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FINITE CLONES CONTAINING ALL PERMUTATIONS

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ABSTRACT. Let A be a finite set with $|A| > 2$. We describe all clones on A containing the set S_A of all permutations of A among its unary operations. (A clone on A is a composition closed set of finitary operations on A containing all projections). With a few exceptions such a clone C is either essentially unary or cellular *i.e.* there exists a monoid M of self-maps of A containing S_A such that either $C = \bar{M}$ (= all essentially unary operations agreeing with some $f \in M$) or $C = \bar{M} \cup \Gamma_h$ where $1 < h \leq |A|$ and Γ_h consists of all finitary operations on A taking at most h values. The exceptions are subclones of Burle's clone or of its variant (provided $|A|$ is even).

1. **Introduction.** Let A be a finite non-empty set. Without loss of generality we shall assume that $A = \mathbf{k} := \{0, 1, \ldots, k-1\}$. For a positive integer n an n-ary operation on $\mathbf{k}$ is a map $f: \mathbf{k}^n \longrightarrow \mathbf{k}$. The set of all n-ary operations on $\mathbf{k}$ is denoted $O^{(n)}$. Put $O := \bigcup_{n=1}^{\infty} O^{(n)}$. A clone on $\mathbf{k}$ is a composition-closed subset of O containing all the projections or, equivalently, the set of all term operations of an algebra on $\mathbf{k}$ (for a more precise definition *cf.* 2.0 below). A clone is thus a multivariable analogy of a transformation monoid or a permutation group on $\mathbf{k}$ whereby the projections play the role of $\mathrm{id}_{\mathbf{k}}$. The clones on $\mathbf{k}$, ordered by $\subseteq$, form an algebraic lattice $\underset{\sim}{L}_k$. The meet of an arbitrary set of clones on $\mathbf{k}$ is their intersection. For $F \subseteq O$ denote $\bar{F}$ the least clone containing F.

Already in 1941 E. Post [Po 41] completely described $\underset{\sim}{L}_2$. Note that $\underset{\sim}{L}_2$ is the lattice of clones of boolean (or switching or truth functions and so pertains to the propositional logic, electrical circuits and discrete optimization). The lattice $\underset{\sim}{L}_2$ is countably infinite and quite exceptional among the lattices $\underset{\sim}{L}_k$ and their variants (the lattices of clones of partial operations, multioperations or delayed operations). Indeed, $|\underset{\sim}{L}_k| = 2^{\aleph_0}$ for $k > 2$ [Ja-Mu 59]; this has been recently refined by exhibiting an interval of $\underset{\sim}{L}_k$ order isomorphic to the boolean lattice $(P(\mathbb{N}), \subseteq)$ of all subsets of $\mathbb{N} := \{0, 1, \ldots\}$ [Ha-Ro 86, 88, 88a] and so *e.g.* $\underset{\sim}{L}_k$ contains a chain order isomorphic to the set $\mathbb{R}$ (of the reals) and an antichain of size $2^{\aleph_0}$. The lattices $\underset{\sim}{L}_k$ are in general unknown and so on the whole the efforts have been concentrated on special parts of $\underset{\sim}{L}_k$, mostly the top (all coatoms or dual atoms are known, *cf.* [Ja 58], [Ro 65, 70]), some clones covered by coatoms [La 82] and all such clones for $k = 3$ [La 82a], or the bottom (some atoms are known for $k > 3$ and all atoms for $k = 3$ [Cs 83]).

The *foundation* of a clone C is the set $C^{(1)} := C \cap O^{(1)}$ of its unary operations. Clearly $C^{(1)}$ is a submonoid of the (full) symmetric semigroup $\underset{\sim}{U} := \langle O^{(1)}; \circ, \mathrm{id}_{\mathbf{k}} \rangle$.

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The foundation may carry a lot of information about C. For example, the foundations were used as the main tool in the classification of clones of boolean functions [Po 41].

P. P. Pálfy completely described the clones whose foundation consists of permutations or constants [Pa 84], a result which provided a starting point for the tame congruence theory [Ho-Mck 88].

At the 1988 Ames conference on Algebraic Logic and Universal Algebra in Theoretical Computer Science, S. Comer asked us about the characterization of clones C on $\mathbf{k}$ whose foundation contains all permutations on $\mathbf{k}$. This problem belongs to the area arising from Slupecki's remarkable 1939 result [Sl 39] which may be formulated as follows. For $k > 2$ the only maximal clone (= coatom of $\underset{\sim}{L}_{\mathbf{k}}$) with foundation $O^{(1)}$ is the Slupecki clone M_{k-1} of all essentially unary operations or non-surjective operations (*i.e.* missing at least one value from $\mathbf{k}$). This result has been improved. Call $B \subseteq O^{(1)}$ *basic* if the Slupecki clone M_{k-1} is the only maximal clone whose foundation contains B. It is known that the symmetric group S_k of all permutations on $\mathbf{k}$ is basic [Sa 60] Theorem 11.1, [Sa 60a]. The alternating group A_k and $O^{(1)} \setminus S_k$ are also basic [Sa 62]; [Ia 58]. A characterization of basic sets is in [Ro 70a]. (For $k = 2$ the analog of the Slupecki clone is the clone of all linear (mod 2) operations. We mention in passing that for $|A| = \aleph_0$ there are exactly two maximal clones with foundation $O^{(1)}$ and each clone with foundation $O^{(1)}$ extends to one of them [Ga 64,64a,65] but the situation seems to be much more complex for $|A| > \aleph_0$ [Da-Ro 85]. Moreover, for any clone C we may ask the same question: What are the clones covered by C in $\underset{\sim}{L}_A$ with foundation $C^{(1)}$?)

A. I. Mal'tsev improved Slupecki's result [Ma A 67] as follows. For $0 < h < k$ let M_h consist of all operations f that are essentially unary or with $|\operatorname{im} f| \leq h$ (*e.g.* M_1 is the clone $\overline{O^{(1)}}$ of all essentially unary operations while M_{k-1} is the above Slupecki clone). Then $M_2 \subset M_3 \subset \cdots \subset M_{k-1} \subset O$ is the unique increasing maximal (*i.e.* unrefinable) chain in $\underset{\sim}{L}_{\mathbf{k}}$ starting from M_2. Burle [Bu 67] showed that

$$M_1 \subset B' \subset M_2 \subset \cdots \subset M_{k-1} \subset O$$

where $\{M_1, B', M_2, \ldots, M_{k-1}, O\}$ is the interval of all clones with foundation $O^{(1)}$, $B' := M_1 \cup B$ and B is the following set of all quasilinear operations on $\mathbf{k}$. Call $f \in O^{(n)}$ *quasilinear* if there are $\phi_0 \colon \mathbf{2} \longrightarrow \mathbf{k}$ and $\phi_i \colon \mathbf{k} \longrightarrow \mathbf{2}$ $(i = 1, \ldots, n)$ such that

$$f(x_1, \ldots, x_n) = \phi_0(\phi_1(x_1)) \dotplus \cdots \dotplus \phi_n(x_n)) \tag{1.1}$$

holds for all $x_1, \ldots, x_n \in \mathbf{k}$ where $\dotplus$ denotes the sum mod 2 on $\mathbf{2}$. The clone B' is a maximal *TC* or abelian clone [Be-McK 84].

We determine the clones whose foundation contains $S_{\mathbf{k}}$. They can be described as follows. For $h = 1, \ldots, k-1$ set

$$\Gamma_h := \{f \in O : |\operatorname{im} f| \leq h\}, V := \{f \in O^{(1)} : |\operatorname{im} f| \leq 2\},$$

and let V_e consist of all $f \in$V such that $|f^{-1}(a)|$ is even for all $a \in \mathbf{k}$ (notice that V_e is nonempty only for k even and then consists of the constant maps and those f with $\ker f$ having two blocks of even size). Finally denote by B_e the set of all quasilinear operations having a representation (1.1) with all $\phi_1, \ldots, \phi_n \in V_e$. Our main result is:

THEOREM. *Let $k > 2$, $\mathbf{k} := \{0, \ldots, k-1\}$ and C be an essential clone containing the set S_k of all permutations of* $\mathbf{k}$. *Then either*

(i) *there exists a submonoid M of $\langle O^{(1)}; \circ, \mathrm{id}_{\mathbf{k}} \rangle$ containing S_k such that*

a) $C = \bar{M} \cup \Gamma_i$ *for some* $2 \leq i < k$ *or*

b) $C = \bar{M} \cup B$ *or*

(ii) *k is even and* $C = \bar{S}_k \cup B_e$.

Denote by $\underset{\sim}{V}$ the set of all $\bar{M}$ such that M is a submonoid of $\langle O^{(1)}; \circ, \mathrm{id}_{\mathbf{k}} \rangle$ containing $S_k \cup V$. The set $\underset{\sim}{V}$ is described in Lemma 2.2 in terms of number-theoretical partitions of k (corresponding to $\ker f$ for $f \in M$). The diagram of the interval $[\bar{S}_k, O]$ of $\underset{\sim}{L}_{\mathbf{k}}$ is on Figure 1 for k odd and on Figure 2 for k even. Its main part is the direct product of the chain $\overline{S_k \cup V} \subset \overline{S_k} \cup B \subset \overline{S_k} \cup \Gamma_2 \subset \cdots \overline{S_k} \cup \Gamma_{k-1}$ and the lattice $(\underset{\sim}{V}, \subseteq)$. For k even we just insert $\overline{S_k \cup V_e}$ and $\bar{S}_k \cup B_e$ near the bottom.

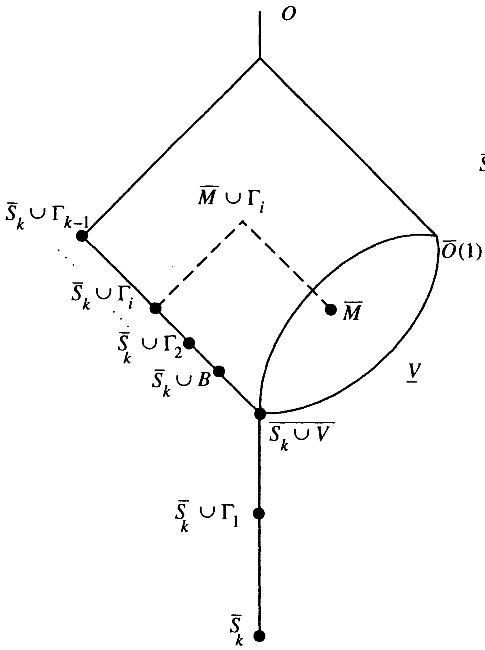

FIGURE 1 (k odd)

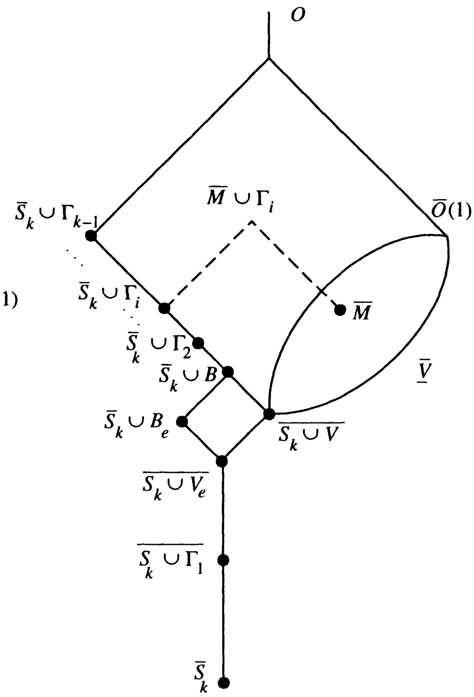

FIGURE 2 (k even)

The elementary proof is essentially combinatorial and based on the techniques from [Ma A67].

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2. **Preliminaries.**

2.0. For $1 \leq i \leq n$ the i-th n-ary projection e_i^n is defined by $e_i^n(x_1,\ldots,x_n) \approx x_i$. (Here and in the sequel the symbol means that both sides are equal for all $x_1,\ldots, x_n \in \mathbf{k}$).

The following definition of a clone, due essentially to Mal'tsev [Ma A 66], is based on a monoid $*$ on O and three unary operations ζ, τ and Δ on O. First we define a binary operation $*$ on O. For $f \in O^{(n)}, g \in O^{(m)}$ and $r := m+n-1$ define $f * g \in O^{(r)}$ by

$$(f * g)(x_1,\ldots,x_r) :\approx f\big(g(x_1,\ldots,x_m), x_{m+1},\ldots,x_r\big).$$

It is easy to see that $\langle O; *, e_1^1\rangle$ is a monoid (*i.e.* $*$ is associative and $e_1^1 * f = f * e_1^1 = f$ for all $f \in O$ where e_1^1 is the identity selfmap of $\mathbf{k}$). For $n > 1$ define $\zeta f \in O^{(n)}, \tau f \in O^{(n)}$ and $\Delta f \in O^{(n-1)}$ by

$$(\zeta f)(x_1,\ldots,x_n) \approx f(x_2,x_3,\ldots,x_n,x_1),$$
$$(\tau f)(x_1,\ldots,x_n) \approx f(x_2,x_1,x_3,\ldots,x_n),$$
$$(\Delta f)(x_1,\ldots,x_{n-1}) \approx f(x_1,x_1,x_2,\ldots,x_{n-1}),$$

while for $n = 1$ put $\zeta f = \tau f = \Delta f := f$.

The algebra $\underset{\sim}{P}_{\mathbf{k}} := \langle O; *, \zeta, \tau, \Delta, e_1^2\rangle$ (where e_1^2—the first binary projection—is a nullary operation, *i.e.* a distinguished element) is called *Mal'tsev's postiterative algebra on O*. A *clone* on $\mathbf{k}$ is a subuniverse of $\underset{\sim}{P}_{\mathbf{k}}$, *i.e.* a submonoid C of $\langle O, *\rangle$ containing the binary projection e_1^2 and satisfying $\zeta(C) \subseteq C, \tau(C) \subseteq C$ and $\Delta(C) \subseteq C$. It is known (*cf.* [Ma A 66]) that clones coincide with the sets of term operations of universal algebras on $\mathbf{k}$.

2.1. Let E_k denote the set of all equivalence relations on $\mathbf{k}$ and S_k the set of all permutations of $\mathbf{k}$. For $\varepsilon \in E_k$ and $\pi \in S_k$ set

$$\varepsilon^{(\pi)} := \big\{(x,y) \in \mathbf{k}^2 : \big(\pi(x),\pi(y)\big) \in \varepsilon\big\}.$$

Call a subset T of E_k *symmetric* if $T^{(\pi)} \subseteq T$ for all $\pi \in S_k$ (*i.e.* if $\varepsilon^{(\pi)} \in T$ whenever $\varepsilon \in T$ and $\pi \in S_k$). Consider $\varepsilon \in E_k$. Order the blocks (*i.e.* equivalence classes) $B_1,\ldots,B_\ell$ of ε so that $b_j := |B_j|$ $(j = 1,\ldots,\ell)$ satisfy $b_1 \geq \cdots \geq b_\ell$. Clearly $\varepsilon^\# := (b_1,\ldots,b_\ell)$ is a partition of k (*i.e.* an integer sequence $(b_1,\ldots,b_\ell)$ such that $b_1 \geq \cdots \geq b_\ell > 0$ and $b_1 + \cdots + b_\ell = k$). Denote P_k the set of all partitions of k and for $\beta_1,\beta_2 \in P_k$ put $\beta_1 \preceq \beta_2$ if $\beta_i = \varepsilon_i^\#$ $(i = 1,2)$ where $\varepsilon_1 \subseteq \varepsilon_2$ (here the inclusion is between binary relations and means that each block of ε is included in a block of ε_2). Clearly $(P_k, \preceq)$ is an ordered set. As usual, an up-set (or order filter) in an ordered set $(P, \leq)$ is a subset Q of P such that $\beta \in Q$ whenever $\beta \geq \gamma$ for some $\gamma \in Q$. For a map $f\colon \mathbf{k} \longrightarrow B$ put $\ker f := \{(a,a') \in \mathbf{k}^2 : f(a) = f(a')\}$. For $T \subseteq E_k$ put

$$M_T := \{f \in O^{(1)} : \ker f \in T\},$$

and for a subset P of P_k put

$$Q_P := \{f \in O^{(1)} : (\ker f)^\# \in P\}.$$

Denote by ω the least element $\{(x,x) : x \in \mathbf{k}\}$ of E_k. We need the following easy and most likely known:

LEMMA 2.2. *The following are equivalent for a subset S of $O^{(1)}$:*

(i) S is a subsemigroup of the symmetric semigroup $\langle O^{(1)}, \circ\rangle$ containing S_k.

(ii) $S = M_T$ for a symmetric subset T of E_k such that $\omega \in T$ and $T \setminus \{\omega\}$ is an up-set of $(E_k, \subseteq)$.

(iii) $S = Q_P$ for a set P of partitions of k such that $\underline{1} := (1, 1, \ldots, 1) \in P$ and $P \setminus \{\underline{1}\}$ is an up-set of $(P_k, \preceq)$.

PROOF. (i) $\Rightarrow$ (ii). Put $T := \{\ker f : f \in S\}$. Let $f \in S$ and $\pi \in S_k$. Then $\pi \in S_k \subseteq S$ and so $g := f \circ \pi \in S$. Put $\theta := \ker f$ and $\tau := \ker g$. Now for all $x, y \in E_k$

$$(x, y) \in \tau \Leftrightarrow f\big(\pi(x)\big) = f\big(\pi(y)\big) \Leftrightarrow (x, y) \in \theta^{(\pi)}.$$

Thus $\tau = \theta^{(\pi)}$ and T is symmetric. Clearly $\omega \in T$ due to $e_1 \in S_k \subseteq S$. Let $h \in O^{(1)}$ be such that $\ker h = \theta$. Then there exists $\ell \in S_k$ such that $h = \ell \circ f$ and hence $h \in S$ proving $M = M_T$. It remains to prove that $T' := T \setminus \{\omega\}$ is an up-set. Let $\theta \in T'$ and let $B_1, \ldots, B_\ell$ be the blocks of θ. Without loss of generality we may assume that $b = |B_1| > 1$ and $B_1 = \{0, \ldots, b-1\}$. Further for $i = 1, \ldots, \ell$ denote by b_i the least element of B_i (in the natural order on $\mathbf{k}$, *e.g.* $b_1 = 0$). Let $1 \leq i < j \leq \ell$ and let θ' be obtained from θ by fusing the blocks B_i and B_j. Define $f \in O^{(1)}$ as follows:

a) Put $f(x) := 0$ for every $x \in B_i$ and $f(x) := 1$ for every $x \in B_j$, b) $f(x) := b_i$ for every $x \in B_1$ provided $i > 1$ and c) $f(x) = b_m$ for all $m \in \{2, \ldots, \ell\} \setminus \{i, j\}$ and every $x \in B_m$. Clearly $\ker f = \theta$ and so $f \in M$. Finally let $g \in O^{(1)}$ be defined by setting $g(x) := b_m$ for all $m \in \{1, \ldots, \ell\}$ and every $x \in B_m$. Again $\ker g = \theta$ and so $g \in M$. Consider $h := g \circ f$. For $x \in B_i$ we have $h(x) = g(0) = b_1$ and for $x \in B_j$ we have $h(x) = g(1) = b_1$, hence $g(x) = 0$ for all $x \in B_i \cup B_j$. If $i > 1$ then for all $x \in B_1$ we have $h(x) = g(b_i) = b_i \neq 0$ and for $m \in \{2, \ldots, \ell\} \setminus \{i, j\}$ and $x \in B_m$ we have $h(x) = g(b_m) = b_m$. Since all the values $b_0, \ldots, b_{j-1}, b_{j+1}, \ldots, b_\ell$ are distinct, we have $\ker h = \theta'$. In view of $h \in M$ we have $\theta' \in T$ as required. If $i = 1$ then for $m \in \{2, \ldots, \ell\} \setminus \{j\}$ and $x \in B_m$ we have $f(x) = b_m$ and $h(x) = g\big(f(x)\big) = g(b_m) = b_m$. Again $\ker h = \theta'$ and so $\theta' \in T$.

(ii) $\Rightarrow$ (iii). Evident.

(iii) $\Rightarrow$ (i). Clearly $S_k \subseteq Q_P$. Let $f, g \in Q_P$. Put $\phi := \ker f; \delta := \ker g$ and $h := f \circ g$. If $\delta^\# = \underline{1}$ (*i.e.* $g \in S_k$), then $(\ker h)^\# = \phi^\#$ and so $h \in Q_P$. Thus let $\delta^\# \neq \underline{1}$. In view of $\delta \subseteq \ker h$, we have that $(\ker h)^\# \in P$ (because $P \setminus \{\underline{1}\}$ is an up-set) and so $h \in Q_P$. ■

2.3. An n-ary operation f on $\mathbf{k}$ depends on its i-th variable (or the i-th variable is essential) if

$$f(a_1, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n)$$

for some $a_1, \ldots, a_n, b_i \in \mathbf{k}$. If f does not depend on its i-th variable, the i-th variable is *fictitious* (also called *non-essential* or *dummy*). The operation f is *essential* if it depends on at least two variables. Clearly f depends at most on its i-th variable if $f(x_1, \ldots, x_n) \approx g(x_i)$ for some $g \in O^{(1)}$. A clone C is *unary* if all its operations depend on at most one variable. The essentially unary clones containing S_k are of the form

$$\bar{S} = \{s * e_i^n : s \in S, 1 \leq i \leq n\}$$

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where $S \subseteq O^{(1)}$ satisfies the conditions of Lemma 2.2.

Consider a non unary clone C with $C \supseteq S_k$. It will turn out that the maximum size of $\operatorname{im} f$ (*i.e.* the maximum number of values f takes) of essential operations $f \in C$ determines the nonunary part of C. Following [Ma I 73] for $1 \leq h \leq k$ the h-cell is the set

$$\Gamma_h := \{f \in O : |\operatorname{im} f| \leq h\}$$

of all at most h-valued operations on $\mathbf{k}$; *e.g.* Γ_1 is the set of all constant operations on $\mathbf{k}$ while $\Gamma_k = O$. A clone C on $\mathbf{k}$ is *cellular* if $C = \bar{S} \cup \Gamma_h$ for some $1 \leq h \leq k$ and $S \subseteq O^{(1)}$. The following lemma describes the cellular clones containing S_k.

For $h = 1, \ldots, k$ put

$$U_h := \{f \in O^{(1)} : |\operatorname{im} f| \leq h\}$$

(*e.g.* U_1 is the set of all constant selfmaps of $\mathbf{k}$ while $U_k = O^{(1)}$).

LEMMA 2.4. *A clone C on $\mathbf{k}$ containing S_k is cellular if and only if $C = \bar{Q}_P \cup \Gamma_h$ where $1 \leq h \leq k$ and P is an up-set of $(P_k, \preceq)$ consisting of all $(b_1, \ldots, b_\ell) \in P_k$ with either $1 \leq \ell \leq h$ or $\ell = k$.*

PROOF. $(\Rightarrow)$. Let $C = \bar{S} \cup \Gamma_h$ for some $1 \leq h \leq k$ and $S_k \subseteq S \subseteq O^{(1)}$ and let $C \supseteq S_k$. Note that Γ_h contains the set U_h . We may assume that S is a subsemigroup of $\langle O^{(1)}; \circ \rangle$ containing $S_k \cup U_h$. Now it suffices to apply Lemma 2.2.

$(\Leftarrow)$. Let C satisfy the condition. We must show that C is a clone. It is easy to see that $\zeta C = \tau C = C$, $\Delta C \subseteq C$ and $e_1^2 \in Q_P \subseteq C$. Let $f, g \in C$. 1) Let $f \in \overline{Q_P}$. If $g \in \overline{Q_P}$ then $f * g \in \overline{Q_P} \subseteq C$. Thus let $g \in \Gamma_h$. Then $|\operatorname{im}(f * g)| \leq |\operatorname{im} g| \leq h$ proves $f * g \in \Gamma_h \subseteq C$. 2) Let $f \in \Gamma_h$. From $|\operatorname{im}(f * g)| \leq |\operatorname{im} f| \leq h$ we get $f * g \in \Gamma_h \subseteq C$. ∎

REMARK 2.5. It is easy to see that for a cellular clone on $\mathbf{k}$ containing S_k the up-set P and integer h from Lemma 2.4 are unique.

2.6. Our aim is to show that the clones on $\mathbf{k}$ containing S_k are (i) unary, (ii) cellular and (iii) the Burle's clone and, if k is even, a particular subclone of it. Note that the clones from (i) are fully described in Lemma 2.2, the clones from (ii) in Lemma 2.4 and the two clones listed in (iii) will be discussed in §4. To prove this claim it suffices to consider the clone $C := \overline{\{f\} \cup S_k}$ for an arbitrary essential n-ary operation f on $\mathbf{k}$. Set $\ell := |\operatorname{im} f|$. For $2 < \ell \leq k$ we show that C is cellular. For $\ell = 2$ we obtain a cellular clone, Burle's clone or its particular subclone (if k is even). The key is the following Iablonskii's basic lemma [Ia 58], in Mal'tsev's formulation [Ma A 67] §2, (it has another part, due to Salomaa, which is not needed here). For $f, g \in O^{(n)}$ call g an *isomer* of f if there is a permutation π of $\{1, \ldots, n\}$ such that $g(x_1, \ldots, x_n) \approx f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

LEMMA 2.7. *Let $f \in O^{(n)}$ be essential and $\ell := |\operatorname{im} f| > 2$. Then there are $p_1, \ldots, p_n$, $q_1, \ldots, q_n \in \mathbf{k}$ and an isomer g of f such that*

$$g(p_1, \ldots, p_n) = a_0, \quad g(q_1, p_2, \ldots, p_n) = a_1, \quad g(p_1, q_2, \ldots, q_n) = a_2 \tag{2.1}$$

where $\operatorname{im} f = \{a_0, \ldots, a_{\ell-1}\}$. ∎

For the key Theorem 2.9 we need the following statement whose proof and that of Theorem 2.9 are essentially taken from [Ma A 67] §3.

LEMMA 2.8. *If $h \in O^{(2)}$ satisfies*

$$h(0,0) = 0, \quad h(1,0) = 1, \quad h(0,1) = h(1,1) \tag{2.2}$$

then $G := \overline{\{h\} \cup U_2 \cup S_k}$ contains Γ_2.

PROOF. We show that there exists a binary operation $h' \in G$ whose restriction to $\mathbf{2}$ is the disjunction (*i.e.* $h'(a,b) = \max(a,b)$ for all $a, b \in \mathbf{2}$). Set $a := h(0,1)$. 1) Suppose $a > 0$. Define $m \in O^{(1)}$ by setting $m(0) := 0$) and $m(x) := 1$ otherwise. Clearly $m \in U_2 \subseteq G$; hence $h'(x,y) :\approx m\big(h(x,y)\big)$ belongs to G and $\vee$ is the restriction of h' to $\mathbf{2}$. 2) Thus let $a = 0$. Define $n \in O^{(1)}$ by setting $n(0) := 1$ and $n(x) := 0$ otherwise. Again $n \in G$ and a direct verification shows that $h'(x,y) :\approx n\Big(h\big(n(x), y\big)\Big) \in G$ and $h'|\mathbf{2} = \vee$. Clearly $n|\mathbf{2}$ is the usual negation $'$. It is well known that the algebra $\langle \mathbf{2}; \vee, ' \rangle$ is primal (*i.e.* complete) and so every boolean function $b \colon \mathbf{2}^n \longrightarrow \mathbf{2}$ extends to some $b^* \in G$ (*i.e.* b^* agrees with b on $\mathbf{2}^n$). Now let $c \in O^{(m)}$ satisfy $\operatorname{im} c \subseteq \mathbf{2}$. Define the following elements of $\mathbf{2}^k$:

$$a(0) := (1,0,\dots,0), \quad a(1) := (0,1,0,\dots,0), \dots, a(k-1) := (0,\dots,0,1).$$

Moreover let $d \colon \mathbf{2}^{mk} \longrightarrow \mathbf{2}$ be defined by $d\big(a(x_1),\dots,a(x_m)\big) := c(x_1,\dots,x_m)$ for all $x_1,\dots,x_m \in \mathbf{k}$ and $d(b_1,\dots,b_{mk}) := 0$ otherwise. As observed above, d extends to some $d^* \in G$. A straight-forward verification shows that

$$c(x_1,\dots,x_m) \approx d^*\big(n_0(x_1),\dots,n_{k-1}(x_1),\dots,n_0(x_m),\dots,n_{k-1}(x_m)\big)$$

proving that $c \in G$. Thus G contains all c with $\operatorname{im} c \subseteq \mathbf{2}$ and, in view of $S_k \subseteq G$ also Γ_2. ∎

THEOREM 2.9. *If f is essential and $\ell := |\operatorname{im} f| > 2$, then the clone*

$$D := \overline{\{f\} \cup U_2 \cup S_k}$$

contains Γ_ℓ.

PROOF. Let $a_0,\dots,a_{\ell-1}, p_1,\dots,p_n, q_1,\dots,q_n$ and g be as in Lemma 2.7. For $i = 1,\dots,n$ define $m_i \in O^{(1)}$ by setting $m_i(0) := p_i$ and $m_i(x) := q_i$ otherwise. Set $t := g(q_1,\dots,q_n)$ and define $r \in O^{(1)}$ by setting $r(a_0) := 0$, $r(a_2) := \min(t,1)$ and $r(x) := 1$ otherwise. Clearly $m_1,\dots,m_n, r \in U_2$ and so $h \in O^{(2)}$ defined by

$$h(x_1,x_2) :\approx r\Big(g\big(m_1(x_1), m_2(x_2),\dots,m_n(x_2)\big)\Big)$$

belongs to D. A straight-forward check shows that h satisfies (2.2) and so $\Gamma_2 \subseteq D$ by Lemma 2.8.

By induction on $i = 2,\dots,\ell$, we prove that $\Gamma_i \subseteq D$. Suppose $2 \leq i < \ell$ and $\Gamma_i \subseteq D$. Let $z \in \Gamma_{i+1}$ be a p-ary with $\operatorname{im} z = \{a_0,\dots,a_i\}$. Put $Z_j := z^{-1}(a_j)$ for all $j = 0,\dots,i$. By assumption for $j = 3,\dots,\ell$, we have $g(r_{j1},\dots,r_{jn}) = a_j$ for some $r_{j1},\dots,r_{jn} \in \mathbf{k}$. Let s_1 map $Z_0 \cup Z_2$ onto p_1, Z_1 onto q_1 and Z_l onto r_{l1} for $\ell = 3,\dots,i$. Similarly for

$j = 2, \dots, n$ let s_j map $Z_0 \cup Z_1$ onto p_j, Z_2 onto q_j and Z_ℓ onto $r_{\ell j}$ for $\ell = 3, \dots, i$. Clearly $s_1, \dots, s_n \in \Gamma_i \subseteq D$. A straight verification shows

$$z(x_1, \dots, x_p) \approx g\big(s_1(x_1, \dots, x_p), \dots, s_n(x_1, \dots, x_p)\big)$$

and so $z \in D$. Thus $\Gamma_{i+1} \subseteq D$. This concludes the inductive step and hence the proof of the theorem. ∎

2.10. We eliminate right away the case $\ell = k$. Indeed for $\operatorname{im} f = \mathbf{k}$, Salomaa [Sa 62] showed that $C = O$. This is also an easy consequence of a general completeness criterion [Ro 65, 70] cf also [Ro 70a]. The proof below also applies if we add the following combinatorial fact. If f is essential and idempotent (*i.e.* $f(x, \dots, x) \approx x$) then $f(a_1, \dots, a_n) = f(b_1, \dots, b_n)$ for some $a_i, b_i \in \mathbf{k}$, $a_i \neq b_i$ $(i = 1, \dots, n)$. (Lemma 2.7 may be used for the proof.)

Similarly in the case $\ell = 1$ we have directly $C = \overline{S_k} \cup \Gamma_1$. In the sequel we assume $1 < \ell < k$.

2.11. In view of Theorem 2.9 for $2 < \ell < k$ it suffices to show that $U_2 \subseteq C = \overline{\{f\} \cup S_k}$ for every essential $f \in O$ with $|\operatorname{im} f| = \ell$. This is done in §3 while §4 is devoted to the special case $\ell = 2$. In the sequel it will be convenient to put

$$U := \big\{|h^{-1}(a)| : h \in C^{(1)}, \operatorname{im} h = \{a, b\}\big\}.$$

Note that $i \in U$ if for some $h \in C^{(1)}$ the equivalence $\ker h$ has exactly two blocks of size i and $k - i$; in particular $i \in U \Leftrightarrow k - i \in U$. Our aim is to show that $U = \{1, \dots, k-1\}$. We need two lemmas.

LEMMA 2.12. *The clone C contains all unary constant operations.*

PROOF. Define $r \in O^{(1)}$ by $r(x) :\approx f(x, \dots, x)$. As $\operatorname{im} r \subseteq \operatorname{im} f$ clearly $r \in C^{(1)} \setminus S_k$. Denote by P the set of partitions of k from Lemma 2.2.(iii) corresponding to $C^{(1)}$. As $\ker r \in P$, the set $P \setminus \{(1, \dots, 1)\}$ is nonempty which implies $(k) \in P$. ∎

LEMMA 2.13. *The set U is nonempty.*

PROOF. We may assume that f depends on its first variable. This means that there exist $c_2, \dots, c_n \in k$ such that $r \in O^{(1)}$ defined by $r(x) :\approx f(x, c_2, \dots, c_n)$ is non-constant. Now by Lemma 2.12 all constants are in C and thus $r \in C$. Proceeding as in the proof of Lemma 2.11 we obtain $U \neq \emptyset$. ∎

LEMMA 2.14. *If $\ell > 2$ then $a \in U$ for some $1 < a < k-1$.*

PROOF. We argue the contrapositive. Suppose $U = \{1, k-1\}$. Let $p_1, \dots, p_n$, $q_1, \dots, q_n$ and g be as in Lemma 2.7. Define $\phi_1, \dots, \phi_n \in O^{(1)}$ by setting $\phi_1(0) := q_1$, $\phi_j(1) := q_j$ $(j = 2, \dots, n)$ and $\phi_t(x) := p_t$ otherwise; notice that every ϕ_t is either constant or $\ker \phi_t$ has two blocks of sizes 1 and $k-1$, and so $\phi_1, \dots, \phi_n \in C$. It follows that $h \in O^{(1)}$, defined by $h(x) \approx g\big(\phi_1(x), \dots, \phi_n(x)\big)$ belongs to C. From (2.1) we get $h(0) = a_1$, $h(1) = a_2$ and $h(x) = a_0$ otherwise. If we fuse the blocks $\{a_1\}$ and $\{a_2\}$, we get $2 \in U$, a contradiction. ∎

In the sequel let k' represent the largest integer not exceeding $\frac{1}{2}k$.

3. Essential operations with more than 2 values.

3.1. In this section $f \in O^{(n)}$ is essential, $|\operatorname{im} f| = \ell$ where $2 < \ell < k$ and $C := \overline{\{f\} \cup S_k}$. The set U was defined in 2.11 as the set of all $|h^{-1}(a)|$ where $h \in C^{(1)}$ and $\operatorname{im} h = \{a, b\}$. We have:

LEMMA 3.2. *If $0 < r \leq k'$ and $r \in U$ then*

$$1, 2, \ldots, 2r - 1 \in U. \tag{3.1}$$

PROOF. Fix $s \in \{r, k - r\}$ and $0 < t < r$. Let $a_0, \ldots, a_{\ell-1}, p_1, \ldots, p_n, q_1, \ldots, q_n$ and g be as in Lemma 2.7. Set

$$A_0 := \{0, \ldots, t-1\}, \quad A_1 := \{t, \ldots, k-s-1\},$$
$$A_2 := \{k-s, \ldots, k-t-1\}, \quad A_3 := \{k-t, \ldots, k-1\}.$$

Note that

$$|A_0| = t, \quad |A_1| = k - s - t, \quad |A_2| = s - t, \quad |A_3| = t \tag{3.2}$$

Next define $g_1 \in O^{(1)}$ by setting $g_1(x) := p_1$ for $x \in A_0 \cup A_2$ and $g_1(x) := q_1$ otherwise. For $j = 2, \ldots, n$ define $g_j \in O^{(1)}$ by setting $g_j(x) := p_j$ for $x \in A_0 \cup A_1$ and $g_j(x) = q_j$ otherwise. Note that

$$|A_0 \cup A_2| = t + k - s - t = k - s, \quad |A_0 \cup A_1| = t + k - t - k + s = s$$

and so for all $j = 1, \ldots, n$ the equivalence $\ker g_j$ has exactly two blocks of size s and $k-s$; whence by our choice of s and $r \in U$ we have $g_1, \ldots, g_n \in C^{(1)}$. Now define $h \in O^{(1)}$ by

$$h(x) :\approx g\big(g_1(s), g_2(x), \ldots, g_n(x)\big).$$

As $g \in C$ clearly $h \in C^{(1)}$. Set $d := g(q_1, \ldots, q_n)$. We need:

CLAIM 1. *There exists $r \in C^{(1)}$ with $r(A_i) = \{a_i\}$ $(i = 0, 1, 2)$ and $r(A_3) = \{a_j\}$ for some $0 \leq j \leq 2$.*

PROOF (OF THE CLAIM). If $d \in \{a_0, a_1, a_2\}$, choose $r := h$. Thus let $d \notin \{a_0, a_1, a_2\}$. Then $A_0, \ldots, A_3$ are the blocks of $\ker h$ and $4 \leq \ell < k$. Let r satisfy $r(A_0 \cup A_3) := \{a_0\}$ and $r(A_i) = \{a_i\}$ for $i = 1, 2$. From Lemma 2.2 we have $r \in C^{(1)}$. ∎

We distinguish three cases according to $j = 0, 1, 2$ in Claim 1.

(i) Let $j = 0$. According to (3.2) the equivalence $\ker r$ has 3 blocks of sizes $2t$, $k-s-t$ and $s - t$. Applying again Lemma 1.2 we can fuse the first and the last block to obtain $u \in C^{(1)}$ having $\ker u$ with two blocks of sizes $s + t$ and $k - s - t$. We have obtained $s + t \in U$.

CLAIM 2. *If $s+t \in U$ for all $s \in \{r, k-r\}$ and $0 < t < r$ then (3.1) holds.*

PROOF (OF THE CLAIM). Choose $s = r$ and $t = 1, \dots, r-1$ to get

$$(3.3) \qquad r+1, \dots, 2r-1 \in U.$$

Similarly, for $s = k-r$ and $t = 1, \dots, r-1$ we have $k-r+t \in U$ and so $r-t \in U$ proving

$$(3.4) \qquad 1, 2, \dots, r-1 \in U.$$

Together with $r \in U$ this proves Claim 2. ∎

(ii) Let $j = 1$. Then $\ker r$ has 3 blocks of sizes t, $k-s$ and $s-t$. Fusing the last two blocks we obtain $k-t \in U$. Thus $t \in U$ for all $0 < t < r$. Fusing the first two blocks we get $k-s+t \in U$. The choice $s = k-r$ gives $r+t \in U$ for all $0 < t < r$. Together this yields (3.1).

(iii) Finally let $j = 2$. Then $\ker r$ has 3 blocks of sizes t, $k-s-t$ and s. Fusing the first and the last block we get $t+s \in U$ and so Claim 2 applies. ∎

LEMMA 3.3. $U = \{1, \dots, k-1\}$.

PROOF. According to Lemma 2.14 the set U contains an element $1 < a < k-1$. By the symmetry of U the set

$$V := U \cap \{1, \dots, k'\}.$$

is nonempty. Denote by v the largest element of V. If $v = k'$ we are done. Thus let $1 < v < k'$. By Lemma 3.2 we have $1, \dots, 2v-1 \in U$. However, $v+1 \leq 2v-1$, hence $v+1 \in U$ in contradiction to the choice of v. ∎

4. **Two-valued operations.**

4.11. As mentioned in 2.11 the case $\ell = 2$ requires special treatment. In this section $C := \overline{S_k \cup \{f\}}$ where $f \in O^{(n)}$ is a 2-valued essential operation.

The following operations—introduced in [Bu 67]—are exceptional. Denote by $\dot{+}$ the sum mod 2 on **2**. (The operation $\dot{+}$, called *exclusive or* in logic, satisfies $0 \dot{+} 0 = 1 \dot{+} 1 = 0$, $0 \dot{+} 1 = 1 \dot{+} 0 = 1$.) For $n > 1$ an operation $g \in O^{(n)}$ is *quasilinear* if there exist a map $\phi_0 \colon \mathbf{2} \longrightarrow \mathbf{k}$ and maps $\phi_1, \dots, \phi_n \colon \mathbf{k} \longrightarrow \mathbf{2}$ such that

$$(4.1) \qquad g(x_1, \dots, x_n) \approx \phi_0\big(\phi_1(x_1) \dot{+} \cdots \dot{+} \phi_n(x_n)\big).$$

The following lemma is an adaption of Lemma 2.7 to nonquasilinear essential 2-valued operations.

LEMMA 4.2. *If $f \in O^{(n)}$ is essential, nonquasilinear and $|\operatorname{im} f| = 2$, then*

$$(4.2) \qquad \begin{aligned} f(a_1, \dots, a_n) &= f(b_1, \dots, b_{i-1}, a_i, b_{i+1}, \dots, b_n) \\ &= f(b_1, \dots, b_n) \neq f(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n). \end{aligned}$$

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for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{k}$ *and* $1 \leq i \leq n$.

PROOF. Let f satisfy the assumptions. It is immediate that so do τf and ζf (*cf.* 2.0). If f has a fictitious variable, say the first, then Δf also satisfies the assumptions. Using repeatedly ζ, τ and Δ we can get rid of all fictitious variables obtaining an operation g satisfying the assumptions and depending on all its variables. If (4.2) holds for g then it holds also for f and so for simplicity we assume that already f depends on all its variables. For notational ease assume $\operatorname{im} f = \mathbf{2}$.

For $1 \leq i \leq n$ and $\underline{c} = (c_1, \ldots, c_n) \in \mathbf{k}^n$ define $f^i_{\underline{c}} \in O^{(1)}$ by setting

$$f^i_{\underline{c}}(x) :\approx f(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots c_n).$$

CLAIM 1. *If there exist* $\underline{a} = (a'_1, a_2, \ldots, a_n) \in \mathbf{k}^n$ *and* $\underline{b} = (b'_1, b_2, \ldots, b_n) \in \mathbf{k}^n$ *such that* $\ker f^1_{\underline{a}} \neq \ker f^1_{\underline{b}}$, *then there are* $a_1, b_1 \in \mathbf{k}$ *such that (4.2) holds for* $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ *and* $i = 1$.

PROOF (OF THE CLAIM). At least one of $\ker f^1_{\underline{a}}$ and $\ker f^1_{\underline{b}}$ has exactly two blocks. Choose the notation so that $\ker f^1_{\underline{a}}$ has two blocks A and B. We claim that

$$f^1_{\underline{a}}(c) \neq f^1_{\underline{a}}(d) \quad \text{and} \quad f^1_{\underline{b}}(c) = f^1_{\underline{b}}(d) \tag{4.3}$$

for suitable $c, d \in \mathbf{k}$. Indeed, if (4.3) does not hold, then for every $c \in A$ and every $d \in B$ we have $f^1_{\underline{b}}(c) \neq f^1_{\underline{b}}(d)$ and so $\ker f^1_{\underline{b}} \subseteq \ker f^1_{\underline{a}}$, in particular $\ker f^1_{\underline{b}}$ has at least two blocks. As it has at most two blocks, we deduce the contradiction $\ker f^1_{\underline{a}} = \ker f^1_{\underline{b}}$. Thus (4.3) is proved. Put $\alpha := f^1_{\underline{b}}(c)$. In view of $\alpha \in \{f^1_{\underline{a}}(c), f^1_{\underline{a}}(d)\}$ we can set $\{a_1, b_1\} = \{c, d\}$ so that $\alpha = f_{\underline{a}}(a_1)$. Now (4.2) is proved for $i = 1$ as

$$\begin{aligned} f(a_1, \ldots, a_n) &= f^1_{\underline{a}}(a_1) = \alpha = f^1_{\underline{b}}(a_1) = f(a_1, b_2, \ldots, b_n) \\ &= f^1_{\underline{b}}(b_1) = f(b_1, \ldots, b_n) \neq f^1_{\underline{a}}(b_1) = f(b_1, a_2, \ldots, a_n). \end{aligned}$$

■

If an isomer of f satisfies the assumptions of Claim 1 then (4.2) holds for a suitable $1 \leq i \leq n$. Thus we are left with the case of f with the following property:

(E) There exist 2-block equivalence relations $\varepsilon_1, \ldots, \varepsilon_n$ on $\mathbf{k}$ such that $\ker f^i_{\underline{a}} = \varepsilon_i$ for all $\underline{a} \in \mathbf{k}^n$ and $i = 1, \ldots, n$.

For $i = 1, \ldots, n$ denote by B_{i0} and B_{i1} the two blocks of ε_i and define $\phi_i \colon \mathbf{k} \longrightarrow \mathbf{2}$ setting $\phi_i(x) := j$ for all $j \in \mathbf{2}$ and all $x \in B_{ij}$.

CLAIM 2. *There is a boolean function g (i.e. a map $g \colon \mathbf{2}^n \longrightarrow \mathbf{2}$) such that*

$$f(x_1, \ldots, x_n) = g\big(\phi_1(x_1), \ldots, \phi_n(x_n)\big) \tag{4.4}$$

holds for all $x_1, \ldots, x_n \in \mathbf{2}$.

PROOF (OF THE CLAIM). We start by showing that for every $j = (j_1, \ldots, j_n) \in \mathbf{2}^n$ the operation f is constant on the cartesian product $P = P(j) := B_{1j_1} \times \cdots \times B_{nj_n}$. Indeed, let $(c_1, \ldots, c_n), (d_1, \ldots, d_n) \in P$.

We show that $\gamma_P := f(c_1,\ldots,c_n) = f(d_1,c_2,\ldots,c_n)$. Indeed, put $\underline{c} = (c_1,\ldots,c_n)$. Since $\ker f^1_{\underline{c}} = \varepsilon_1$ and c_1, d_1 both belonging to the block B_{1j_1} of ε_1, we obtain

$$\gamma_P = f^1_{\underline{c}}(c_1) = f^1_{\underline{c}}(d_1) = f(d_1,c_2,\ldots,c_n).$$

Continuing in this fashion we get $\gamma_P = f(d_1,\ldots,d_n)$ and so f is constant on P. To get (4.4) it suffices to define $g\colon \mathbf{2}^n \longrightarrow \mathbf{2}$ by setting $g(j) := \gamma_{P(j)}$ for all $j \in \mathbf{2}^n$. ∎

CLAIM 3. *For all* $x_1,\ldots,x_n \in \mathbf{2}$

$$g(x_1,\ldots,x_n) = x_1 \dotplus \cdots \dotplus x_n \dotplus g(0,\cdots,0).$$

PROOF (OF THE CLAIM). Put $c := g(0,\ldots,0)$. To every $j = (j_1,\ldots,j_n) \in \mathbf{2}^n$ assign

$$|j| = j_1 + 2j_2 + \cdots + 2^{n-1}j_n.$$

Suppose

$$g(j) \neq j_1 \dotplus \cdots \dotplus j_n \dotplus c \tag{4.5}$$

holds for some $j = (j_1,\ldots,j_n) \in \mathbf{2}^n$. Let $j \in \mathbf{2}^n$ satisfy (4.5) with the least possible $|j|$. Due to our choice of c clearly $|j| > 0$. Denote by i the first index such that $j_i = 1$ and set $j' := (0,\ldots,0,j_{i+1},\ldots,j_n)$. Clearly $|j'| = |j| - 2^{i-1}$ and so by the minimality of $|j|$ we have

$$g(j') = j_{i+1} \dotplus \cdots \dotplus j_n \dotplus c.$$

According to (4.5) we have $g(j) \neq 1 \dotplus j_{i+1} \dotplus \cdots \dotplus j_n \dotplus c$, and so in view of $g \in \mathbf{2}$ we have $g(j') = g(j)$. Choose $c_m \in B_{m0}$ for all $1 \leq m \leq i$ and $c_m \in B_{m,j_m}$ for $i < m \leq n$. Put $\underline{c} := (c_1,\ldots,c_n)$ and consider $f^i_{\underline{c}}$. For $x \in B_{i0}$ we have $f^i_{\underline{c}}(x) = g(j')$ and for $x \in B_{i1}$ the value of $f^i_{\underline{c}}(x)$ is $g(j)$. Since $g(j') = g(j)$ the function $f^i_{\underline{c}}$ is constant, in contradiction to the property (E). ∎

Combining Claims 2 and 3 we obtain that f is quasilinear (with ϕ_0 the identity map on $\mathbf{2}$). This contradicts our assumption and settles the last case of f satisfying (E). ∎

LEMMA 4.3. *If f satisfies the assumption of Lemma 4.2, then* $U = \{1,\ldots,k-1\}$.

PROOF. We start with the following:

CLAIM 1. *If* $0 < d \leq k'$ *and* $d \in U$, *then* $d+1,\ldots,2d+1 \in U$.

PROOF (OF THE CLAIM). For notational simplicity assume that (4.2) holds for $i = 1$. Fix z so that $k - 2d - 1 \leq z < k - d$. Put

$$\begin{aligned} &A_0 := \{0,\ldots,k-d-z-1\}, \quad A_1 := \{k-d-z,\ldots,d-1\} \\ &A_2 := \{d,\ldots,d+z-1\}, \quad A_3 := \{d+z,\ldots,k-1\} \end{aligned} \tag{4.6}$$

and define $g_1,\ldots,g_n \in O^{(1)}$ as follows: (i) $g_1(x) = a_1$ for all $x \in A_0 \cup A_1$, (ii) $g_j(x) = a_j$ for all $2 \leq j \leq n$ and $x \in A_0 \cup A_2$ and (iii) $g_j(x) = b_j$ otherwise. Note that each $\ker g_j$ has either one block (if g_i is constant) or exactly two blocks of sizes d and $k-d$. Thus $g_1,\ldots,g_n \in C$ and so $h(x) :\approx f\big(g_1(x),\ldots,g_n(x)\big)$ belongs to C. According to (4.2) the equivalence $\ker h$ has exactly two blocks A_2 and $\mathbf{k}\setminus A_2$. As $|A_2| = z$, we have $z, k-z \in U$. The above restriction $k-2d-1 \leq z < k-d$ translates into $d < k-z \leq 2d+1$. ∎

CLAIM 2. *If* $0 < d \leq k'$ *and* $d \in U$ *then* $1, \ldots, d-1 \in U$.

PROOF (OF THE CLAIM). Let $0 < z < d$. Put

$$A_0 := \{0, \ldots, d-z-1\}, \quad A_1 := \{d-z, \ldots, d-1\},$$
$$A_2 := \{d, \ldots, d+z-1\}, \quad A_3 := \{d+z, \ldots, k-1\}. \tag{4.7}$$

Define $g_1, \ldots g_n \in O^{(1)}$ by setting (i) $g_1(x) := a_1$ for all $x \in A_0 \cup A_1$, (ii) $g_j(x) = a_j$ for all $2 \leq j \leq n$ and $x \in A_0 \cup A_2$ and (iii) $g_j(x) = b_j$ otherwise. Again h defined by $h(x) \approx f\big(g_1(x), \ldots, g_n(x)\big)$ is such that $\ker h$ has exactly two blocks A_2 and $\mathbf{k} \setminus A_2$ proving $z \in U$. ∎

PROOF OF THE LEMMA. From Lemma 2.13 we know that $U \neq \emptyset$ and so there is $d \in U$, $d \leq k'$. By Claim 2 we have $1, \ldots, d \in U$. Suppose to the contrary that $U \neq \{1, \ldots, k-1\}$ and denote m the least element of $\{1, \ldots, k-1\} \setminus U$. Then $1 < m \leq k'$. Now $m-1 \in U$ and so by Claim 1 also $m, \ldots, 2m-1 \in U$, a contradiction. ∎

Now we have:

PROPOSITION 4.1. *Let f be an essential operation with $|\operatorname{im} f| = 2$. If f is not quasilinear then $C := \overline{\{f\} \cup S_k}$ contains Γ_2.*

PROOF. We have $U_2 \subseteq C$ by Lemma 4.3. Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, and i be as in Lemma 4.2. We may assume that $f(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) = 0$ while the other three values in (4.2) are 1. Define $u_i \in O^{(1)}$ by $u_i(0) := b_i$ and $u_i(x) := a_i$ otherwise. For $1 \leq j \leq n, j \neq i$ put $u_j(0) := a_j$ and $u_j(x) := b_j$ otherwise. Clearly $u_1, \ldots, u_n \in U_2 \subseteq C$. Define $h \in O^{(2)}$ by

$$h(x_1, x_2) \approx f\big(u_1(x_2), \ldots, u_{i-1}(x_2), \quad u_i(x_1), \quad u_{i+1}(x_2), \ldots, u_n(x_2)\big).$$

Clearly $h \in C$ and $h(0,0) = 0$, $h(1,0) = h(0,1) = h(1,1) = 1$. Now it suffices to apply Lemma 2.8. ∎

4.5. We turn to the remaining case of a quasilinear f.

In the sequel $f \in O^{(n)}$ is an essential quasilinear operation. We may assume that $\operatorname{im} f = \mathbf{2}$. Clearly in the expression (4.1) of f the map $\phi_0 \colon \mathbf{2} \longrightarrow \mathbf{2}$ is non-constant and thus either $\phi_0(x) = x$ for $x = 0, 1$ or $\phi_0(x) = x \dotplus 1$ for $x = 0, 1$. In the second case replace ϕ_n by ϕ_n' where $\phi_n'(x) := \phi_n(x) \dotplus 1$ for all $x \in \mathbf{k}$. We have obtained that

$$f(x_1, \ldots, x_n) \approx \phi_1(x_1) \dotplus \cdots \dotplus \phi_n(x_n). \tag{4.8}$$

Without loss of generality we may assume that f depends exactly on its first ℓ variables (*i.e.* $\phi_1, \ldots, \phi_\ell$ are non-constant while $\phi_{\ell+1}, \ldots, \phi_n$ are constant). Proceeding as above we get

$$f(x_1, \ldots, x_n) \approx \phi_1(x_1) \dotplus \cdots \dotplus \phi_\ell(x_\ell).$$

Denote by C the clone generated by f and the constant selfmaps of $\mathbf{k}$.

We start with the following:

FACT 4.6. Let $1 \leq m < \ell$. Then
(i)

$$\phi_{m+1}(c_{m+1}) \dotplus \cdots \dotplus \phi_n(c_n) = 0 \tag{4.9}$$

for some $c_{m+1}, \ldots, c_n \in \mathbf{k}$, and
(ii) C contains the operation

$$f_m(x_1, \ldots, x_m) :\approx \phi_1(x_1) \dotplus \cdots \dotplus \phi_m(x_m) \tag{4.10}$$

PROOF. We argue the contrapositive. Suppose (i) does not hold. Then $\phi_{m+1}(x_1) \dotplus \cdots \dotplus \phi_n(x_{n-m})$ is the $(n-m)$-ary constant operation with value 1 and so

$$f(x_1, \ldots, x_n) \approx \phi_1(x_1) \dotplus \cdots \dotplus \phi_m(x_m) \dotplus 1$$

contradicting the fact that f depends on its first ℓ variables. (ii) Denote by $c_{m+1}, \ldots, c_n$ the elements from (i). Clearly

$$f_m(x_1, \ldots, x_m) \approx f(x_1, \ldots, x_m, c_{m+1}, \ldots, c_n)$$

belongs to C. ∎

As before U stands for the set of all positive integers u such that for some $f \in C^{(1)}$ the equivalence ker f has two blocks of sizes u and $k - u$. We have:

LEMMA 4.7. *Let f, C and U be as in 4.5.*
(i) If $a, b \in U$ satisfy $0 < a \leq b < k - 1$ then

$$b - a, \quad b - a + 2, \ldots, c \in U \tag{3.9}$$

where $c := a + b$ if $a + b \leq k$ and $c := 2k - a - b$ if $a + b > k$,
(ii) if u, $u + 1 \in U$ then $1 \in U$,
(iii) if $1 \in U$ then $2 \in U$, and
(iv) if $1, 2 \in U$ then $3 \in U$.

PROOF. Define $g \in O^{(2)}$ by setting

$$g(x_1, x_2) :\approx \phi_1(x_1) \dotplus \phi_2(x_2).$$

According to Fact 4.6, the operation g belongs to C. As neither ϕ_1 nor ϕ_2 is constant, we have $\operatorname{im} \phi_1 = \operatorname{im} \phi_2 = \mathbf{2}$. Fix $\alpha_i, \beta_i \in \mathbf{k}$ $(i = 1, 2)$ so that

$$\phi_1(\alpha_1) = \phi_2(\alpha_2) = 0, \quad \phi_1(\beta_1) = \phi_2(\beta_2) = 1.$$

(i) Let e satisfy $\max(a + b - k, 0) \leq e \leq a$. Define $\psi_1, \psi_2 \in O^{(1)}$ by setting
  1) $\psi_1(x) := \alpha_1$ for all $0 \leq x < a$, $\psi_2(x) := \alpha_2$ for all $0 \leq x < a - e$ and $a \leq x < k - b + e$
  2) $\psi_j(x) := \beta_j$ otherwise $(j = 1, 2)$.

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Notice that $\ker\psi_1$ has blocks of sizes a and that $k-a$ and $\ker\psi_2$ has blocks of sizes $k-b$ and b; and so $\psi_1,\psi_2 \in C$ (due to $a,b\in U$). The unary operation

$$h(x) :\approx g\big(\psi_1(x),\psi_2(x)\big)$$

belongs to C. Note that

$$h(x) = \begin{cases} \phi_1(\alpha_1) \dotplus \phi_2(\alpha_2) = 0 \dotplus 0 = 0 & \text{for all } 0 \leq x < a-e \\ \phi_1(\alpha_1) \dotplus \phi_2(\beta_2) = 0 \dotplus 1 = 1 & \text{for all } a-e \leq x < a \\ \phi_1(\beta_1) \dotplus \phi_2(\alpha_2) = 1 \dotplus 0 = 1 & \text{for all } a \leq x < k-b+e \\ \phi_1(\beta_1) \dotplus \phi_2(\beta_+2) = 1 \dotplus 1 = 0 & \text{for all } k-b+e \leq x < k \end{cases}$$

and so $|h^{-1}(0)| = a+b-2e$. Thus $a+b-2e \in U$. Choosing e in its range (which depends on whether $a+b \leq k$ or $a+b > k$) we obtain (3.9).

(ii) Define $\mu_1,\mu_2 \in O^{(1)}$ by setting $\mu_1(x) := \alpha_1$ for $0 \leq x < u$, $\mu_2(x) := \alpha_2$ for $u < x < k-1$ and $\mu_j(x) = \beta_j$ otherwise. Straight verification shows that $s(x) :\approx g\big(\mu_1(x),\mu_2(x)\big)$ satisfies $s^{-1}(0) = \{u\}$ and so $1 \in U$.

(iii) Define $\nu_1,\nu_2 \in O^{(1)}$ by setting $\nu_1(0) := \alpha_1$, $\nu_2(1) := \alpha_2$ and $\nu_j(x) := \beta_j$ otherwise. Then $\nu_1,\nu_2 \in C$ and $s(x) :\approx g\big(\nu_1(x),\nu_2(x)\big)$ has $s^{-1}(1) = \{0,1\}$ proving $2 \in U$.

(iv) Define $\varepsilon_1,\varepsilon_2 \in O^{(1)}$ by setting $\varepsilon_1(0) = \varepsilon_1(1) = \alpha_1$, $\varepsilon_2(3) = \alpha_2$ and $\varepsilon_j(x) = \beta_j$ otherwise and proceed as above. ■

LEMMA 4.8. *If f, C and U are as in 4.5 then either (i) $U = \{1,\ldots,k-1\}$ or (ii) $U = \{2,4,\ldots,k-2\}$ and k is even.*

PROOF. Let $k=3$. Since $U \neq \emptyset$ by Lemma 2.13, we have $2 \in U$ which implies $U = \{1,2\}$ and (i) holds. Thus let $k > 3$. According to Lemma 4.7(iv) we have $U \neq \{1,k-1\}$ and so $a \in U$ for some $1 < a \leq k'$. Suppose U does not contain all even numbers not exceeding k'. Denote m the least even number $\leq k'$ such that $m \in U$ while $m+2 \notin U$. Choosing $a = b = m$ in Lemma 4.7(i) we get $2,4,\ldots,2m \in U$. As $m+2 \notin U$ we have $2m \leq m$ in contradiction to $m = 2$. It follows that U contains all even positive numbers $\leq k'$. We proceed by cases.

A. Let $k = 4\ell+1$. We have $2,4,\ldots,2\ell \in U$ and so $k-2\ell = 2\ell+1 \in U$. In Lemma 4.7 (i) choose $a = b = 2\ell+1$ to obtain $c = 2k-2(2\ell+1) = 4\ell$ and so $2,4,\ldots,4\ell \in U$. If we add the elements of the form $k-u$ we get $1,2,\ldots,4\ell-1 \in U$, proving (i).

B. Let $k = 4\ell+3$. We have $2,\ldots,2\ell \in U$, hence $2\ell+3 = k-2\ell \in U$. Choosing $a = b = 2\ell+3$ in Lemma 4.7(i) we get $2,4,\ldots,4\ell \in U$. Now U contains $k-2,\ldots,k-4\ell$ and so $3,5,\ldots,4\ell+1 \in U$. Finally choosing $a=2$ and $b=3$ in Lemma 4.7(i) we get $1 \in U$, and so (i) holds.

C. Let $k = 2\ell$. As all positive even numbers not exceeding ℓ are in U we have $U \supseteq \{2,4,\ldots,2\ell-2\}$. If we have equality we have (ii). Thus assume that U also contains some odd number o. We may assume that it does not exceed $k' = \ell$. If $o = 1$, then by Lemma 4.7(v) also $3 \in U$ and so we may assume $3 \leq o \leq \ell$. Suppose that U

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does not contain all odd numbers between 3 and ℓ. Denote by u the least integer such that $1, 3, 5, \ldots, 2u+1 \in U$, $2u+1 \leq \ell$ while $2u+3 \notin U$. Choosing $a = 2u+1$ and $b = 2u+2$ in Lemma 4.7(i) we get $c = 4u+3$ (as $a+b = 4u+3 \leq 2\ell - 3 < k$) and so $4u+3 \leq 2u+3$ leading to $u = 0$ whereas $u \geq 1$. This contradiction shows that U contains all odd numbers between 3 and ℓ. By Lemma 4.7(iii) we have $1 \in U$ proving (ii). ∎

The two cases in Lemma 4.8 lead to the clones investigated in the next section.

5. Clones of quasilinear operations containing S.

5.1. Call a selfmap ϕ of $\mathbf{k}$ *even* if $|\phi^{-1}(a)|$ is even for all $a \in \mathbf{k}$, *i.e.* if $\ker f$ consists of blocks of even size. Put

$$T := \{\phi \in O^{(1)} : \operatorname{im} \phi \subseteq \mathbf{2}\}.$$

Recall that $f \in O^{(n)}$ is quasilinear (4.1) if

$$f(x_1, \ldots, x_n) \approx \phi_0\big(\phi_1(x) \dotplus \cdots \dotplus \phi_n(x)\big) \tag{5.1}$$

where $\phi_0\colon \mathbf{2} \longrightarrow \mathbf{k}$ and $\phi_1, \ldots, \phi_n \in T$. Denote by B the set of all quasilinear operations. Call $f \in B$ even if it can be expressed as in (5.1) with all $\phi_1, \ldots, \phi_n$ even. Denote by B_e the set of even quasilinear operations and $Q := \{e_i^n : 1 \leq i \leq n < \omega\}$ the clone of all projections. We have:

LEMMA 5.2. *$Q \cup B$ and $Q \cup B_e$ are clones.*

PROOF. Let C be one of $Q \cup B$ and $Q \cup B_e$ and let ζ, τ, Δ and $*$ be as in 2.0. a) It is easy to see that $\zeta C = \tau C = C$. b) Let $n > 1$ and $f \in C$ be given by (5.1). Put $\phi_1'(x) :\approx \phi_1(x) \dotplus \phi_2(x)$ and $\phi_i' := \phi_{i+1}$ $(i = 1, \ldots, n-1)$. Then

$$(\Delta f)(x_1, \ldots, x_{n-1}) \approx \phi_0\big(\phi_1'(x_1) \dotplus \cdots \dotplus \phi_{n-1}'(x_{n-1})\big).$$

Clearly $\phi_i' \in T$ and so $\Delta f \in B$ settling $\Delta C \subseteq C$ in the case $C = Q \cup B$. Let ϕ_1 and ϕ_2 be even. It suffices to verify that $|\phi_1'^{-1}(0)|$ is even. For $i, j \in \mathbf{2}$ put $A_{ij} := \phi_1^{-1}(i) \cap \phi_2^{-1}(j)$ and $\alpha_{ij} := |A_{ij}|$. Clearly

$$\alpha_{00} + \alpha_{01} = |\phi_1^{-1}(0)| \equiv 0 \pmod 2, \quad \alpha_{01} + \alpha_{11} = |\phi_2^{-1}(1)| \equiv 0 \pmod 2$$

and so

$$|\phi_1'^{-1}(0)| = \alpha_{00} + \alpha_{11} \equiv \alpha_{01} + \alpha_{11} \equiv 0 \pmod 2$$

proving that ϕ_1' is even and $\Delta f \in C$ in the case $C = Q \cup B_e$. c) Set $f \in C^{(n)}$ and $g \in C^{(m)}$. Put $r := m + n - 1$ and $h := f * g$. Suppose that at least one of f and g is a projection. Then it is easy to check that $h \in C$ (to express h in the form (5.1) choose ϕ_i to be the constant c_0 with value 0 whenever h does not depend on its i-th variable). Thus suppose that neither f nor g is a projection. Let f be given by (5.1) and g by

$$g(x_1, \ldots, x_m) \approx \psi_0\big(\psi_1(x_1) \dotplus \cdots \dotplus \psi_m(x_m)\big).$$

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Now

$$h(x_1,\ldots,x_r) \approx \phi_0\Big(\phi_1\Big(\psi_0\big(\psi_1(x_1) \dotplus \cdots \dotplus \psi_m(x_m)\big)\Big) \dotplus \phi_2(x_{m+1}) \dotplus \cdots \dotplus \phi_n(x_r)\Big)$$
$$= \phi_0\Big(\chi\big(\psi_1(x_1) \dotplus \cdots \dotplus \psi_m(x_m)\big) \dotplus \phi_2(x_{m+1}) \dotplus \cdots \dotplus \phi_n(x_r)\Big)$$

where $\chi := \phi_1 \circ \psi_0 \colon \mathbf{2} \longrightarrow \mathbf{2}$. Note that either 1) χ is constant, or 2) $\chi(x) = x$ for all $x \in \mathbf{2}$, or 3) $\chi(x) = 1 \dotplus x$ for all $x \in \mathbf{2}$.

CASE 1. Let χ be constant. Put $c_0(x) := 0$ for all $x \in \mathbf{k}$. Then $h(x_1,\ldots,x_r) \approx \phi_0\big(\chi(x_1) \dotplus c_0(x_2) \dotplus \cdots \dotplus c_0(x_m) \dotplus \phi_{m+1}(x_{m+1}) \dotplus \cdots \dotplus \phi_n(x_r)\big)$, and so $h \in C$.

CASE 2. Let $\chi(x) = x$ for all $x \in \mathbf{2}$. Then clearly $h \in C$.

CASE 3. Let $\chi(x) = x \dotplus 1$ for all $x \in \mathbf{2}$. Setting $\phi_0'(x) := \phi_0(x \dotplus 1)$ for all $x \in \mathbf{2}$ we get

$$h(x_1,\ldots,x_r) \approx \phi_0'\big(\psi_1(x_1) \dotplus \cdots \dotplus \psi_m(x_m) \dotplus \phi_2(x_{m+1}) \dotplus \cdots \dotplus \phi_n(x_r)\big),$$

and so again $h \in C$. ∎

Put $V := \{\phi \in O^{(1)} : |\operatorname{im}\phi| \leq 2\}$.

LEMMA 5.3. *The set $\bar{M} \cup B$ is a clone for every $V \subseteq M \subseteq O^{(1)}$.*

PROOF. Notice that $V \subseteq B$. It suffices to check that $\bar{M} \cup B$ is closed under $*$. Clearly this holds for $\bar{M}$ and by Lemma 5.2 also for B. Let $f \in B^{(n)}$ be given by (5.1) and let $g \in \bar{M}$ be m-ary. Then

$$g(x_1,\ldots,x_m) \approx g'(x_i)$$

for some $1 \leq i \leq m$ and some *unary* operation $g' \in \bar{M}$. Put $r := m + n - 1$. We have

$$(f * g)(x_1,\ldots,x_r) \approx \phi_0\Big(c_0(x_1) \dotplus \cdots \dotplus c_0(x_{i-1}) \dotplus \phi_1\big(g'(x_i)\big)$$
$$\dotplus c_0(x_{i+1}) \dotplus \cdots \dotplus c_0(x_m) \dotplus \phi_2(x_{m+1}) \dotplus \cdots \dotplus \phi_n(x_r)\Big),$$

(where again c_0 maps $\mathbf{k}$ onto $\{0\}$). If $i > 1$ then $(g * f)(x_1,\ldots,x_r) \approx g'(x_i)$ while for $i = 1$

$$(g * f)(x_1,\ldots,x_r) \approx g'\Big(\phi_0\big(\phi_1(x_1) \dotplus \cdots \dotplus \phi(x_n) \dotplus c_0(x_{n+1}) \dotplus \cdots \dotplus c_0(x_r)\big)\Big). \quad ∎$$

We derive a Slupecki type criterion for $Q \cup B_e$. Denote by V_e the set of all even maps from V (*i.e.* $V_e := \{\phi \in O^{(1)} : |\operatorname{im}\phi| \leq 2, |\phi^{-1}(a)|$ even for all $a \in \operatorname{im}\phi\}$.

We have:

PROPOSITION 5.4. *Let f be a quasilinear and essential operation. Then:*
(i) $\overline{V_e \cup \{f\}} = Q \cup B_e$ provided f is even, and
(ii) $\overline{V \cup \{f\}} = Q \cup B$ otherwise.

PROOF. (i) By Lemma 5.2 the set $Q \cup B_e$ is a clone; and, in view of $V_e \subseteq B_e$ and $f \in B_e$ the clone $D := \overline{V \cup \{f\}}$ is a subclone of $Q \cup B_e$. For $\supseteq$ it suffices to prove $D \supseteq B_e$.

We may assume that im$f$ = **2** (if not, replace f by $\psi \circ f$ for a suitable $\psi \in V_e$) and that f depends exactly on its first ℓ variables, *i.e.*

$$f(x_1,\ldots,x_n) \approx \phi_1(x_1) \dotplus \cdots \dotplus \phi_\ell(x_\ell),$$

for some $\phi_i\colon \mathbf{k} \longrightarrow \mathbf{2}$ $(i = 1,\ldots,\ell)$. Notice that the existence of an even f implies k^n is even and so k is even. It follows that all constant selfmaps of **k** belong to V_e. Applying 4.5–4.6 (for $m = 2$) we obtain that

$$g(x,y) :\approx \phi_1(x) \dotplus \phi_2(y)$$

belongs to D. As ϕ_1 is non-constant, we have $\phi_1(c) = 0$ and $\phi_1(d) = 1$ for some $c, d \in \mathbf{k}$. There is $\lambda \in V_e$ with $\lambda(0) = c$ and $\lambda(1) = d$. Put $\mu := \phi_1 \circ \lambda$. Similarly, $\phi_2(c') = 0$ and $\phi_2(d') = 1$ for some $c', d' \in \mathbf{k}$. The map ν mapping $A := \mu^{-1}(0)$ onto $\{c'\}$ and $B := \mu^{-1}(1)$ onto $\{d'\}$ clearly belongs to V_e. The operation

$$g_2(x_1,x_2) :\approx \phi_1\big(\lambda(x_1)\big) \dotplus \phi_2\big(\nu(x_2)\big) \approx \mu(x_1) \dotplus \mu(x_2) \tag{5.2}$$

belongs to D and agrees with $\dotplus$ on **2** (due to $\mu(x) = x$ for $x = 0, 1$). For $m > 2$ define g_m inductively by setting $g_m := g_{m-1} * g_2$. Clearly all g_m belong to D. By induction on $m \geq 2$ we show that

$$g_m(x_1,\ldots,x_m) \approx \mu(x_1) \dotplus \cdots \dotplus \mu(x_m). \tag{5.3}$$

The equation (5.2) shows the validity of (5.3) for m=2. Let $m > 2$ and suppose (5.3) holds for m-1. By the definition of g_m, (5.2), (5.3) and $\mu(x) = x$ for $x = 0, 1$ we get

$$\begin{aligned} g_m(x_1,\ldots,x_m) &\approx \mu\big(\mu(x_1) \dotplus \cdots \dotplus \mu(x_{m-1})\big) \dotplus \mu(x_m) \\ &\approx \mu(x_1) \dotplus \cdots \dotplus \mu(x_{m-1}) \dotplus \mu(x_m), \end{aligned}$$

concluding the induction step.

Finally let $f \in B_e$ be an arbitrary n-ary operation. Then (5.1) holds for some $\phi_0\colon \mathbf{2} \longrightarrow \mathbf{k}$ and even $\phi_1,\ldots,\phi_n \in T$. From $\mu(x) = x$ for $x = 0, 1$ it is immediate that

$$f(x_1,\ldots,x_n) \approx \phi_0\Big(\mu\big(\phi_1(x_1)\big) \dotplus \cdots \dotplus \mu\big(\phi_n(x_m)\big)\Big) \approx \phi_0\Big(g_n\big(\phi_1(x_1),\ldots,\phi_n(x_m)\big)\Big);$$

and so $f \in D$ proving the required $B_e \subseteq D$.

(ii) The proof is virtually the same as that of (i) but simpler since we can drop all restrictions to even operations. ∎

REMARK 5.5. Let $\phi \in O^{(1)}$ be not even and satisfy $1 < |\operatorname{im}\phi| < k$. Set $D := \{\phi\} \cup B_e$. Using $V_e \subseteq B_e$ it is easy to show that D contains some $\psi \in V \setminus V_e$. Now D contains some g_2 of the form (5.2) and proceeding as in the proof of Lemma 4.7 one can show that $V \subseteq D$. Applying Proposition 5.4(ii) we get $D \supseteq Q \cup B$.

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