than $5h_n+3d$. Therefore,

$$p(5h_n+3d) \geq 5h_n+3d+f_n+3h_n+2d-c = \tfrac{8}{5}(5h_n+3d)+\tfrac{1}{5}d+f_n-c.$$

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So, from here on, assume also that $B_n^2\hat{1}B_n\hat{1}B_n^2$ is not a subword of B_{n+1}. Therefore, we can write

$$B_{n+1} = B_n\hat{1}\bigg(\prod_{i=1}^{y}(B_n^2\hat{1})^{\beta_i}(B_n\hat{1})^2\bigg)(B_n^2\hat{1})^{\kappa}B_n$$

for some $y \geq 1$ (since u is non-empty) and $\beta_i \geq 1$ and $\beta_1 = \beta < \kappa$.

Suppose first that $\beta_y < \beta$. Set $m = \min\{\beta_i\}$. Take i such that $\beta_i = m$, then $\beta_i \leq \beta_y < \beta$ so $i > 1$. Then, $B_n\hat{1}(B_n^2\hat{1})^{\beta_{i-1}}(B_n\hat{1})^2(B_n^2\hat{1})^{\beta_i}B_n\hat{1} \in \mathcal{L}$. This means $B_n\hat{1}(B_n^2\hat{1})^m(B_n\hat{1})^2(B_n^2\hat{1})^m B_n\hat{1} \in \mathcal{L}$ as $m \leq \beta_{i-1}$.

$B_n\hat{1}(B_n^2\hat{1})^{\beta_y}(B_n\hat{1})^2(B_n^2\hat{1})^{\kappa}B_n$ is a suffix of B_{n+1}, so $B_n\hat{1}(B_n^2\hat{1})^m(B_n\hat{1})^2(B_n^2\hat{1})^m B_n^2 \in \mathcal{L}$ since $m \leq \beta_y$ and $m < \beta < \kappa$. Therefore, $B_n\hat{1}(B_n^2\hat{1})^m(B_n\hat{1})^2(B_n^2\hat{1})^m B_n \in \mathcal{L}^{RS}$.

This gives at least $(2m+2)h_n + (m+2)d - c$ right-special words which are not suffixes of B_{n+1}, all of length less than $(4m+4)h_n + (2m+3)d$. Therefore,

$$\begin{aligned} p((4m+4)h_n+(2m+3)d) &\geq (4m+4)h_n+(2m+3)d+f_n+(2m+2)h_n+(m+2)d-c \\ &= \tfrac{3}{2}((4m+4)h_n+(2m+3)d)+f_n+\tfrac{1}{2}d-c. \end{aligned}$$

So, we may assume that $\beta_y \geq \beta$.

$B_{n+1}\hat{1}B_{n+1}$ has the subword $(B_n^2\hat{1})^{\kappa}B_n\hat{1}B_n\hat{1}(B_n^2\hat{1})^{\beta}B_n\hat{1}$. As $\beta < \kappa$, $B_n\hat{1}(B_n^2\hat{1})^{\beta}B_n\hat{1}B_n\hat{1}(B_n^2\hat{1})^{\beta}B_n\hat{1} \in \mathcal{L}$.

$B_{n+1}B_{n+1}$ has the subword $B_n\hat{1}(B_n^2\hat{1})^{\beta_y}(B_n\hat{1})^2(B_n^2\hat{1})^{\kappa}B_nB_n$ which has $B_n\hat{1}(B_n^2\hat{1})^{\beta_y}(B_n\hat{1})^2(B_n^2\hat{1})^{\beta}B_nB_n$ as a subword.

As $\beta_y \geq \beta$, then $B_n\hat{1}(B_n^2\hat{1})^{\beta}(B_n\hat{1})^2(B_n^2\hat{1})^{\beta}B_n \in \mathcal{L}^{RS}$. Since $\beta < \kappa$, this gives at least $2(\beta+1)h_n + (\beta+2)d - c$ right-special words which are not suffixes of B_{n+1}, all of length less than $(4\beta+4)h_n + (2\beta+3)d$. Therefore,

$$\begin{aligned} p((4\beta+4)h_n+(2\beta+3)d) &\geq (4\beta+4)h_n+(2\beta+3)d+f_n+2(\beta+1)h_n+(\beta+2)d-c \\ &= \tfrac{3}{2}((4\beta+4)h_n+(2\beta+3)d)+f_n+\tfrac{1}{2}d-c. \qquad \square \end{aligned}$$

LEMMA 4.20. *If $\alpha = 1$ and $\gamma > 1$, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - c$.*

Proof. In this case, $B_{n+1}B_{n+1}$ contains the subword $B_n^2\hat{1}(B_n\hat{1})^{\gamma-1}(B_n^2\hat{1})^{\beta+1}B_n$ and $B_{n+1}\hat{1}B_{n+1}$ contains $B_n^2\hat{1}(B_n\hat{1})^{\gamma}(B_n^2\hat{1})^{\beta}B_n\hat{1}$. Therefore, $(B_n\hat{1})^{\gamma}(B_n^2\hat{1})^{\beta}B_n^2$ appears in $B_{n+1}B_{n+1}$ and $(B_n\hat{1})^{\gamma}(B_n^2\hat{1})^{\beta}B_n\hat{1}$ in $B_{n+1}1B_{n+1}$. As B_{n+1} has $\hat{1}B_n\hat{1}B_n$ as a suffix (as $\gamma > 1$), every suffix of $(B_n\hat{1})^{\gamma}(B_n^2\hat{1})^{\beta}B_n$ longer than $01^{c-1}B_n\hat{1}B_n$ is not a suffix of B_{n+1} and is right-special. This gives at least $(\gamma+2\beta-1)h_n + (\gamma+\beta-1)d - c$ right-special words which are not suffixes of B_{n+1} of length less than $(\gamma+2\beta+1)h_n + (\gamma+\beta)d$. Therefore, as $\gamma + 2\beta - 3 \geq 2 + 2 - 3$,

$$\begin{aligned} &p((\gamma+2\beta+1)h_n+(\gamma+\beta)d) \\ &\quad \geq (\gamma+2\beta+1)h_n+(\gamma+\beta)d+f_n+(\gamma+2\beta-1)h_n+(\gamma+\beta-1)d-c \\ &\quad = \tfrac{3}{2}((\gamma+2\beta+1)h_n+(\gamma+\beta)d)+\tfrac{1}{2}(\gamma+2\beta-3)h_n+\tfrac{1}{2}(\gamma+\beta-2)d+f_n-c \\ &\quad > \tfrac{3}{2}((\gamma+2\beta+1)h_n+(\gamma+\beta)d)+f_n-c. \qquad \square \end{aligned}$$

LEMMA 4.21. *If $\alpha > \gamma \geq 1$, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - c$.*

Proof. Lemma 4.18 states there are at least $\alpha(h_n + d) - c$ right-special words which are not suffixes of B_{n+1} all of length less than $(\alpha + \gamma + 1)(h_n + d)$. Since $\alpha \geq \gamma + 1$,

$$\begin{aligned} p((\alpha+\gamma+1)(h_n+d)) &\geq (\alpha+\gamma+1)(h_n+d) + f_n + \alpha(h_n+d) - c \\ &= \tfrac{3}{2}(\alpha+\gamma+1)(h_n+d) + \tfrac{1}{2}(\alpha-\gamma-1)(h_n+d) + f_n - c \\ &\geq \tfrac{3}{2}(\alpha+\gamma+1)(h_n+d) + f_n - c. \end{aligned}$$ □

LEMMA 4.22. *If $\alpha > 1$ and $\gamma > 1$ and $\beta > 1$, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - 2c$.*

Proof. The word $(B_n\widehat{1})^\gamma B_n^2\widehat{1}B_n\widehat{1}$ is a subword of $B_{n+1}B_{n+1}$ since $\alpha > 1$. The word $(B_n\widehat{1})^{\alpha+\gamma}B_n^2\widehat{1}B_n^2$ is a subword of $B_{n+1}\widehat{1}B_{n+1}$ since $\beta > 1$. Therefore $(B_n\widehat{1})^\gamma B_n^2\widehat{1}B_n \in \mathcal{L}^{RS}$.

Since $\gamma > 1$, B_{n+1} has suffix $\widehat{1}B_n\widehat{1}B_n$ so there are at least $(\gamma+1)h_n + \gamma d - c$ right-special words which are not suffixes of B_{n+1} with length less than $(\gamma+3)h_n + (\gamma+1)d$. By Lemma 4.18, there are at least $\alpha(h_n+d) - c$ right-special words which are not suffixes of B_{n+1} and with length less than $(\alpha+\gamma+1)(h_n+d)$ and all with suffix $\widehat{1}B_n\widehat{1}B_n$ and which do not contain B_n^2 so there is no overlap with the right-special words already identified.

Therefore there are at least $(\gamma+1+\alpha)h_n + (\gamma+\alpha)d - 2c$ right-special words which are not suffixes of B_{n+1} all of length less than $(\gamma+1+\alpha)h_n + (\gamma+\alpha)d$ (as $\alpha \geq 2$ implies $\gamma+1+\alpha \geq \gamma+3$). Then

$$\begin{aligned} p((\gamma+\alpha+1)h_n + (\gamma+\alpha)d) &\geq (\gamma+\alpha+1)h_n + (\gamma+\alpha)d \\ &\quad + f_n + (\gamma+1+\alpha)h_n + (\gamma+\alpha)d - 2c \\ &= 2((\gamma+\alpha+1)h_n + (\gamma+\alpha)d) + f_n - 2c. \end{aligned}$$ □

LEMMA 4.23. *If $\alpha > 1$ and $\gamma > 1$ and $B_n^2\widehat{1}B_n^2\widehat{1} \in \mathcal{L}$ then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - c$.*

Proof. If the word $(B_n^2\widehat{1})^2 \in \mathcal{L}$ then necessarily $B_n\widehat{1}(B_n^2\widehat{1})^2B_n\widehat{1} \in \mathcal{L}$ since somewhere to the right of $(B_n^2\widehat{1})^2$ in B_{n+1} must be $B_n\widehat{1}$ as $\gamma > 1$. Then $B_n\widehat{1}B_n^2\widehat{1}B_n \in \mathcal{L}^{RS}$ which gives at least $2h_n + d - c$ right-special words of length less than $4h_n + 2d$ which are not suffixes of B_{n+1}. Then

$$p(4h_n + 2d) \geq 4h_n + 2d + f_n + 2h_n + d - c = \tfrac{3}{2}(4h_n + 2d) + f_n - c.$$ □

4.5.1. *The $1 < \alpha \leq \gamma$ and $B_n^2\widehat{1}B_n^2 \notin \mathcal{L}$ case.* From here on, we assume $B_n^2\widehat{1}B_n^2 \notin \mathcal{L}$. Therefore we can write

$$B_{n+1} = \bigg(\prod_{t=1}^{L}((B_n\widehat{1})^{\alpha_t}B_n^2\widehat{1})\bigg)(B_n\widehat{1})^{\gamma-1}B_n$$

for some $\alpha_t \geq 1$ and $L \geq 1$ where $\alpha_1 = \alpha$ and we write $\alpha_{L+1} = \gamma - 1$.

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LEMMA 4.24. *If* $1 < \alpha \leq \gamma$ *and* $\alpha_t < \gamma - 1$ *for some* $t \geq 2$ *then there exists* $q_n \geq h_n$ *such that* $p(q_n) \geq 1.5q_n + f_n - c$.

Proof. Observe that $B_n\widehat{1}(B_n\widehat{1})^{\alpha_k}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_{k+1}}B_n^2 \in \mathcal{L}$ for all $1 \leq k \leq L$ since, in the case when $k > 1$, it is a subword of B_{n+1} and, in the case when $k = 1$, it is a subword of $B_n\widehat{1}B_{n+1}$ which is a subword of $B_{n+1}\widehat{1}B_{n+1}$.

If $\alpha_{t+1} < \alpha_{k+1}$ for some $1 \leq t, k \leq L$ then the word $B_n\widehat{1}(B_n\widehat{1})^{\alpha_k}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_{k+1}}B_n^2$ has the subword $B_n\widehat{1}(B_n\widehat{1})^{\alpha_k}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_{t+1}}B_n\widehat{1}$. As $B_n\widehat{1}(B_n\widehat{1})^{\alpha_t}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_{t+1}}B_n^2 \in \mathcal{L}$, this implies that the word $B_n\widehat{1}(B_n\widehat{1})^{\min(\alpha_t,\alpha_k)}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_{t+1}}B_n \in \mathcal{L}^{RS}$.

Since B_{n+1} ends in $B_n^2(\widehat{1}B_n)^{\gamma}$, if $\alpha_{t+1} < \gamma - 1$ then suffixes of our right-special word which are longer than $01^{c-1}B_n\widehat{1}(B_n\widehat{1})^{\alpha_{t+1}}B_n$ are not suffixes of B_{n+1}. This gives at least $(\min(\alpha_t, \alpha_k) + 2)(h_n + d) - d - c$ right-special words which are not suffixes of B_{n+1} which have length less than $(\min(\alpha_t, \alpha_k) + \alpha_{t+1} + 4)(h_n + d) - 2d$.

By hypothesis $\min\{\alpha_t : t \geq 2\} < \gamma - 1$. Let t such that $\alpha_{t+1} = \min\{\alpha_t : t \geq 2\}$. Then there exists k such that $\alpha_{t+1} < \alpha_{k+1}$ since the last α_k is followed by $\gamma - 1 > \alpha_{t+1}$. Write $m = \min(a_t, a_k)$. Then $a_{t+1} \leq m$ since it is chosen to be minimal. We then have, as $m - \alpha_{t+1} \geq 0$,

$$\begin{aligned} p((\alpha_{t+1} + m + 4)(h_n + d) - 2d) &\geq (\alpha_{t+1} + m + 4)(h_n + d) - 2d + f_n + (m + 2)(h_n + d) - d - c \\ &= \tfrac{3}{2}((\alpha_{t+1} + m + 4)(h_n + d) - 2d) + \tfrac{1}{2}(m - \alpha_{t+1})(h_n + d) + f_n - c \\ &\geq \tfrac{3}{2}((\alpha_{t+1} + m + 4)(h_n + d) - 2d) + f_n - c. \end{aligned}$$ □

LEMMA 4.25. *If* $1 < \alpha < \gamma$ *and* $\alpha_t \geq \gamma - 1$ *for all* $t \geq 2$ *then there exists* $q_n \geq h_n$ *such that* $p(q_n) \geq 1.5q_n + f_n - c$.

Proof. The word $B_{n+1}B_{n+1}$ contains $B_n(\widehat{1}B_n)^{\gamma-1}\widehat{1}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha-1}B_n^2$ as a subword, so, as $\alpha < \gamma$, the word $(B_n\widehat{1})^{\alpha}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha-1}B_n0 \in \mathcal{L}$.

In the case $L = 1$, the word $B_{n+1}\widehat{1} = (B_n\widehat{1})^{\alpha}B_n^2(\widehat{1}B_n)^{\gamma}\widehat{1}$, so $(B_n\widehat{1})^{\alpha}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha-1}B_n\widehat{1} \in \mathcal{L}$ since $\alpha < \gamma$. In the case when $L > 1$, since $\alpha_L \geq \gamma - 1 \geq \alpha$, the word $B_{n+1}\widehat{1}$ ends in $(B_n\widehat{1})^{\alpha_L}B_n^2(\widehat{1}B_n)^{\gamma}\widehat{1}$ which has $(B_n\widehat{1})^{\alpha}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha-1}B_n\widehat{1}$ as a subword.

So, $(B_n\widehat{1})^{\alpha}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha-1}B_n \in \mathcal{L}^{RS}$. As B_{n+1} has $(\widehat{1}B_n)^{\gamma}$ as a suffix, our word gives at least $(\alpha + 1)(h_n + d) - d - c$ right-special words which are not suffixes of B_{n+1} all of length less than $(2\alpha + 1)(h_n + d) - d$. Then,

$$\begin{aligned} p((2\alpha + 1)(h_n + d) - d) &\geq (2\alpha + 1)(h_n + d) - d + f_n + (\alpha + 1)(h_n + d) - d - c \\ &= \tfrac{3}{2}((2\alpha + 1)(h_n + d) - d) + \tfrac{1}{2}h_n + f_n - c. \end{aligned}$$ □

LEMMA 4.26. *If* $\alpha = \gamma > 1$ *and* $\alpha_t \geq \gamma - 1$ *for all* $t \geq 2$ *and for some* $t \geq 2$, $\alpha_t \geq \gamma$ *with* $\alpha_t \neq 2\gamma$, *then there exists* $q_n \geq h_n$ *such that* $p(q_n) \geq 1.5q_n + f_n - 2c$.

Proof. First consider the case when $\alpha_t > 2\gamma$. As $B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_t}B_n^2$ is a subword of B_{n+1} and $\alpha_t \geq 2\gamma + 1$, this means $(B_n\widehat{1})^{2\gamma+2}B_n^2 \in \mathcal{L}$. Then, $(B_n\widehat{1})^{2\gamma+1}B_n \in \mathcal{L}^{RS}$. Since B_{n+1} has $B_n^2\widehat{1}(B_n\widehat{1})^{\gamma-1}B_n$ as a suffix, there are at least $(\gamma + 1)h_n + (\gamma + 1)d - c$ right-special suffixes of our word all of length less than $(2\gamma + 2)h_n + (2\gamma + 1)d$. Then,

$$p((2\gamma+2)h_n+(2\gamma+1)d) \geq (2\gamma+2)h_n+(2\gamma+1)d+f_n+(\gamma+1)h_n+(\gamma+1)d-c$$
$$= \tfrac{3}{2}((2\gamma+2)h_n+(2\gamma+1)d)+f_n+\tfrac{1}{2}d-c.$$

Next consider when $\alpha_t < 2\gamma$. Then, $B_n\widehat{1}(B_n\widehat{1})^{\gamma-1}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_t}B_n^2$ is a subword of B_{n+1} since $\alpha_{t-1} \geq \gamma-1$. As the word $(B_n\widehat{1})^{\gamma}B_n^2\widehat{1}(B_n\widehat{1})^{2\gamma}$ is a subword of $B_{n+1}\widehat{1}B_{n+1}$, this means $(B_n\widehat{1})^{\gamma}B_n^2\widehat{1}(B_n\widehat{1})^{\alpha_t}B_n \in \mathcal{L}^{RS}$. Since $\alpha_t > \gamma-1$ and B_{n+1} has $B_n^2\widehat{1}(B_n\widehat{1})^{\gamma-1}B_n$ as a suffix, our word gives at least $(\gamma+2+\alpha_t+1-\gamma-1)h_n+(\gamma+\alpha_t-\gamma)d-c$ right-special suffixes which are not suffixes of B_{n+1}, all of length less than $(\gamma+\alpha_t+3)h_n+(\gamma+\alpha_t+1)d$. Then as $\alpha_t \geq \gamma$,

$$\begin{aligned}
&p((\gamma+\alpha_t+3)h_n+(\gamma+\alpha_t+1)d)\\
&\quad\geq (\gamma+\alpha_t+3)h_n+(\gamma+\alpha_t+1)d+f_n+(\alpha_t+2)h_n+(\alpha_t+1)d-2c\\
&\quad= \tfrac{3}{2}((\gamma+\alpha_t+3)h_n+(\gamma+\alpha_t+1)d)+\tfrac{1}{2}(\alpha_t-\gamma+1)h_n+\tfrac{1}{2}(\alpha_t-\gamma+1)d+f_n-2c\\
&\quad\geq \tfrac{3}{2}((\gamma+\alpha_t+3)h_n+(\gamma+\alpha_t+1)d)+f_n-2c. \qquad\square
\end{aligned}$$

LEMMA 4.27. *If $\alpha=\gamma>1$ and $\alpha_t \in \{\gamma-1, 2\gamma\}$ for all t and $\alpha_t = 2\gamma$ for some t, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n+f_n-2c$.*

Proof. Here, we can write

$$B_{n+1} = B_n\widehat{1}\bigg(\prod_{i=1}^{s}((B_n\widehat{1})^{\gamma-1}B_n^2\widehat{1})^{y_i}(B_n\widehat{1})^{2\gamma}B_n^2\widehat{1}\bigg)((B_n\widehat{1})^{\gamma-1}B_n^2\widehat{1})^{z}(B_n\widehat{1})^{\gamma-1}B_n$$

for some $s \geq 1$ and $y_i, z \geq 0$ and $y_1 \geq 1$ (as $\alpha=\gamma>1$). Rearranging the grouping and writing $D_n = (B_n\widehat{1})^{\gamma}B_n$,

$$\begin{aligned}
B_{n+1} &= B_n\widehat{1}\bigg(\prod_{i=1}^{s}((B_n\widehat{1})^{\gamma-1}B_n\ B_n\widehat{1})^{y_i}(B_n\widehat{1})^{2\gamma}B_n\ B_n\widehat{1}\bigg)((B_n\widehat{1})^{\gamma-1}B_n\ B_n\widehat{1})^{z}(B_n\widehat{1})^{\gamma-1}B_n\\
&= \bigg(\prod_{i=1}^{s}((B_n\widehat{1})^{\gamma}B_n)^{y_i}(B_n\widehat{1})^{2\gamma+1}B_n\bigg)((B_n\widehat{1})^{\gamma}B_n)^{z+1}\\
&= \bigg(\prod_{i=1}^{s}D_n^{y_i}D_n\widehat{1}D_n\bigg)D_n^{z+1} = D_n^{y_1+1}\widehat{1}\bigg(\prod_{i=2}^{s}D_n^{y_i+2}\widehat{1}\bigg)D_n^{z+2}.
\end{aligned}$$

Write $k_n = \ell en(D_n)$.

First consider when $y_1 > z$. Since $B_{n+1}B_{n+1}$ has the subword $D_n^{z+2}D_n^{y_1+1}\widehat{1}$ and D_n has 0 as a prefix (as B_n does), then $D_n^{y_1+z+2} \in \mathcal{L}^{RS}$. Since D_n has B_n as a suffix, this word disagrees with B_{n+1} on suffixes longer than $1^{c-1}D_n^{z+2}$. We then have at least y_1k_n-c right-special words of length less than $(y_1+z+2)k_n$ which are not suffixes of B_{n+1}.

Lemma 4.18 states there are at least $\gamma h_n+\gamma d = k_n-h_n$ right-special words of length less than $(2\gamma+1)h_n+2\gamma d$ which are not suffixes of B_{n+1} and which do not contain B_n^2 as a subword, and hence do not overlap with the words above.

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Then as $y_1 \geq z+1$ (and $k_n \geq 2h_n$ since $\gamma > 1$),

$$\begin{aligned} p((y_1+z+2)k_n) &\geq (y_1+z+2)k_n + f_n + y_1k_n - c + k_n - h_n - c \\ &= \tfrac{3}{2}(y_1+z+2)k_n + \tfrac{1}{2}(y_1-z)k_n - h_n + f_n - 2c \\ &\geq \tfrac{3}{2}(y_1+z+2)k_n + \tfrac{1}{2}k_n - h_n + f_n - 2c \geq \tfrac{3}{2}(y_1+z+2)k_n + f_n - 2c. \end{aligned}$$

Now consider the case when $y_i < z$ for some $1 \leq i \leq s$. Set $m = \min\{y_i : 1 \leq i \leq s\}$ so that $m < z$ and take i minimal such that y_i is minimal.

Since B_{n+1} has $D_n^{y_s+2}\widehat{1}D_n^{z+2}$ as a suffix, then $D_n^{m+2}\widehat{1}D_n^{z+2} \in \mathcal{L}$. When $i > 1$, as $D_n^{y_{i-1}+1}\widehat{1}D_n^{y_i+2}\widehat{1}$ is a subword of B_{n+1}, then $D_n^{m+2}\widehat{1}D_n^{m+2}\widehat{1} \in \mathcal{L}$ as $y_{i-1} \geq y_i + 1$ as i was taken minimal. Then, $D_n^{m+2}\widehat{1}D_n^{m+2} \in \mathcal{L}^{RS}$ as $m < z$. As this word disagrees with suffixes of B_{n+1} on words longer than $1^{c-1}D_n^{m+2}$, this gives at least $(m+2)k_n + d - c$ right-special words of length less than $2(m+2)k_n + d$ which are not suffixes of B_{n+1}. Then,

$$\begin{aligned} p(2(m+2)k_n + d) &\geq 2(m+2)k_n + d + f_n + (m+2)k_n + d - c \\ &= \tfrac{3}{2}(2(m+2)k_n + d) + \tfrac{1}{2}d + f_n - c. \end{aligned}$$

When $i = 1$, as B_{n+1} has $D_n^{y_s+2}\widehat{1}D_n^{z+2}$ as a suffix (or $D_n^{y_1+1}\widehat{1}D_n^{z+2}$ in the case $s = 1$), we have $D_n^{m+1}\widehat{1}D_n^{z+2} \in \mathcal{L}$. The word $B_{n+1}\widehat{1}B_{n+1}$ has the subword $D_n^{z+2}\widehat{1}D_n^{y_1+1}\widehat{1}$ which has $D_n^{m+1}\widehat{1}D_n^{m+1}\widehat{1}$ as a subword. As $m < z$, this means $D_n^{m+1}\widehat{1}D_n^{m+1} \in \mathcal{L}^{RS}$. This word disagrees with suffixes of B_{n+1} on words longer than $1^{c-1}D_n^{m+1}$, so there at least $(m+1)k_n + d - c$ right-special words of length less than $2(m+1)k_n + d$ which are not suffixes of B_{n+1}. Then,

$$\begin{aligned} p(2(m+1)k_n + d) &\geq 2(m+1)k_n + d + f_n + (m+1)k_n + d - c \\ &= \tfrac{3}{2}(2(m+1)k_n + d) + \tfrac{1}{2}d + f_n - c. \end{aligned}$$

From here on, assume that $y_i \geq z$ for all i. We are left with the case when $y_1 = z$.

Since $B_{n+1}\widehat{1}B_{n+1}$ has the subword $D_n^{z+2}\widehat{1}D_n^{y_1+1}\widehat{1} = D_n^{z+2}\widehat{1}D_n^{z+1}\widehat{1}$ (as $y_1 = z$) and B_{n+1} has suffix $D_n^{y_s+2}\widehat{1}D_n^{z+2}$ which has the subword $D_n^{z+2}\widehat{1}D_n^{z+1}D_n$ (as $y_s \geq z$), this gives $D_n^{z+2}\widehat{1}D_n^{z+1} \in \mathcal{L}^{RS}$. This word disagrees with suffixes of B_{n+1} on words longer than $1^{c-1}D_n^{z+1}$ meaning there are at least $(z+2)k_n + d - c$ right-special words of length less than $(2z+3)k_n + d$ which are not suffixes of B_{n+1}. Then,

$$\begin{aligned} &p((2z+3)k_n + d) \geq (2z+3)k_n + d + f_n + (z+2)k_n + d - c \\ &= \tfrac{3}{2}((2z+3)k_n + d) + \tfrac{1}{2}k_n + \tfrac{1}{2}d + f_n - c \geq \tfrac{3}{2}((2z+3)k_n + d) + f_n - c. \end{aligned}$$ □

LEMMA 4.28. *If $\alpha = \gamma > 1$ and $\alpha_t = \gamma - 1$ for all $t \geq 2$ and B_{n+2} has $B_{n+1}\widehat{1}B_{n+1}$ as a suffix, then there exists $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - 2c$.*

Proof. We are left with $B_{n+1} = (B_n\widehat{1})^{\gamma}B_n^2\widehat{1}((B_n\widehat{1})^{\gamma-1}B_n^2\widehat{1})^{L-1}(B_n\widehat{1})^{\gamma-1}B_n = ((B_n\widehat{1})^{\gamma}B_n)^{L+1}$.

Since $B_{n+1}B_{n+1}$ must occur somewhere in B_{n+2} and not as a suffix, B_{n+1} $B_{n+1}\widehat{1} \in \mathcal{L}$, and since $B_{n+1}B_{n+1}\widehat{1} = ((B_n\widehat{1})^{\gamma}B_n)^{2L+2}\widehat{1}$, we have $((B_n\widehat{1})^{\gamma}B_n)^{2L+1} \in \mathcal{L}^{RS}$.

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Since $h_{n+1} = (L+1)((\gamma+1)h_n + \gamma d)$, our right-special word has length

$$(2L+1)((\gamma+1)h_n + \gamma d) = 2h_{n+1} - ((\gamma+1)h_n + \gamma d).$$

Since B_{n+2} has $B_{n+1}\widehat{1}B_{n+1}$ as a suffix, this word disagrees with B_{n+2} on suffixes of length at least $h_{n+1} + c$. Therefore, there are at least $h_{n+1} - ((\gamma+1)h_n + \gamma d) - c$ right-special suffixes of our word which are not suffixes of B_{n+2}.

Lemma 4.18 states there are also at least $\gamma(h_n + d) - c$ right-special words which do not have B_n^2 as a subword, and hence do not overlap with those above nor with suffixes of B_{n+2}, all of length at most $(2\gamma+1)(h_n+d)$. Then as $\gamma > 1$,

$$\begin{aligned} &p(2h_{n+1} - ((\gamma+1)h_n + \gamma d)) \\ &\quad \geq 2h_{n+1} - ((\gamma+1)h_n + \gamma d) + f_n + h_{n+1} - ((\gamma+1)h_n + \gamma d) - c + \gamma(h_n + d) - c \\ &\quad = \tfrac{3}{2}(2h_{n+1} - ((\gamma+1)h_n + \gamma d)) + \tfrac{1}{2}(\gamma-1)h_n + \tfrac{1}{2}\gamma d + f_n - 2c \\ &\quad > \tfrac{3}{2}(2h_{n+1} - ((\gamma+1)h_n + \gamma d)) + f_n - 2c. \end{aligned}$$

□

4.5.2. *Proof of Proposition 4.17*

Proof of Proposition 4.17. Lemma 4.19 gives $q_n \geq h_n$ such that $p(q_n) \geq 1.5q_n + f_n - 2c$ when $\alpha = \gamma = 1$. Lemma 4.20 takes care of $\alpha = 1$ and $\gamma > 1$. When $\alpha > \gamma \geq 1$, Lemma 4.21 gives such a q_n.

We are left with the case when $\gamma \geq \alpha > 1$. Lemma 4.22 covers $\alpha, \gamma > 1$ and $\beta > 1$, so we proceed with $\beta = 1$. Lemma 4.23 covers the situation when $B_n^2\widehat{1}B_n^2\widehat{1} \in \mathcal{L}$, so we can assume that word does not appear from here on, so B_{n+1} is of the form written above Lemma 4.24. Lemma 4.24 handles when $\alpha_t < \gamma - 1$, so we may assume $\alpha_t \geq \gamma - 1$ for all t.

Lemma 4.25 then covers the case when $\alpha < \gamma$, so we may proceed with $\alpha = \gamma$. Then Lemma 4.26 shows that if $\alpha_t \geq \gamma$ with $\alpha_t \neq 2\gamma$ for some t, then we have such a q_n, so we may assume $\alpha_t \in \{\gamma - 1, 2\gamma\}$ for all t. Lemma 4.27 handles the case when $\alpha_t = 2\gamma$ for some t, so we can assume $\alpha_t = \gamma - 1$ for all t.

By hypothesis, $a_{n+1,z_{n+1}} = 1$, meaning that B_{n+2} ends with $B_{n+1}1B_{n+1}$. Lemma 4.28 then guarantees the existence of such a q_n.

There are then $q_n \geq h_n$ with $p(q_n) \geq 1.5q_n + f_n - 2c$ for infinitely many, and hence all sufficiently large n. □

4.6. *Proof of Theorem 4.2*

Proof of Theorem 4.2. Set $C = 3c$. Every $n \geq N$ satisfies one of: (1) $a_{n,1} = 1$ and $a_{n,z_n} \geq 2$; (2) $a_{n,1} \geq 2$ and $a_{n,z_n} = 1$; (3) $a_{n,1} = a_{n,z_n} = 1$ and $a_{n,j} \geq 3$ for some j; (4) $a_{n,1}, a_{n,z_n} \geq 2$; or (5) $a_{n,1} = a_{n,z_n} = 1$, $a_{n,j} \leq 2$ and $a_{n,j} = 2$ for some j. At least one of those cases happens infinitely often. For cases (1)–(3), Proposition 4.7 gives the result.

For case (4), by Proposition 3.10, there exists a rank-one subshift generating the same language such that $a_{n,1} \geq 2$ and $a_{n,z_n} \geq 2$, and $a_{n+1,z_{n+1}} \geq 2$ for infinitely many n. Proposition 4.13 applied to that subshift gives the claim.

If cases (1)–(4) all do not happen infinitely often, then for all sufficiently large n, we are in case (5) in which case Proposition 4.17 gives the claim. □

5. *Low complexity weakly mixing rank-one subshifts*

Consider the following class of rank-one subshifts.

Definition 5.1. Let $L_n > 1$ and $\gamma_n > 1$ for all n. Define the rank-one subshift with $B_1 = 0$ and

$$B_{n+1} = ((B_n 1)^{\gamma_n} B_n)^{L_n}.$$

Observe that $h_{n+1} = L_n((\gamma_n + 1)h_n + \gamma_n)$ and $B_{n+1}B_{n+1} = ((B_n 1)^{\gamma_n} B_n)^{2L_n}$ and

$$B_{n+1}1B_{n+1} = ((B_n 1)^{\gamma_n} B_n)^{L_n-1}(B_n 1)^{2\gamma_n+1} B_n((B_n 1)^{\gamma_n} B_n)^{L_n-1}.$$

5.1. *Right-special words*

LEMMA 5.2. *Let $w \in \mathcal{L}^{RS}$ with $1B_n$ as a suffix. Then w is a suffix of $(B_n 1)^{2\gamma_n} B_n$ or w is a suffix of B_{n+1} or w has B_{n+1} as a proper suffix.*

Proof. Observe that $1B_n$ is always preceded by B_n, so w shares a suffix with $B_n 1 B_n$.

First consider when w has $0B_n1B_n$ as a suffix. As B_n1 is always preceded by B_n or 1, in this case, w shares a suffix with $B_n^2 1 B_n$. Since $1B_n0$ only appears as a prefix of $1B_n^2$, having $w0 \in \mathcal{L}$ would then mean $B_n^2 1 B_n^2 \in \mathcal{L}$, but that word is not in $\mathcal{L}$ since $\gamma_n > 1$.

So w has $1B_n1B_n$ as a suffix (or else is a suffix of B_n1B_n which is a suffix of B_{n+1}) and therefore shares a suffix with $B_n1B_n1B_n$. Following the same logic, if w shares a suffix with $0(B_n1)^t B_n$, then $w0$ shares a suffix with $0(B_n1)^t B_n0$ which can only occur as a subword of $B_n^2 1(B_n1)^{t-1}B_n^2$, requiring that $t \geq \gamma_n$.

So, w shares a suffix with $B_n1(B_n1)^{\gamma_n-1}B_n = (B_n1)^{\gamma_n}B_n$. As B_n1 is always preceded by 1 or B_n, we have two cases to consider (if w is a suffix of $(B_n1)^{\gamma_n}B_n$, then it is a suffix of B_{n+1}).

First consider when w has $1(B_n1)^{\gamma_n}B_n$ as a suffix. The only occurrence of that word is in $B_{n+1}1B_{n+1}$ and it is always preceded by $(B_n1)^{\gamma_n}B_n$, so w must share a suffix with $(B_n1)^{2\gamma_n+1}B_n$. Since $1(B_n1)^{2\gamma_n}B_n1 \notin \mathcal{L}$ as $(B_n1)^{2\gamma_n+1}$ is always preceded by B_n (as $L_n > 1$) and since $w1 \in \mathcal{L}$, either w is a suffix of $(B_n1)^{2\gamma_n}B_n$ or w has $0(B_n1)^{2\gamma_n}B_n$ as a suffix. Since $0(B_n1)^{2\gamma_n}B_n0 \notin \mathcal{L}$ because $B_n(B_n1)^{2\gamma_n}B_nB_n = B_n^2\widehat{1}(B_n\widehat{1})^{2\gamma_n-1}B_n^2 \notin \mathcal{L}$ as $\gamma_n > 1$, it must be that w is a suffix of $(B_n1)^{2\gamma_n}B_n$.

Now consider when w has $0(B_n1)^{\gamma_n}B_n$ as a suffix. Then w shares a suffix with $B_n(B_n1)^{\gamma_n}B_n$. Since $(B_n1)^{\gamma_n}B_n$ is always preceded by $(B_n1)^{\gamma_n}B_n$ or 1, then w shares a suffix with $((B_n1)^{\gamma}B_n)^2$. Then $w1$ shares a suffix with $((B_n1)^{\gamma}B_n)^2 1$ and since $((B_n1)^{\gamma}B_n)^2 1$ is always a suffix of $B_{n+1}1$, this shows that w shares a suffix with B_{n+1}. Then either w is a suffix of B_{n+1} or w has B_{n+1} as a proper suffix. □

LEMMA 5.3. *Let $w \in \mathcal{L}^{RS}$ with $0B_n$ as a suffix. Then w is a suffix of $((B_{n-1}1)^{\gamma_{n-1}} B_{n-1})^{L_{n-1}-1}B_n$ and $n > 1$.*

Proof. Since $0B_1 = 00$ and $000 = B_1^3 \notin \mathcal{L}$, we have $n > 1$.

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Every occurrence of B_n appears either as $1B_n1$ or $1B_nB_n1$. The word $0B_n$ is not a subword of $1B_n1$ and occurs as a subword of $1B_nB_n1$ at $L_{n-1}+1$ distinct starting locations.

The word $0B_n1$ only appears as a suffix of $1B_nB_n1$ since it must appear somewhere in $1B_nB_n1$ and the only appearance of B_n1 in that word is as a suffix as $B_n1 = ((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}(B_{n-1}1)^{\gamma_{n-1}+1}$, and $(B_{n-1}1)^{\gamma_{n-1}+1}$ is not a subword of $B_nB_n = ((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{2L_{n-1}}$.

So, $w1$ shares a suffix with B_nB_n1 so w shares a suffix with B_nB_n. Since B_n0 must be a prefix of B_n^2 and $B_n^3 \notin \mathcal{L}$, then $B_nB_n0 \notin \mathcal{L}$. As $w0 \in \mathcal{L}$, w is then a proper suffix of B_nB_n.

Suppose w has $0((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n$ as a suffix. As that word only appears as a subword of B_nB_n when the leading 0 is the tail 0 of the first $(B_{n-1}1)^{\gamma_{n-1}}B_{n-1}$ in the first B_n of B_n^2, the word $0((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n0 \notin \mathcal{L}$ as $0((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n$ must be a suffix of B_nB_n and hence be followed by a 1. However, then $w0 \notin \mathcal{L}$.

Suppose that w has $1((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n$ as a suffix. As $1B_{n-1}$ is always preceded by B_{n-1}, then w would share a suffix with $B_{n-1}1((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n$ but that contains $(B_{n-1}1)^{\gamma_{n-1}+1}$ as a subword which is not a subword of $B_nB_n = (B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{2L_{n-1}+}$.

Therefore, w must be a suffix of $((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n$. □

PROPOSITION 5.4. *Let $w \in \mathcal{L}^{RS}$ with $\ell en(w) > 1$. Then there exists a unique n such that exactly one of the following holds (and for $m \neq n$, none of them hold):*

- *w is a suffix of B_{n+1} and $h_n < \ell en(w) \leq h_{n+1}$;*
- *w is a suffix of $(B_n1)^{2\gamma_n}B_n$ and $(\gamma_n+1)h_n + \gamma_n < \ell en(w) \leq (2\gamma_n+1)h_n + 2\gamma_n$;*
- *w is a suffix of $((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n$ and $h_n < \ell en(w) \leq h_n(2 - 1/L_{n-1})$ and $n > 1$.*

In all three cases, $h_n < \ell en(w) \leq h_{n+1}$.

Proof. As $11 \notin \mathcal{L}$, w must end in 0. Let n be the largest integer such that w has B_n as a proper suffix (such n exists since $B_1 = 0$). Then w has either $0B_n$ or $1B_n$ as a suffix.

Lemma 5.2 states that if w has $1B_n$ as a suffix, then either w is a suffix of $(B_n1)^{2\gamma_n}B_n$ or is a suffix of B_{n+1}, which are the second and first cases of the proposition, respectively, or else w has B_{n+1} as a proper suffix which would contradict the choice of n.

Lemma 5.3 states that if w has $0B_n$ as a suffix, then $n > 1$ and w is a suffix of $((B_{n-1}1)^{\gamma_{n-1}}B_n)^{L_{n-1}-1}B_n$. This puts us in the third case as $(\gamma_{n-1}+1)h_{n-1} + \gamma_{n-1} = (1/L_{n-1})h_n$.

Suffixes of $(B_n1)^{2\gamma_n}B_n$ of length less than or equal to $(\gamma_n+1)h_n + \gamma_n$ are suffixes of $(B_n1)^{\gamma_n}B_n$ which is a suffix of B_{n+1}, but all suffixes longer than that are not suffixes of B_{n+1} as B_{n+1} has $0(B_n1)^{\gamma_n}B_n$ as a suffix. Suffixes of $((B_{n-1}1)^{\gamma_{n-1}}B_{n-1})^{L_{n-1}-1}B_n$ of length at least h_n+1 have $0B_n$ as a suffix, so are not suffixes of B_{n+1} as B_{n+1} has $1B_n$ as a suffix. Clearly there is no overlap between the second and third cases as the second has $1B_n$ as a suffix and the third has $0B_n$ as a suffix. Therefore, the length restrictions make the cases a partition of $\mathcal{L}^{RS}$.

Since $(2 - 1/L_{n-1})h_n < (2\gamma_n+1)h_n + 2\gamma_n < 2((\gamma_n+1)h_n + \gamma_n) \leq L_n((\gamma_n+1)h_n + \gamma_n) = h_{n+1}$, in all three cases, $h_n < \ell en(w) \leq h_{n+1}$. □

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5.2. *The complexity function*

PROPOSITION 5.5. *The complexity function satisfies* $p(h_2+1)=h_2(1+(1/L_1))+1$ *and for* $q>h_2$*, choosing* n *to be the unique integer such that* $h_n<q\leq h_{n+1}$,

$$p(q+1)-p(q)=\begin{cases} 2 & \text{when } h_n<q\leq(2-(1/L_{n-1}))h_n,\\ 1 & \text{when } (2-(1/L_{n-1}))h_n<q\leq(\gamma_n+1)h_n+\gamma_n,\\ 2 & \text{when } (\gamma_n+1)h_n+\gamma_n<q\leq(2\gamma_n+1)h_n+2\gamma_n,\\ 1 & \text{when } (2\gamma_n+1)h_n+2\gamma_n<q\leq h_{n+1}.\end{cases}$$

Proof. In Proposition 5.4, there is no overlap among n since $h_n<\ell en(w)\leq h_{n+1}$ for all three cases.

Recall that $p(q+1)-p(q)=|\{w\in\mathcal{L}^{RS}:\ell en(w)=q\}|$.

Let q and n such that $h_n<q\leq h_{n+1}$. There is exactly one suffix of B_{n+1} of length q. There is a suffix of the second form in Proposition 5.4 of length q precisely when $(\gamma_n+1)h_n+\gamma_n<q\leq(2\gamma_n+1)h_n+2\gamma_n$. There is a suffix of the third form in Proposition 5.4 of length q precisely when $h_n<q\leq(2-(1/L_{n-1}))h_n$ and $n>1$.

For $1<q\leq h_2$, Proposition 5.4 applies with $n=1$ and the third case is vacuous. Then, $p(q+1)-p(q)=1$ for $1<q\leq(\gamma_1+1)h_1+\gamma_1=2\gamma_1+1$. For $2\gamma_1+1<q\leq(2\gamma_1+1)h_1+2\gamma_1=4\gamma_1+1$, we have $p(q+1)-p(q)=2$ and for $4\gamma_1+1<q\leq h_2$, $p(q+1)-p(q)=1$. Therefore, as $p(2)=3$ and $h_2=L_1((\gamma_1+1)h_1+\gamma_1)=L_1(2\gamma_1+1)$,

$$\begin{aligned}&p(h_2+1)\\&=p(2)+(p(2\gamma_1+1)-p(2))+(p(4\gamma_1+1)-p(2\gamma_1+1)+(p(h_2+1)-p(4\gamma_1+1))\\&=3+(2\gamma_1-1)+2(2\gamma_1)+(h_2-4\gamma_1)=h_2+2\gamma_1+2=h_2+\frac{1}{L_1}h_2+1.\end{aligned}$$

□

THEOREM 5.6. *The transformations in Definition 5.1 satisfy* $p(h_{n+1})=(1+1/L_n)h_{n+1}$. *If* $\gamma_n/h_n\to 0$*, then they also satisfy*

$$\liminf\frac{p(q)}{q}=1+\liminf\frac{1}{\max(L_{n-1},\gamma_n+1)},$$
$$\limsup\frac{p(q)}{q}=\frac{3}{2}+\limsup\frac{1}{4\min(L_{n-1},\gamma_n+1)-2}.$$

Proof. For $n\geq 2$, by Proposition 5.5,

$$p\bigg(\bigg(2-\frac{1}{L_{n-1}}\bigg)h_n+1\bigg)-p(h_n+1)=2\bigg(1-\frac{1}{L_{n-1}}\bigg)h_n,$$
$$p((\gamma_n+1)h_n+\gamma_n+1)-p\bigg(\bigg(2-\frac{1}{L_{n-1}}\bigg)h_n+1\bigg)=\bigg(\gamma_n-1+\frac{1}{L_{n-1}}\bigg)h_n+\gamma_n,$$
$$p((2\gamma_n+1)h_n+2\gamma_n+1)-p((\gamma_n+1)h_n+\gamma_n+1)=2(\gamma_nh_n+\gamma_n),$$
$$p(h_{n+1}+1)-p((2\gamma_n+1)h_n+2\gamma_n+1)=h_{n+1}-(2\gamma_n+1)h_n-2\gamma_n,$$

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and therefore

$$\begin{aligned}
p(h_{n+1}+1) - p(h_n+1)&\\
&= \left(2 - \frac{2}{L_{n-1}} + \gamma_n - 1 + \frac{1}{L_{n-1}} + 2\gamma_n - 2\gamma_n - 1\right)h_n + \gamma_n + 2\gamma_n - 2\gamma_n + h_{n+1}\\
&= h_{n+1} + \left(\gamma_n - \frac{1}{L_{n-1}}\right)h_n + \gamma_n = h_{n+1} + (\gamma_n+1)h_n + \gamma_n - h_n - \frac{1}{L_{n-1}}h_n\\
&= h_{n+1} + \frac{1}{L_n}h_{n+1} - h_n - \frac{1}{L_{n-1}}h_n,
\end{aligned}$$

which implies that

$$\begin{aligned}
p(h_{n+1}+1) &= p(h_2+1) + \sum_{m=2}^{n}(p(h_{m+1}+1) - p(h_m+1))\\
&= 1 + \left(1 + \frac{1}{L_1}\right)h_2 + \sum_{m=2}^{n}\left(\left(1+\frac{1}{L_m}\right)h_{m+1} - \left(1 + \frac{1}{L_{m-1}}\right)h_m\right)\\
&= 1 + \left(1 + \frac{1}{L_n}\right)h_{n+1}.
\end{aligned}$$

Since $p(h_{n+1}+1) - p(h_{n+1}) = 1$, then $p(h_{n+1}) = (1 + 1/L_n)h_{n+1}$.

Combining this with our initial observations,

$$p\left(\left(2 - \frac{1}{L_{n-1}}\right)h_n + 1\right) - 1 = \left(1 + \frac{1}{L_{n-1}}\right)h_n + 2\left(1 - \frac{1}{L_{n-1}}\right)h_n = \left(3 - \frac{1}{L_{n-1}}\right)h_n, \tag{†}$$

$$\begin{aligned}
p((\gamma_n+1)h_n + \gamma_n + 1) - 1 &= \left(3 - \frac{1}{L_{n-1}}\right)h_n + \left(\gamma_n - 1 + \frac{1}{L_{n-1}}\right)h_n + \gamma_n\\
&= (\gamma_n+2)h_n + \gamma_n,
\end{aligned}$$

$$p((2\gamma_n+1)h_n + 2\gamma_n + 1) - 1 = (\gamma_n+2)h_n + \gamma_n + 2\gamma_n h_n + 2\gamma_n = (3\gamma_n+2)h_n + 3\gamma_n, \tag{‡}$$

and so

$$\begin{aligned}
\frac{p(h_n)}{h_n} &= 1 + \frac{1}{L_{n-1}},\\
\frac{p((2 - 1/L_{n-1})h_n + 1) - 1}{(2 - 1/L_{n-1})h_n} &= \frac{3 - 1/L_{n-1}}{2 - 1/L_{n-1}} = \frac{3}{2} + \frac{1/2\,1/L_{n-1}}{2 - 1/L_{n-1}} = \frac{3}{2} + \frac{1}{4L_{n-1} - 2},\\
\frac{p((\gamma_n+1)h_n + \gamma_n + 1) - 1}{(\gamma_n+1)h_n + \gamma_n} &= \frac{(\gamma_n+2)h_n + \gamma_n}{(\gamma_n+1)h_n + \gamma_n} = 1 + \frac{1}{\gamma_n + 1 + \gamma_n/h_n},\\
\frac{p((2\gamma_n+1)h_n + 2\gamma_n + 1) - 1}{(2\gamma_n+1)h_n + 2\gamma_n} &= \frac{(3\gamma_n+2)h_n + 3\gamma_n}{(2\gamma_n+1)h_n + 2\gamma_n} = \frac{3}{2} + \frac{1/2}{2\gamma_n + 1 + 2\gamma_n/h_n}.
\end{aligned}$$

Now observe that, since $1 \le p(q+1) - p(q) \le 2$ for all q, the function $p(q)$ is increasing when $p(q+1) - p(q) = 2$ and decreasing when $p(q+1) - p(q) = 1$. Therefore, the lim inf and lim sup are attained along sequences of the four above-mentioned values.

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Provided $\gamma_n/h_n \to 0$, then

$$\liminf_q \frac{p(q)}{q} = \liminf_n \min\left(1 + \frac{1}{L_{n-1}}, 1 + \frac{1}{\gamma_n + 1}\right)$$

and

$$\limsup_q \frac{p(q)}{q} = \limsup_n \max\left(\frac{3}{2} + \frac{1}{4L_{n-1} - 2}, \frac{3}{2} + \frac{1}{4\gamma_n + 2}\right). \qquad \square$$

5.3. *Complexity nearing 1.5q*

THEOREM 5.7. *Let $\epsilon > 0$ and $f(q) \to \infty$. Then there exists $\gamma_n = \gamma > 1$ and $L_n \to \infty$ such that the transformation in Definition 5.1 satisfies*

$$\limsup \frac{p(q)}{q} < \frac{3}{2} + \epsilon \quad \textit{and} \quad p(h_n) < h_n + f(h_n).$$

Proof. Choose $\gamma > 1$ such that $1/(4\gamma + 2) < \epsilon$.

Given h_n, choose q_n such that for all $q \geq q_n$, we have $f(q) > (\gamma + 1)h_n + \gamma$. Then choose L_n such that $L_n((\gamma + 1)h_n + \gamma) \geq q_n$. Then by Theorem 5.6,

$$p(h_{n+1}) = \left(1 + \frac{1}{L_n}\right)h_{n+1} = h_{n+1} + (\gamma + 1)h_n + \gamma < h_{n+1} + f(q_n) \leq h_{n+1} + f(h_{n+1}).$$

Since $L_n \to \infty$, lim sup $p(q)/q = 3/2 + 1/(4\gamma + 2) < 3/2 + \epsilon$. $\square$

6. *Weak mixing for rank-one transformations*

THEOREM 6.1. *Let T be a rank-one transformation with bounded spacers (there exists k such that $s_{n,i} \leq k$ for all $0 \leq i < r_n$ and all n) and $\kappa > 0$ such that for all sufficiently large n,*

$$|\{s_{n,i} = 0 : 0 \leq i < r_n\}| \geq \kappa(r_n + 1) \quad \textit{and} \quad |\{s_{n,i} = 1 : 0 \leq i < r_n\}| \geq \kappa(r_n + 1).$$

Then T is weakly mixing on a finite measure space.

We adapt the proof that Chacon's transformation is weakly mixing from [**Sil08**].

LEMMA 6.2. [**Sil08**, Lemma 2.7.3] *For any measurable set A and $\epsilon > 0$, there exists N such that for all $n \geq N$, there exists $Q \subseteq \{0, \ldots, h_n - 1\}$ such that $\mu(A \triangle \bigcup_{q \in Q} I_{n,q}) < \epsilon$.*

LEMMA 6.3. [**Sil08**, Lemma 3.7.3] *For any positive measure set A and $\epsilon > 0$, there exists N such that for all $n \geq N$, there exists $0 \leq a < h_n$ such that $\mu(A \cap I_{n,a}) > (1 - \epsilon)\mu(I_{n,a})$.*

LEMMA 6.4. *Let I a level and A a measurable set such that $\mu(A \cap I) \geq (3/4)\mu(I)$. For any $0 < \delta < 1$, there exists N such that for all $n \geq N$, if $I = \bigsqcup_{q \in Q} I_{n,q}$ is the partition of I into sublevels in C_n, then $|\{q \in Q : \mu(A \cap I_{n,q}) \geq \delta\mu(I_{n,q})\}| \geq (1/2)|Q|$.*

Proof. Choose $\alpha > 0$ such that $\alpha < (1/4)(1 + (1/\delta))^{-1}$, so that $\alpha/\delta + \alpha + 1/4 < 1/2$. Let $A_1 = A \cap I$. By Lemma 6.2, there exists N such that for any $n \geq N$, there is

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$Q' \subseteq Q$ such that if we set $I' = \bigsqcup_{a\in Q'} I_{n,a}$, then $\mu(A_1 \triangle I') < \alpha\mu(I)$. Now observe that $\mu(I'\triangle I) \le \mu(I'\triangle A_1) + \mu(A_1\triangle I) < \alpha\mu(I) + (1/4)\mu(I)$.

Set $Q'' = \{a \in Q' : \mu(I_{n,a} \setminus A_1) < \delta\mu(I_{n,a})\}$ and $I'' = \bigsqcup_{a\in Q''} I_{n,a}$. Since $\delta\mu(I_{n,a}) \le \mu(I_{n,a} \setminus A_1)$ for $a \in Q' \setminus Q''$,

$$\delta\mu(I'\triangle I'') = \delta\mu(I' \setminus I'') = \sum_{a\in Q'\setminus Q''} \delta\mu(I_{n,a}) \le \sum_{a\in Q'\setminus Q''} \mu(I_{n,a} \setminus A_1) \le \mu(I' \setminus A_1) < \alpha\mu(I),$$

so $\mu(I''\triangle I) \le \mu(I''\triangle I') + \mu(I'\triangle I) < (\alpha/\delta)\mu(I) + (\alpha + (1/4))\mu(I) < (1/2)\mu(I)$. Then $\mu(I'' \cap I) \ge (1/2)\mu(I)$, which means $|Q''| \ge (1/2)|Q|$. □

LEMMA 6.5. *If T is on a finite measure space and there exists $\kappa > 0$ and $\{t_{n,\ell}\}$ such that for any two levels I and J in C_n, with J being ℓ levels below I, $\mu(T^{t_{n,\ell}} I \cap I) \ge \kappa^\ell\mu(I)$ and $\mu(T^{t_{n,\ell}} I \cap J) \ge \kappa^\ell\mu(J)$, then T is weakly mixing.*

Proof. Let A and B be any positive measure sets. By Lemma 6.3, there exist levels I_1 and J_1 in some column C_N such that $\mu(A \cap I_1) > (3/4)\mu(I_1)$ and $\mu(B \cap J_1) > (3/4)\mu(J_1)$. Let $0 \le \ell < h_N$ such that I_1 is ℓ levels above J_1 (interchanging the roles of A and B if necessary).

Set $\delta = \kappa^\ell/3$. By Lemma 6.4, there exists $n > N$ such that if $I_1 = \bigcup_{q\in Q_1} I_{n,q}$ and $J_1 = \bigcup_{q\in Q_2} I_{n,q}$, then $|\{q \in Q_1 : \mu(A \cap I_{n,q}) \ge (1-\delta)\mu(I_{n,q})\}| \ge (1/2)|Q_1|$ and $|\{q \in Q_2 : \mu(B \cap I_{n,q}) \ge (1-\delta)\mu(I_{n,q})\}| \ge (1/2)|Q_2|$. Since I_1 is ℓ levels above J_1, $q \in Q_1$ if and only if $q - \ell \in Q_2$ and $|Q_1| = |Q_2|$. Therefore,

$$|\{q \in Q_1 : \mu(A \cap I_{n,q}) < (1-\delta)\mu(I_{n,q}) \text{ or } \mu(B \cap I_{n,q-\ell}) < (1-\delta)\mu(I_{n,q})\}| < \tfrac{1}{2}|Q_1| + \tfrac{1}{2}|Q_2| = |Q_1|,$$

meaning there exists $q \in Q_1$ such that $I = I_{n,q}$ and $J = I_{n,q-\ell}$ satisfy $\mu(A \cap I) \ge (1-\delta)\mu(I)$ and $\mu(B \cap J) \ge (1-\delta)\mu(J)$.

By hypothesis, $\mu(T^{t_{n,\ell}} I \cap I) \ge \kappa^\ell\mu(I) = 3\delta\mu(I)$ and $\mu(T^{t_{n,\ell}} I \cap J) \ge \kappa^\ell\mu(J) = 3\delta\mu(I)$. Set $A_1 = A \cap I$ and $B_1 = B \cap J$, so that $\mu(I \setminus A_1) < \delta\mu(I)$ and $\mu(J \setminus B_1) < \delta\mu(I)$. Then

$$\mu(T^{t_{n,\ell}} A_1 \cap B_1) \ge \mu(T^{t_{n,\ell}} I \cap J) - \mu(I \setminus A_1) - \mu(J \setminus B_1) \ge 3\delta\mu(I) - \delta\mu(I) - \delta\mu(I) = \delta\mu(I) > 0$$

and similarly

$$\mu(T^{t_{n,\ell}} A_1 \cap A_1) \ge \mu(T^{t_{n,\ell}} I \cap I) - \mu(I \setminus A_1) - \mu(I \setminus A_1) \ge 3\delta\mu(I) - \delta\mu(I) - \delta\mu(I) = \delta\mu(I) > 0.$$

Hence, for all positive measure sets A and B, there exists t such that $\mu(T^t A \cap A) \ge \mu(T^t A_1 \cap A_1) > 0$ and $\mu(T^t A \cap B) > 0$, which is equivalent to weak mixing [**Fur81**]. □

LEMMA 6.6. *Let $\kappa > 0$ and $n \in \mathbb{N}$, and set $t_{n,\ell} = \sum_{t=0}^{\ell-1} h_{n+t}$. Assume*

$$|\{s_{n,i} = 0 : 0 \le i < r_n\}| \ge \kappa(r_n + 1) \quad \textit{and} \quad |\{s_{n,i} = 1 : 0 \le i < r_n\}| \ge \kappa(r_n + 1).$$

Let I and J be levels in C_n with J being ℓ levels below I. Then

$$\mu(T^{t_{n,\ell}} I \cap I) \ge \kappa^\ell \mu(I) \quad \textit{and} \quad \mu(T^{t_{n,\ell}} I \cap J) \ge \kappa^\ell \mu(J).$$

Proof. Write $I = I_{n,a}$ for some $0 \le a < h_n$. As $T^{h_n} I_{n,a} \supset \bigsqcup_{i<r_n : s_{n,i}=0} I_{n,a}^{[i+1]}$, applying this twice,

$$T^{h_{n+1}+h_n} I_{n,a} \supset \bigsqcup_{i_0 : s_{n,i_0}=0} T^{h_{n+1}} I_{n,a}^{[i+1]} \supset \bigsqcup_{i_0 : s_{n,i_0}=0} \quad \bigsqcup_{i_1 : s_{n+1,i_1}=0} I_{n,a}^{[i_0+1][i_1+1]},$$

where $I_{n,a}^{[i][j]}$ has the obvious meaning: it is the jth sublevel of the ith sublevel of $I_{n,a}$ meaning $I_{n,a}^{[i][j]}$ is a level in C_{n+2}. Continuing this process:

$$T^{\sum_{t=0}^{\ell-1} h_{n+t}} I_{n,a} \supset \bigsqcup_{i_0 : s_{n,i_0}=0} \quad \bigsqcup_{i_1 : s_{n+1,i_1}=0} \cdots \bigsqcup_{i_{\ell-1} : s_{n+\ell-1,i_{\ell-1}}=0} I_{n,a}^{[i_0+1][i_1+1]\cdots[i_{\ell-1}+1]}.$$

Therefore,

$$\begin{aligned}
\mu(T^{\sum_{t=0}^{\ell-1} h_{n+t}} I_{n,a} \cap I_{n,a}) &\ge \sum_{i_0 : s_{n,i_0}=0} \cdots \sum_{i_{\ell-1} : s_{n+\ell-1,i_{\ell-1}}=0} \mu(I_{n,a}^{[i_0+1]\cdots[i_{\ell-1}+1]}) \\
&\ge \bigg(\prod_{t=0}^{\ell-1} \kappa(r_{n+t} + 1)\bigg)\mu(I_{n+\ell,a}) \\
&= \bigg(\prod_{t=0}^{\ell-1} \kappa(r_{n+t} + 1)\bigg)\bigg(\prod_{t=0}^{\ell-1} \frac{1}{r_{n+t} + 1}\bigg)\mu(I_{n,a}) = \kappa^\ell \mu(I_{n,a}).
\end{aligned}$$

Similarly, $T^{h_n} I_{n,a} \supset \bigsqcup_{i<r_n : s_{n,i}=1} I_{n,a-1}^{[i+1]}$, so

$$T^{\sum_{t=0}^{\ell-1} h_{n+t}} I_{n,a} \supset \bigsqcup_{i_0 : s_{n,i_0}=1} \quad \bigsqcup_{i_1 : s_{n,i_1}=1} \cdots \bigsqcup_{i_{\ell-1} : s_{n+\ell-i,i_{\ell-1}}=1} I_{n,a-\ell}^{[i_0+1][i_1+1]\cdots[i_{\ell-1}+1]}.$$

As $J = I_{n,a-\ell}$, since J is ℓ levels below I in C_n,

$$\mu(T^{\sum_{t=0}^{\ell-1} h_{n+t}} I_{n,a} \cap J) \ge \sum_{i_0 : s_{n,i_0}=1} \cdots \sum_{i_{\ell-1} : s_{n+\ell-1,i_{\ell-1}}=1} \mu(I_{n,a-\ell}^{[i_0+1]\cdots[i_{\ell-1}+1]}) \ge \kappa^\ell \mu(I_{n,a-\ell}).$$

□

PROPOSITION 6.7. *Let T be a rank-one transformation. If there exists a constant k such that $s_{n,i} \le k$ for all $0 \le i \le r_n$ for all sufficiently large n, then T is on a finite measure space.*

Proof. Writing S_n for the spacers added above the nth column C_n, we have $\mu(S_n) = \sum_{i=0}^{r_n} s_{n,i}\mu(I_{n+1}) \le k(r_n + 1)\mu(I_{n+1}) = k\mu(I_n) = (k/h_n)\mu(C_n)$. Since $h_n \ge \prod_{j=1}^{n-1}(r_j + 1) \ge 2^{n-1}$, then $\mu(C_{n+1}) \le (1 + (k/2^{n-1}))\mu(C_n)$ so $\lim \mu(C_n) \le \mu(C_0) \prod_{n=0}^{\infty}(1 + (k/2^n)) < \infty$. □

Proof of Theorem 6.1. Lemmas 6.5 and 6.6 and Proposition 6.7. □

6.1. *Weak mixing for low complexity transformations*

COROLLARY 6.8. *The subshifts in Definition 5.1 are weakly mixing (on finite measure spaces) provided that* $\limsup \gamma_n < \infty$.

Proof. Since $B_{n+1} = ((B_n 1)^{\gamma_n} B_n)^{L_n}$, we have $|\{0 \le i < r_n : s_{n,i} = 0\}| = L_n - 1$ and $|\{0 \le i < r_n : s_{n,i} = 1\}| = L_n \gamma_n$. As $r_n + 1 = L_n(\gamma_n + 1)$, this means

$$\frac{|\{0 \le i < r_n : s_{n,i} = 0\}|}{r_n + 1} = \frac{L_n - 1}{L_n(\gamma_n + 1)} \ge \frac{1}{\gamma_n + 1} \frac{1}{2}.$$

Likewise, $|\{i : s_{n,i} = 1\}|/(r_n + 1) = \gamma_n/(\gamma_n + 1)$. As γ_n is bounded, Theorem 6.1 gives weak mixing. □

THEOREM 6.9. *For every* $\epsilon > 0$*, there exists a weakly mixing rank-one transformation (on a probability space) such that the associated subshift has complexity* $\limsup p(q)/q < 1.5 + \epsilon$.

For any $f(q) \to \infty$*, the subshifts can be made to satisfy* $p(q) < q + f(q)$ *infinitely often.*

Proof. Corollary 6.8 and Theorem 5.7. □

THEOREM 6.10. *For every* $\epsilon > 0$*, there exists a subshift with complexity satisfying* $\limsup p(q)/q < 1.5 + \epsilon$ *and* $\liminf p(q)/q < 1 + \epsilon$ *such that the associated rank-one transformation is weakly mixing (on a probability space) and has minimal self-joinings (hence also has trivial centralizer and is mildly mixing).*

Proof. For $\epsilon > 0$, let $\gamma > 1$ such that $1/(\gamma + 1) < \epsilon$. Then the transformation in Definition 5.1 with $\gamma_n = \gamma$ and $L_n = \gamma + 1$ satisfies, by Theorem 5.6,

$$\liminf \frac{p(q)}{q} = 1 + \frac{1}{\gamma + 1} < 1 + \epsilon \qquad \text{and} \qquad \limsup \frac{p(q)}{q} = \frac{3}{2} + \frac{1}{4\gamma - 2} < \frac{3}{2} + \epsilon,$$

and Corollary 6.8 gives weak mixing. As $\{r_n\}$ is bounded, Ryzhikov's theorem [**Ryz13**] gives minimal self-joinings (the transformations are non-rigid since the $s_{n,i}$ are not constant over $0 \le i < r_n$, and hence are not 'flat' in the sense of [**Ryz13**, Theorem 2]). □

Remark 6.11. The examples with $p(q) < q + f(q)$ such that $L_n \to \infty$ are most likely not mildly mixing, and hence do not have minimal self-joinings. In essence, any alternative construction of those examples (where $f(q)/q \to 0$ so $L_n \to \infty$) which has bounded spacers necessarily involves constructing $B'_{n+1} = ((B_n 1)^{\gamma} B_n)^{\ell_n}$ with ℓ_n uniformly bounded followed by $B_{n+1} = (B'_{n+1})^{L_n/\ell_n}$. As the second step involves adding no spacers, the construction is 'flat' and therefore should admit a rigid factor.

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6.2. *Totally ergodic subshifts with* $\limsup p(q)/q = 1.5$

THEOREM 6.12. *For any $f(q) \to \infty$, there exists a totally ergodic rank-one subshift (on a probability space) satisfying $p(q) < 1.5q + f(q)$ for all sufficiently large q and $p(h_n) < h_n + f(h_n)$ for all $n \geq 2$.*

Proof. Let $f^*(q) = \inf\{f(q') : q' \geq q\}$. Then $f^*(q)$ is non-decreasing and $f^*(q) \to \infty$.

Set $\gamma_1 = L_1 = 2$. Given γ_{n-1} and L_{n-1} (and therefore h_n), choose γ_n such that $(1/2)h_n < f^*(\gamma_n)$. Then choose $L_n = m_n!$ for some $m_n > n$ such that $(\gamma_n + 1)h_n + \gamma_n < f^*(L_n)$.

As $h_{n+1} = L_n((\gamma_n + 1)h_n + \gamma_n)$, we then have $(1/L_n)h_{n+1} < f^*(L_n) \leq f^*(h_{n+1})$. Theorem 5.6 gives that

$$p(h_n) = \left(1 + \frac{1}{L_{n-1}}\right)h_n < h_n + f^*(h_n) \leq h_n + f(h_n).$$

The count (†) in the proof of Theorem 5.6 gives that

$$\begin{aligned} p\left(\left(2 - \frac{1}{L_{n-1}}\right)h_n + 1\right) &= \left(3 - \frac{1}{L_{n-1}}\right)h_n + 1 = \frac{3}{2}\left(2 - \frac{1}{L_{n-1}}\right)h_n + \frac{1}{2}\frac{1}{L_{n-1}}h_n + 1 \\ &\leq \frac{3}{2}\left(2 - \frac{1}{L_{n-1}}\right)h_n + \frac{1}{2}f^*(h_n) + 1 \\ &< \frac{3}{2}\left(\left(2 - \frac{1}{L_{n-1}}\right)h_n + 1\right) + f^*\left(\left(2 - \frac{1}{L_{n-1}}\right)h_n + 1\right) \end{aligned}$$

and the count (‡) in the proof of Theorem 5.6 gives

$$\begin{aligned} p((2\gamma_n + 1)h_n + 2\gamma_n + 1) &= (3\gamma_n + 2)h_n + 3\gamma_n + 1 = \frac{3}{2}((2\gamma_n + 1)h_n + 2\gamma_n) + \frac{1}{2}h_n + 1 \\ &\leq \frac{3}{2}((2\gamma_n + 1)h_n + 2\gamma_n) + f^*(\gamma_n) + 1 \\ &< \frac{3}{2}((2\gamma_n + 1)h_n + 2\gamma_n + 1) + f^*((2\gamma_n + 1)h_n + 2\gamma_n + 1). \end{aligned}$$

As $p(q) - 1.5q$ is maximized at one of these two lengths in each range $h_n < q \leq h_{n+1}$, for all $q > h_2$,

$$p(q) < 1.5q + f^*(q) \leq 1.5q + f(q).$$

It remains to show total ergodicity (as Proposition 6.7 puts it on a finite measure space).

Let A be a positive measure set and $t \in \mathbb{N}$ such that $T^t A = A$. For $\epsilon > 0$ and $n > t$ such that $2t/(\gamma_n + 1) < \epsilon$, define the sets

$$Q_n(\epsilon) = \{0 \leq j < h_n : \mu(I_{n,j} \cap A) > (1 - \epsilon)\mu(I_{n,j})\}.$$

If for some fixed $\epsilon > 0$, it holds that $Q_{n,\epsilon} = \emptyset$ for infinitely many n, then $\mu(A) = 0$ (Lemma 6.3), so we can also define $j_n(\epsilon) = \min\{j \in Q_n(\epsilon)\}$ for sufficiently large n.

Observe that for $j \geq t$,

$$\mu(I_{n,j-t} \cap A) = \mu(T^{-t} I_{n,j} \cap A) = \mu(I_{n,j} \cap T^t A) = \mu(I_{n,j} \cap A)$$

and so if $j \in Q_n(\epsilon)$ with $j \geq t$, then $j - t \in Q_n(\epsilon)$. Therefore, $j_n(\epsilon) < t$. Now observe that, for $j > 0$,

$$\mu(T^{h_n} I_{n,j} \cap I_{n,j-1}) \geq \sum_{i<r_n : s_{n,i}=1} \mu(I_{n,j-1}^{[i]}) = \frac{|\{0 \leq i < r_n : s_{n,i} = 1\}|}{r_n + 1} \mu(I_{n,j})$$
$$= \left(1 - \frac{1}{\gamma_n + 1}\right)\mu(I_{n,j})$$

since $s_{n,i} = 1$ for $L_n\gamma_n$ values of i and $r_n + 1 = L_n(\gamma_n + 1)$. Then, for $1 \leq s < t$ and $j \geq s$,

$$\mu(T^{sh_n} I_{n,j} \triangle I_{n,j-s}) \leq \sum_{u=0}^{s-1} \mu(T^{(s-u)h_n} I_{n,j-u} \triangle T^{(s-u-1)h_n} I_{n,j-u-1})$$
$$= \sum_{u=0}^{s-1} \mu(T^{h_n} I_{n,j-u} \triangle I_{n,j-u-1}) < \frac{2s}{\gamma_n + 1}\mu(I_{n,j})$$

and, therefore,

$$\mu(T^{sh_n} I_{n,j} \cap I_{n,j-s}) \geq \left(1 - \frac{2s}{\gamma_n + 1}\right)\mu(I_{n,j}) \geq \left(1 - \frac{2t}{\gamma_n + 1}\right)\mu(I_{n,j}) > (1-\epsilon)\mu(I_{n,j}).$$

Since $L_{n-1} = m_{n-1}!$ and L_{n-1} divides $h_n = L_{n-1}((\gamma_{n-1} + 1)h_{n-1} + \gamma_{n-1})$ and $m_n > n > t$, we have that t divides h_n so $T^{sh_n} A = A$. Then for $1 \leq s < t$ and $0 \leq j < h_n - s$ with $j \in Q_\epsilon(n)$,

$$\mu(I_{n,j+s} \cap A) = \mu(T^{sh_n}(I_{n,j+s} \cap A)) = \mu(T^{sh_n} I_{n,j+s} \cap A)$$
$$\geq \mu(T^{sh_n} I_{n,j+s} \cap I_{n,j} \cap A) \geq \mu(T^{sh_n} I_{n,j+s} \cap I_{n,j}) - \mu(I_{n,j} \setminus A)$$
$$> (1 - \epsilon)\mu(I_{n,j}) - \epsilon\mu(I_{n,j}),$$

meaning that if $j \in Q_n(\epsilon)$ with $j < h_n - s$, then $j + s \in Q_n(2\epsilon)$.

Since $j \in Q_n(\epsilon)$ implies $j \in Q_n(2\epsilon)$, this means that $j_n(\epsilon) + kt + s \in Q_n(2\epsilon)$ for all $k \geq 0$ and $0 \leq s < t$ such that $j_n(\epsilon) + kt + s < h_n$. So $Q_n(2\epsilon)$ contains all $j_n(\epsilon) \leq j < h_n$. Then $|Q_n(2\epsilon)| \geq h_n - t$, so

$$\mu(A) \geq \sum_{j \in Q_n(2\epsilon)} \mu(A \cap I_{n,j}) > |Q_n(2\epsilon)|(1 - 2\epsilon)\mu(I_{n,j})$$
$$\geq \left(1 - \frac{t}{h_n}\right)(1 - 2\epsilon)\mu(C_n) \to 1 - 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $\mu(A) = 1$. □

Remark 6.13. Our proof of weak mixing does not apply when γ_n is unbounded and we strongly suspect our transformations with $\gamma_n \to \infty$ are not weakly mixing.

7. *Attaining specific complexities*

We conclude with a brief discussion of the main open question.

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Question 7.1. For what pairs of values $1 \leq \alpha \leq \beta < 2$ does there exist a weakly mixing (rank-one or not) subshift with $\liminf p(q)/q = \alpha$ and $\limsup p(q)/q = \beta$?

Obviously, the most interesting question is whether there exists a weakly mixing subshift, necessarily not rank-one, with $\beta < 1.5$. We tentatively conjecture that our examples are the best possible.

Conjecture 7.2. Every subshift admitting a weakly mixing (probability) measure has complexity such that $\limsup p(q)/q > 1.5$.

Heinis [Hei02] showed that $\beta \geq 3 - 2/\alpha$ for every subshift with $\limsup p(q)/q < 2$. Our work shows that $\beta \geq 1.5$ is necessary for total ergodicity in the rank-one setting.

The values $\alpha = 1$ and $\beta = 1/(4\gamma + 2)$ for $\gamma \in \mathbb{N}$, $\gamma \geq 2$, are attained by our examples as they have complexity satisfying $\liminf p(q)/q = 1$ provided $L_n \to \infty$, and $\limsup p(q)/q = 1.5 + 1/(4\gamma + 2)$.

Ferenczi [Fer95] showed that the weakly mixing rank-one subshift given by $B_{n+1} = B_n^2 1 B_n^2$ has $\alpha = 1.5$ and $\beta = 5/3$ (this is the example that was the previously known lowest complexity).

Our examples can be adapted to attain more pairs: for all $2 \leq m < M$, by setting $\gamma = M - 1$ and $L = m$, Theorem 5.6 gives a weakly mixing subshift such that

$$\liminf \frac{p(q)}{q} = 1 + \frac{1}{M} \quad \text{and} \quad \limsup \frac{p(q)}{q} = \frac{3}{2} + \frac{1}{4m - 2}.$$

Since $M \geq 3$ and $m \geq 2$, all of these examples satisfy $\alpha \leq 4/3$ and $\beta \leq 5/3$.

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