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ON THE RECOVERY OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper we consider questions of recapturing an analytic function in a Banach space from its values on a uniqueness set. The principal method is to use reproducing kernels to construct a sequence in the Banach space which converges in norm to the given functions. The method works for several classical Banach spaces of analytic functions including some Hardy and Bergman spaces.

0. **Introduction.** In this paper we are considering the problem of reconstructing a vector f in one of the classical Banach spaces of analytic functions from knowledge of partial data about the vector. A general problem of this type can be stated in the following way. Let X be a Banach space and S be a subset of X^*. The set S is said to be *determining* for X if whenever $f \in X$ and $\Gamma(f) = 0$ for all $\Gamma \in S$, then f is a zero vector. Assume that S is a determining set for X and the numbers $\{\Gamma(f) : \Gamma \in S\}$ are known (the data), where f is a vector in X. Can one determine a sequence of vectors $\{f_n\}$ in X, where f_n are constructed using only the data given, and so that the sequence $\{f_n\}$ is a Cauchy sequence in X converging to f? As written the problem is intractable. One reason for this is that for all the spaces X that we consider, the space X^* is identified isometrically with another Banach space of analytic functions Y and the action of a linear functional Γ is prescribed, usually by the formula $\Gamma(f) = \int f\bar{g}$ where g is in Y and prescribing the number $\Gamma(f)$ need not yield the vector g.

Problems like this have a long history, see [10]. The most natural conditions to assume for such Banach spaces X is to assume that the functionals Γ are evaluations at points in the domain of definition of the functions in X. In this case we can identify the action of the functional with integration against a known kernel, and the determining set is a uniqueness sequence for the space. In some cases, Banach spaces are parametrized by a parameter p and form scales of reflexive Banach spaces if $p \neq 1$. Examples are the Hardy spaces H^p and the Bergman spaces A^p on the unit disc. Although there are continuous inclusions between these spaces the uniqueness sets can differ from one value of p to another. For example the zero sequences of the Bergman spaces differ for different values of p (see [6]). But it is known that for the Hardy spaces, if $S = \{\Gamma_n\}$ and $\Gamma_n(f) = f(a_n) = b_n$ where (a_n) is a subsequence of Δ, and if the Blaschke condition fails (*i.e.*, $\sum(1-|a_n|) = \infty$), then S is a determining set for $H^p(\Delta)$. If the space is such that boundary values of functions in X exist a.e. on $\partial\Delta$ then one can consider the following problem. Given a vector f in X and assuming the values of f are known on a set E of $\partial\Delta$ of positive measure, can one find a sequence of vectors in X converging to f (using only the data

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that f is known on E). This problem can be reformulated in the setting introduced above. Riesz [9] proved that in the $H^p(\Delta)$ case such an E is a "uniqueness" set and this will induce a determining sequence.

The most natural way to construct such a sequence $\{f_n\}$ is to use some kind of interpolation. A detailed bibliography on the subject may be found in [10]. Mainly the results in this area dealt with degree of convergence and convergence of interpolating sequences uniformly on compacta.

The first results towards recovering the function from its values on E was done by Carleman [2]. Using the harmonic measure on E Carleman constructed a sequence that converges to f uniformly on compacta. Patil [8] modified this construction and found a sequence which converges to f in H^p $(p > 1)$. In [1], Anderson and Cima extended this result to the case where E is a spherical "cap" part of the boundary of the unit ball in $\mathbb{C}^n$. In general setting the construction of the recapturing sequence is based on the theory of minimal interpolation. Unfortunately, little is known about minimal interpolants. The only exception is the case of $H^p(\Delta)$ (see [3] and [7]). Thus it is important to investigate the possibility of finding explicit and simple reconstruction procedures. In this paper we focus on constructing rational recapturing sequences by using the technique of reproducing kernels.

In Section 1 we prove two results. The first shows that if X is a Hilbert space of analytic functions on Δ with reproducing kernel K and if the determining set is given by evaluation functionals at points of Δ, then we can construct an explicit sequence of functions $\{f_n\}$ in X using only the known data converging in norm to a given f. The second result generalizes the theorem in [1]. Namely, if we are given Ω a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and if $E \subset \partial\Delta$, with the $(2n-1)$ Hausdorff measure of E positive, then we can produce a determining set for any f in $H^2(\Omega)$ known on E.

In Section 2 we use this result to construct a sequence converging to f in X if X is a Banach space of analytic functions with reproducing kernel and

$$\|f\|_X = \left(\int_\Delta |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}, \quad 2 < p < \infty,$$

where $d\mu(x)$ is a finite positive measure on $\bar{\Delta}$ such that $\|f_r(x)\|_X = \|f(rx)\|_X \longrightarrow \|f\|_X$ as $f \longrightarrow 1$. The significant feature of this construction is that it is independent of p. Further, if f is in X_p, then the recapturing sequence converges in X_p. If $1 \leq p \leq 2$ and the sequence $\{a_n\}$ lies in a compact part of Δ we can modify the construction given in Section 2 to obtain a recovering sequence that converges to f (in the corresponding p metric). We do this in Section 3.

In the last section we consider these problems for the Hardy spaces H^p, with $1 \leq p < 2$. We can construct recovering sequences of vectors in these spaces if we require an additional growth condition on the Blaschke sums.

Although we discuss spaces of analytic functions in the unit disk in the complex plane the results of Sections 1 through 3 hold for analytic functions in any bounded star-like domain in $\mathbb{C}^n$ with smooth boundary. (Our problem makes sense for $n > 1$ because a uniqueness set inside the domain contains a countable dense subset which is also a uniqueness set.) All proofs are literally the same. The results of the last section are based on special results and estimates for functions from Hardy spaces and so are not transferable to another domain.

1. **Recovery in Hilbert spaces with reproducing kernels.** Let X be a Hilbert space of analytic functions in Δ with reproducing kernel $K(z,w)$ holomorphic in z and antiholomorphic in w (*e.g.* H^2, A^2 or the Dirichlet space) such that

$$f(w) = \langle f(z), K(z,w)\rangle$$

for all $f \in X$. ($\langle \cdot, \cdot\rangle$ as usual denote the Hermitian inner product in X.) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in Δ which is an X-uniqueness set. Denote

$$L_n = \operatorname{span}\{K(z,a_\ell), \ell = 1,\dots,n\}. \tag{1}$$

As usual, if some point a_α is repeated and has multiplicity m, then the spanning elements in (1) corresponding to a_α are

$$K(z,a_\alpha), \quad \frac{\partial}{\partial \bar{w}} K(z,w)|_{w=a_\alpha}, \dots, \frac{\partial^{(m-1)}}{\partial \bar{w}^{m-1}} K(z,w)|_{w=a_\alpha}.$$

Further, denote by B_n the following $(n\times n)$ matrix

$$B_n = \big(K(a_i,a_j)\big)_{i,j=1}^{n}. \tag{2}$$

Given a function $f \in X$ set

$$\begin{aligned} \xi_n &= \big(f(a_1),\dots,f(a_n)\big) \in \mathbb{C}^n, \\ \eta_n &= (\eta_n^1,\dots,\eta_n^n) = (B_n^T)^{-1}\xi_n, \\ f_n(z) &= \sum_{m=1}^{n} \eta_n^m K(z,a_m) \in L_n. \end{aligned} \tag{3}$$

Let P_n be the orthogonal projection $P_n: X \longrightarrow L_n$.

PROPOSITION 1. *(1) The function f_n interpolates f at $a_1,\dots,a_n$;*
(2) $P_n f = f_n$.

PROOF. Both results are straightforward. Indeed, write $\beta_n = \big(f_n(a_1),\dots,f_n(a_n)\big)$. We have

$$\beta_n = B_n^T \eta_n = \xi_n$$

which implies (1). Furthermore,

$$\langle f - f_n, K(z,a_\alpha)\rangle = f(a_\alpha) - f_n(a_\alpha) = 0, \quad \alpha = 1,\dots,n$$

which implies (2). ∎

THEOREM 1. *f_n converges to f in X as $n \longrightarrow \infty$.*

PROOF. We have by Proposition 1

$$\|f\|_X^2 = \|f_n\|_X^2 + \|f - f_n\|_X^2, \tag{4}$$

which implies that the sequence $\{f_n\}_{n=1}^{\infty}$ is bounded in X and therefore contains a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$, $n_k \longrightarrow \infty$ that converges weakly to some function $g \in X$. Since $g(a_\alpha) = f(a_\alpha)$, $\alpha = 1, 2, \ldots$ and $\{a_n\}$ is the uniqueness set for X we conclude that $g = f$. Since a sequence of convex combinations of f_k converges to f in norm of X (see [4], p. 422), we obtain that f belongs to the closure of the convex hull of $\{f_{n_k}\}_{k=1}^{\infty}$ and, therefore,

$$\|f\|_X \leq \sup_k\{\|f_{n_k}\|_X\} \leq \sup_n\{\|f_n\|_X\}. \tag{5}$$

By (4) we have $\|f\|_X \geq \sup_n\{\|f_n\|_X\}$ which together with (5) implies

$$\|f\|_X = \sup_n\{\|f_n\|_X\}. \tag{6}$$

Further, by Proposition 1

$$\|f - f_n\|_X = \operatorname{dist}(f, L_n) \geq \operatorname{dist}(f, L_{n+1}) = \|f - f_{n+1}\|_X. \tag{7}$$

Using (4) and (7) we conclude that

$$\|f\|_X = \lim_{n \longrightarrow \infty} \|f_n\|_X,$$

which implies that

$$\|f - f_n\|_X \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \blacksquare$$

COROLLARY. *$f \longrightarrow f_n$ uniformly on compacta.*

This result is straightforward because of pointwise evaluation

$$|f(z)| \leq \|f\|_X\big(K(z,z)\big)^{1/2}.$$

REMARK. This construction can be extended to the case when $X = H_2(\Gamma)$. Assume Γ is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and E is a subset of $\partial\Gamma$ of positive $(2n-1)$ Hausdorff measure. For each $n = 0, 2, 4, \ldots$ and with $k(n) = 2^n$ choose a disjoint measurable partition $\{\mathcal{D}_i^{k(n)}\}$ which satisfies the following:

i) $\partial\Gamma = \bigcup_i^{k(n)} \mathcal{D}_i^{k(n)}$
ii) diameter$(\mathcal{D}_i^{k(n)}) \longrightarrow 0$ as $n \longrightarrow \infty$
iii) the $\mathcal{D}_i^{k(n)}$ are nested in that $\mathcal{D}_i^{k(n)}$ is the disjoint union of the $\mathcal{D}_i^{k(n+1)}$ which intersect $\mathcal{D}_i^{k(n)}$.

Let $e_i^n = E \cap \mathcal{D}_i^{k(n)}$. Denote χ_E the characteristic function of E. Then $\chi_E \in L_2(\partial\Gamma)$ and $\varphi_E \in H_2(\Gamma)$, where φ_E is the projection of χ_E to $H_2(\Gamma)$. (This projection is defined by the integration against the Szegö kernel.) It is easy to check that

$$(f, \varphi_E)_{H_2(\Gamma)} = \int_E f(\tau)\, d\tau$$

for all $f \in H_2(\Gamma)$ (we denoted the measure on $\partial\Gamma$ by $d\tau$).

Without loss of generality we may assume that

$$\operatorname{mes}(E_i^n) > 0, \quad i = 1, \dots, k_n$$
$$\operatorname{mes}(E_i^n) = 0, \quad i = k_n + 1, \dots, k(n).$$

In a similar fashion we denote

$$L_n(E) = \operatorname{span}\{\varphi_{E_i^n}, \dots, \varphi_{E_{k_n}^n}\},$$
$$B_n(E) = (\langle \varphi_{E_j^n}), \varphi_{E_i^n} \rangle)_{i,j=1}^{k_n},$$
$$\xi_n = \left(\int_{E_1^n} f(\tau)\, d\tau, \dots, \int_{E_{k_n}^n} f(\tau)\, d\tau \right),$$
$$\eta_n = \left(B_n(E)^T \right)^{-1} \xi_n,$$
$$f_n = \sum_{\ell=1}^{k_n} \eta_n^\ell \varphi_{E_\ell^n}.$$

Then f_n is the orthogonal projection f onto $L_n(E)$ and, therefore, $\|f_n\|_{H_2(\Gamma)}$ are uniformly bounded. Hence there is a subsequence $\{f_{n_k}\}$ that converges weakly in $H_2(\Gamma)$ to some function g and

$$\int_{e_i^n} g(\tau)\, d\tau = \int_{e_i^n} f(\tau)\, d\tau \quad \text{for all } n \text{ and } i = 1, \dots, k_n.$$

The last equality implies that $f|_E = g|_E$ a.e. Since E is a uniqueness set $f = g$ and we apply the reasoning of Theorem 1 to prove that f_n converges to f in $H_2(\Gamma)$. Note that for $\Gamma = \Delta$ this construction differs from Carleman's although, in fact, it also uses the harmonic measure.

2. $2 < p < \infty$. Let $d\mu$ be a nonnegative probability measure on $\bar{\Delta}$. We do not exclude the case when $\mu(\partial\Delta) > 0$ ("Hardy" case). Let X_p, $1 \leq p < \infty$ be a space of analytic function in Δ such that

$$\|f\|_{X_p} = \sup_{0<r<1} \left(\int_{\bar{\Delta}} |f_r(x)|^p \, d\mu(x) \right)^{1/p} < \infty.$$

If $\mu(\partial\Delta) > 0$ we require that f has boundary values on $\operatorname{supp}(\mu) \cap \partial\Delta$. Assume X_p has a reproducing kernel $K(z, w)$, that is

$$f(w) = \int_{\bar{\Delta}} f(z) \overline{K(z, w)}\, d\mu(z)$$

for all $f \in X_p$ and $K(z, w) \in X_p$ for all $w \in \Delta$, $1 \leq p \leq \infty$. If $p = 2$, then X_2 is a Hilbert space. Since the measure μ is finite we have

$$X_{p_1} \subset X_{p_2} \quad \text{if } p_1 > p_2$$

and, therefore, any uniqueness set for X_{p_0} is a uniqueness set for X_p if $p > p_0$.[1]

Let $a = \{a_k\}_{k=1}^{\infty}$ be a uniqueness set for X_2. Denote

(8) $$c(a,n,r) = \sup\{|f(z)| : |z| \le r, \|f\|_{X_2} \le 1, f(a_1) = \cdots = f(a_n) = 0\}.$$

PROPOSITION 2. *$c(a,n,r) \longrightarrow 0$ as $n \longrightarrow \infty$.*

PROOF. First we note that the problem (8) has a solution. Denote this solution f_n^r. There is a subsequence $\{f_{n_k}^r\}$ weakly converging in X_2. If f^* is the weak limit, then f^* vanishes at $a_1, \ldots, a_n, \ldots$ which implies $f^* \equiv 0$. Since the sequence $\{c(a,n,r)\}_{n=1}^{\infty}$ is monotone decreasing as $n \longrightarrow \infty$ the required result follows. ∎

Fix two sequences $r_k \nearrow 1$ and $\varepsilon_k \searrow 0$ as $k \longrightarrow \infty$ and choose a sequence of natural numbers $\{N_k\}_{k=1}^{\infty}$ that satisfies the following conditions:

(1) $N_k \nearrow \infty$ as $k \longrightarrow \infty$

(2) $c(a, N_k, r_k) < \varepsilon_k$.

Now we define the sequence $\{f_n(z)\}$ by

(9) $$f_k(z) = \sum_{\ell=1}^{N_k} \eta_{N_k}^{\ell} K(r_k z, a),$$

where $\eta_{N_k}^{\ell}$ are defined by (3).

THEOREM 2. *If $a = \{a_k\}_{k=1}^{\infty}$ is a uniqueness set for X_2 and $f \in X_p$, $2 \le p < \infty$, then the sequence (9) converges to f in X_p.*

PROOF. Denote $\Delta_\varrho = \{z \in \mathbb{C} : |z| < \varrho\}$. The measure μ defines the measure μ_ϱ on $\bar{\Delta}_\varrho$ by $\mu_\varrho(C) = \mu(\frac{1}{\varrho}C)$ for all measurable subsets C. Note that the function $f_r(z) = f(rz)$, $0 < r < 1$, belongs to $X_p(\Delta_{1/r}, \mu_{1/r})$ and

$$\|f_r\|_{X_p(\Delta_{1/r},\mu_{1/r})} = \|f\|_{X_p}.$$

Moreover, $K_r(z,w) = K(rz, rw)$ is the reproducing kernel for $X_p(\Delta_{1/r}, \mu_{1/r})$. Let us denote $a_n^k = \frac{a_n}{r_k}$. We obviously have

$$f_{r_k}(a_n^k) = f(a_n),$$

and, therefore, the function (9) interpolates f_{r_k} at $a_1^k, \ldots, a_{N_k}^k$. By Proposition 1, it is the orthogonal projection of f_{r_k} onto $\operatorname{span}\{K_{r_k}(\cdot, a_1^k), \ldots, K_{r_k}(\cdot, a_{N_k}^k)\}$. Hence $\|f_k\|_{X_2(\Delta_{\frac{1}{r_k}}, \mu_{\frac{1}{r_k}})} \le \|f_{r_k}\|_{X_2(\Delta_{\frac{1}{r_k}}, \mu_{\frac{1}{r_k}})} \le \|f\|_{X_2} \le \|f\|_{X_p}$.

Now we obtain

$$\begin{aligned}
&\sup_{|z|\le 1} |f_{r_k}(z) - f_k(z)| \\
&\quad \le \sup\{|g(z)| : |z| \le 1, \|g\|_{X_2(\Delta_{\frac{1}{r_k}}, \mu_{\frac{1}{r_k}})} \le 2\|f\|_{X_p}, g(a_1^k) = \cdots = g(a_{N_k}^k) = 0\} \\
&\quad = \sup\{|g(z)| : |z| \le r_k, g \in X_2, \|g\|_{X_2} \le 2\|f\|_p, g(a_1) = \cdots g(a_{N_k}) = 0\} \\
&\quad = 2\|f\|_p \cdot c(z, N_k, r_k) < 2\|f\|_p \varepsilon_k.
\end{aligned}$$

[1] The converse statement is in general false. For example, the Bergman spaces zero sets for functions of A_p are different for different p (see [6]).

Since the total mass of the measure μ is finite the last inequality implies

$$\|f-f_k\| \leq \|f-f_{r_k}\|_{X_p} + \|f_{r_k}-f_k\|_{X_p} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \qquad \blacksquare$$

REMARK. It follows from the proof of Theorem 2 that the same result holds for functions f from the disk-algebra $A(\bar{\Delta})$. In this case f_k converges to f uniformly on $\bar{\Delta}$.

3. **Compact uniqueness sets.** If the sequence $\{a_n\}_{n=1}^{\infty}$ lies in the compact part of Δ we can modify (9) to obtain the sequence that converges to f in X_p for an arbitrary $1 \leq p < \infty$.

Denote

$$d(r,p) = \sup\{|f(z)| : |z| \leq r, \quad \|f\|_{X_p} \leq 1\},$$

where $0 < r < 1$.

Again, let $r_k \nearrow 1$, $\varepsilon_k \searrow 0$ as $k \longrightarrow \infty$ and let $\varrho_k = \frac{1+r_k}{2}$. We may assume that $\sup_n |a_n| < r_1$. Choose a sequence $\{M_k\}_{k=1}^{\infty}$ of natural numbers such that

(1) $M_k \nearrow \infty$ as $k \longrightarrow \infty$;

(2) $c(a, M_k, \frac{2r_k}{1+r_k}) < \frac{\varepsilon_k}{d(\varrho_k, p)}$

where $c(a, M_k, \frac{2r_k}{1+r_k})$ is defined by (8). Proposition 2 provides the possibility of such a choice.

Set

$$B_k = \left(K\left(\frac{a_i}{\varrho_k}, \frac{a_j}{\varrho_k}\right)\right)_{i,j=1}^{M_k}$$

and define

$$g_k(z) = \sum_{\ell=1}^{M_k} \eta_k^{\ell} K\left(\frac{r_k z}{\varrho_k}, \frac{a_\ell}{\varrho_k}\right) \tag{10}$$

where $\eta_k = (\eta_k^1, \ldots, \eta_k^{M_k})$ is defined by $\eta_k = (B_k^T)^{-1}\xi_{M_k}$.

THEOREM 3. *If $\{a_n\}_{n=1}^{\infty}$ is a uniqueness set for X_p ($1 \leq p < \infty$) which lies in the compact part of Δ (i.e. an infinite sequence with a limit point in Δ) and $f \in X_p$, then the sequence (10) converges to f in X_p.*

PROOF. If $f \in X_p$, then

$$\sup_{|z| \leq \varrho_k} |f(z)| \leq \|f\|_{X_p} d(\varrho_k, p)$$

and, therefore,

$$\|f_{r_k}\|_{X_2(\Delta_{\frac{\varrho_k}{r_k}}, \mu_{\frac{\varrho_k}{r_k}})} = \|f\|_{X_2(\Delta_{\varrho_k}, \mu_{\varrho_k})} \leq \|f\|_{X_p} d(\varrho_k, p).$$

As in the proof of Theorem 2, set $a_k^n = \frac{a_n}{r_k}$. Consider the function $f_{r_k}(z)$ as an element of $X_2(\Delta_{\frac{\varrho_k}{r_k}}, \mu_{\frac{\varrho_k}{r_k}})$, then (10) represents the interpolation of f_{r_k} by the linear combination of kernels $K(\frac{r_k}{\varrho_k}z, \frac{r_k}{\varrho_k}a_k^\ell)$, $\ell = 1, \ldots, M_k$ at the points $a_k^1, \ldots, a_k^{M_k}$. Since $K(\frac{r_k}{\varrho_k}z, \frac{r_k}{\varrho_k}w)$ is the reproducing kernel in $X_2(\Delta_{\frac{\varrho_k}{r_k}}, \mu_{\frac{\varrho_k}{r_k}})$, we have by Proposition 1

$$\|g_k\|_{X_2(\Delta_{\frac{\varrho_k}{r_k}}, \mu_{\frac{\varrho_k}{r_k}})} \leq \|f\|_{X_2(\Delta_{\frac{\varrho_k}{r_k}}, \mu_{\frac{\varrho_k}{r_k}})} \leq \|f\|_{X_p} d(\varrho_k, p).$$

Thus

$$\begin{aligned}\sup_{|z|\leq 1} |f_{r_k}(z) - g_k(z)| &\leq 2\|f\|_{X_p} d(\varrho_k, p) c\Big(a, M_k, \frac{r_k}{\varrho_k}\Big) \\ &= 2\|f\|_{X_p} d(\varrho_k, p) c\Big(a, M_k, \frac{2r_k}{1+r_k}\Big) \\ &< 2\|f\|_{X_p} \varepsilon_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.\end{aligned}$$

Finally we have

$$\|f - g_k\|_{X_p} \leq \|f - f_{r_k}\|_{X_p} + \|f_{r_k} - g_k\|_{X_p} \leq \|f - f_{r_k}\| + \sup_{|z|\leq 1} |f_{r_k}(z) - g_k(z)| \longrightarrow 0$$

as $k \longrightarrow \infty$. ■

4. **Recovery in H_p.** Let $X_p = H_p(\Delta)$, $1 \leq p < 2$. In this case we can slightly modify (10) to construct a recovering sequence even in the case when $|a_n| \longrightarrow 1$ as $n \longrightarrow \infty$, provided that $|a_n|$ converges to 1 rather slowly.

Without loss of generality we may assume that the sequence $\{|a_n|\}_{n=1}^{\infty}$ increases and

(11)

$$|a_1| = |a_2| = \cdots = |a_{k_1}| < |a_{k_1+1}| = \cdots = |a_{k_2}| < |a_{k_2+1}| = \cdots = |a_{k_n}| < |a_{k_{n+1}}| = \cdots.$$

Suppose additionally that $\{a_n\}$ satisfies

$$\inf_n (1 - |a_n|) n^{(\frac{1}{2} - \varepsilon)} = c(a) > 0 \tag{12}$$

for some $\varepsilon > 0$.

Let us denote $B_{a_1,\ldots,a_n}(z)$ the Blaschke product vanishing at $a_1, \ldots, a_n$.

LEMMA. *If $|\xi| \geq \frac{1}{2}$, then*

$$|B_{a_1,\ldots,a_n}(\xi)| \leq \exp\Big\{-\frac{1 - |\xi|}{1 + |\xi|} \sum_{\ell=1}^{n} (1 - |a_\ell|)\Big\} \tag{13}$$

PROOF. We have

$$\log\left(|B_{a_1,\dots,a_n}(\xi)|\right) = \frac{1}{2}\sum_{\ell=1}^{n}\log\left|\frac{\xi-a_i}{1-\bar{a}_i\xi}\right|^2 \leq \frac{1}{2}\sum_{\ell=1}^{n}\log\left(\frac{|\xi|+|a_\ell|}{1+|\xi|\,|a_\ell|}\right)^2$$
$$= \frac{1}{2}\sum_{\ell=1}^{n}\log\left(1-\left(1-\left(\frac{|\xi|+|a_\ell|}{1+|\xi|\,|a_\ell|}\right)^2\right)\right)$$
$$= -\frac{1}{2}\sum_{\ell=1}^{n}\sum_{k=1}^{\infty}\frac{(1-|\xi|^2)^k(1-|a_\ell|^2)^k}{k(1+|\xi|\,|a_\ell|)^{2k}}$$
$$\leq -\frac{1}{2}\sum_{\ell=1}^{n}\frac{(1-|\xi|^2)(1-|a_\ell|^2)}{(1+|\xi|\,|a_\ell|)^2}$$
$$= -\frac{1-|\xi|^2}{2}\sum_{\ell=1}^{n}(1-|a_\ell|)\frac{1+|a_\ell|}{(1+|\xi|\,|a_\ell|)^2}.$$

Now we note that for $0 < x < 1$

$$\left(\frac{1+x}{(1+|\xi|x)^2}\right)' = \frac{1-2|\xi|-|\xi|x}{(1+|\xi|x)^3} < 0$$

and, therefore, $\frac{1+|a_\ell|}{(1+|\xi|\,|a_\ell|)^2} \geq \frac{2}{(1+|\xi|)^2}$. ∎

To construct a recovering sequence we shall use the following result.

If $f \in H_p$, $\|f\|_{H_p} \leq 1$, $f(a_1) = \cdots = f(a_k) = 0$, then

$$|f(\xi)| \leq \frac{|B_{a_1,\dots,a_k}(\xi)|}{(1-|\xi|^2)^{1/p}}. \tag{14}$$

Now, choose a sequence $\delta_m \searrow 0$ and let $\{n_m\}_{m=1}^{\infty}$ be a sequence of natural numbers such that

(1) $n_m \nearrow \infty$ as $m \longrightarrow \infty$
(2) $\exp\left\{-\frac{(c(a))^2}{8}n_m^{2\varepsilon} + 2\log n_m\right\} < \delta_m$.

Define $(n_m \times n_m)$-matrix B_m and $\eta_m \in \mathbb{C}^{n_m}$ by

$$B_m = (b_{ij}^m) = \left(\frac{1}{1-(1+\frac{c(a)}{n_m})^2 a_i\bar{a}_j}\right)_{i,j=1}^{n_m} \tag{15}$$
$$\eta_m = (\eta_m^1,\dots,\eta_m^{n_m}) = (B_m^T)^{-1}\xi_{n_m}.$$

where, as before, ξ_{n_m} is the vector of known values $(f(a_1),\dots,f(a_{n_m}))$.

Now we set

$$h_m(z) = \sum_{\ell=1}^{n_m}\eta_m^\ell\frac{1}{1-(1+\frac{c(a)}{n_m})^2|a_{n_m}|z\bar{a}_\ell}. \tag{16}$$

THEOREM 4. *If the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies (11), (12) and $f \in H_p$, $p \geq 1$, then the sequence (16) converges to f in H_p.*

PROOF. Denote $r_m = |a_{n_m}|$, $\varrho_m = \frac{1}{r_m(1+\frac{c(a)}{n_m})}$. It is easy to check that $\varrho_m > 1$. Further, the kernel of $C_{\varrho_m}(z,w) = \frac{1}{1-\frac{z\bar{w}}{\varrho_m^2}}$ is the reproducing kernel for $H_p(\Delta_{\varrho_m})$ and, therefore, the matrix (15) is

$$B_m = \left(C_{\varrho_m}\left(\frac{a_i}{r_m}, \frac{a_j}{r_m}\right)\right)_{i,j=1}^{n_m}$$

and (16) represents the interpolation of the function $f_{r_m}(z)$ by the linear combination of $C_{\varrho_m}(z, \frac{a_\ell}{r_m})$. By Proposition 1

$$\begin{aligned}
\|h_m\|_{H_2(\Delta_{\varrho_m})} &\le \|f_{r_m}\|_{H_2(\Delta_{\varrho_m})}\\
&\le \max_{|\xi|=\varrho_m} |f_{r_m}(\xi)|\\
&= \max_{|\xi|=\frac{1}{1+\frac{c(a)}{n_m}}} |f(\xi)|\\
&\le \|f\|_p \frac{1}{(1-|\xi|^2)^{1/p}}\\
&\le \left(\frac{2}{c(a)}\right)^{1/p} \|f\|_p (n_m)^{1/p}
\end{aligned}$$

Thus, $\|f_{r_m} - h_m\|_{H_2(\Delta_{\varrho_m})} \le \frac{4}{(c(a))^{1/p}} \|f\|_p (n_m)^{1/p}$ and, since $(f_{r_m} - h_m)$ vanishes at $\frac{a_1}{r_m}, \ldots, \frac{a_n}{r_m}$, (13) and (14) imply

$$\begin{aligned}
&\sup_{|z|=1} |f_{r_m}(z) - h_m(z)|\\
&\le \|f_{r_m} - h_m\|_{H_2(\Delta_{\varrho_m})}\\
&\qquad \times \max_{|z|=1}\left\{|\varphi(z)| : \varphi \in H_2(\Delta_{\varrho_m}), \|\varphi\|_{H_2(\Delta_{\varrho_m})} \le 1, \varphi\left(\frac{a_1}{r_m}\right) = \cdots = \varphi\left(\frac{(a_{n_m})}{r_m}\right) = 0\right\}\\
&= \|f_{r_m} - h_m\|_{H_2(\Delta_{\varrho_m})}\\
&\qquad \times \max_{|z|=r_m(1+\frac{c(a)}{n_m})}\left\{|\varphi(z)| : \varphi \in H_2(\Delta),\right.\\
&\qquad\qquad \left.\|\varphi\|_{H_2(\Delta)} \le 1, \varphi\left(a_i\left(1+\frac{c(a)}{n_m}\right)\right) = 0, i \in 1, \ldots, n_m\right\}\\
&\le \frac{4}{\big(c(a)\big)^{1/p}} \|f\|_p (n_m)^{1/p} \frac{1}{\big(1 - |a_{n_m}|(1+\frac{c(a)}{n_m})\big)^{1/2}}\\
&\qquad \times \exp\left\{-\frac{1-(1+\frac{c(a)}{n_m})|a_{n_m}|}{1+(1+\frac{c(a)}{n_m})|a_{n_m}|} \sum_{\ell=1}^{n_m}\left(1 - \left(1+\frac{c(a)}{n_m}\right)|a_\ell|\right)\right\}\\
&= \frac{4}{\big(c(a)\big)^{1/p}} \|f\|_p (n_m)^{1/p} \frac{1}{\big((1+\frac{c(a)}{n_m})(1-|a_{n_m}|) - \frac{c(a)}{n_m}\big)^{1/2}}\\
&\qquad \times \exp\left\{-\frac{(1+\frac{c(a)}{n_m})(1-|a_{n_m}|) - \frac{c(a)}{n_m}}{(1+\frac{c(a)}{n_m})(1+|a_{n_m}|) - \frac{c(a)}{n_m}}\left(\left(1+\frac{c(a)}{n_m}\right)\sum_{\ell=1}(1-|a_\ell|) - c(a)\right)\right\}.
\end{aligned}$$

By (12) we have $(1-|a_n|) \geq \frac{c(a)}{n^{1/2-\varepsilon}}$ for all n which implies that if m is sufficiently large then

$$\sum_{\ell=1}^{n_m}(1-|a_\ell|) \geq c(a)\sum_{\ell=1}^{n_m}\frac{1}{n_m^{1/2-\varepsilon}} \geq \frac{c(a)}{2}n_m^{\varepsilon+1/2},$$

$$\left(1+\frac{c(a)}{n_m}\right)(1-|a_{n_m}|)-\frac{c(a)}{n_m} \geq \frac{c(a)}{2n_m^{1/2-\varepsilon}}.$$

Finally we obtain

$$\begin{aligned}\sup_{|z|=1}|f_{r_m}(z)-h_m(z)| &\leq \frac{8\|f\|_p}{\big(c(a)\big)^{1/2+1/p}}(n_m)^{\frac14+\frac1p-\frac{\varepsilon}{2}}\exp\left\{-\frac{\big(c(a)\big)^2}{8}n_m^{2\varepsilon}\right\}\\ &= \frac{8\|f\|_p}{\big(c(a)\big)^{1/2+1/p}}\cdot\exp\left\{-\frac{\big(c(a)\big)^2}{8}n_m^{2\varepsilon}+\left(\frac14+\frac1p-\frac{\varepsilon}{2}\right)\log n_m\right\}\\ &\leq \frac{8\|f\|_p}{\big(c(a)\big)^{1/2+1/p}}\cdot\delta_m.\end{aligned}$$

The rest of the proof is standard:

$$\|f-h_m\|_p \leq \|f-f_{r_m}\|_p+\frac{8\|f\|_p}{\big(c(a)\big)^{1/2+1/p}}\cdot\delta_m \longrightarrow 0 \quad \text{as } m\longrightarrow\infty. \qquad \blacksquare$$

REMARK. It follows from the proof of Theorem 4 that the condition (12) might be weakened to

$$\inf_n(1-|a_n|)\frac{\sqrt{n}}{\varphi(n)\log n} > 0$$

where φ is any increasing function on $[1,\infty)$, $\varphi(x)\longrightarrow\infty$ as $x\longrightarrow\infty$.

The following construction allows recovery even in the case when the condition (12) does not hold, but the sequence $\{a_n\}_{n=1}^{\infty}$ lies in an internally tangent disk, say

(17) $$a_n \in \Delta(\alpha)=\{z\in\mathbb{C} : |z-\alpha|<1-\alpha\}, \quad 0<\alpha<1,\ n=1,\ldots.$$

Let us denote

(18) $$a_n' = \frac{a_n-\alpha}{1-\alpha}.$$

If $\{a_n\}_{n=1}^{\infty}$ satisfies (17) then it is easily seen that $|a_n'|<1$ and $\sum_1^\infty(1-|a_n'|)=\infty$.

Given the function $f(z)\in H_p(\Delta)$ we define $g(z)$ by

(19) $$g(z)=f(z)\cdot(1-z)^2.$$

The estimate

$$|f(z)| \leq \frac{\|f\|_p}{(1-|z|)^{1/p}}$$

implies that the restriction $g(z)$ to $\Delta(\alpha)$ is a bounded function. This restriction defines the operator $A_\alpha : H_\infty(\Delta) \longrightarrow H_\infty(\Delta)$

$$A_\alpha g(z) = g\big((1-\alpha)z + \alpha\big). \tag{20}$$

The following result by Gabriel [5] is important for our construction:

$$\|A_\alpha g\|_p \le (1+\alpha)\|g\|_p. \tag{21}$$

Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive numbers which decreases to zero as $n \longrightarrow \infty$ and let N_k be defined by the conditions

(1) $N_k \nearrow \infty$ as $k \longrightarrow \infty$

(2)

$$\exp\Big\{-\frac{\alpha_k}{2}\sum_{\ell=1}^{N_k}(1-|a'_\ell|)\Big\} < \alpha_k^2. \tag{22}$$

Again set

$$B_{N_k} = \Big\|\frac{1}{1-\bar a'_i \bar a'_j}\Big\|_{i,j=1}^{N_k},$$

and let η_{N_k} be defined by (3) and

$$h_k^{(z)} = \sum_{\ell=1}^{N_k}\frac{\eta_{N_k}^\ell}{1-\bar a'_\ell(1-\alpha_k)\big((1-\alpha_k)z-\alpha_k\big)}. \tag{23}$$

THEOREM 5. *If $f \in H_p(\Delta)$ and $\{a_n\}_{n=1}^\infty$ and $\{a'_n\}_{n=1}^\infty$ satisfy (17), (18) respectively, then $\{h_k(z)\}_{k=1}^\infty$ converges to $g(z) = f(z)(1-z)^2$ as $k \longrightarrow \infty$.*

PROOF. First we note that for $f \in H_p(\Delta)$ the sequence $\{A_{\alpha_n} f\}_{n=1}^\infty$ converges weakly to $g(z)$ as $n \longrightarrow \infty$ and, by (21), it converges to $g(z)$ in $H_p(\Delta)$. Denote

$$\varphi_k(z) = g_{1-\alpha_k}\big((1-\alpha_k)z + \alpha_k\big) = g\big((1-\alpha_k)^2 z + \alpha_k(1-\alpha_k)\big) = A_{\alpha_k} g_{1-\alpha_k}.$$

We have

$$\begin{aligned}\|g-h_k\|_p &\le \|g - A_{\alpha_k} g\|_p + \|A_{\alpha_k} g - \varphi_k\|_p + \|\varphi_k - h_k\|_p \\ &= \|g - A_{\alpha_k} g\|_p + \|A_{\alpha_k}(g - g_{1-\alpha_k})\|_p + \|\varphi_k - h_k\|_p.\end{aligned} \tag{24}$$

The first term of the right-hand side of (24) tends to zero as we mentioned above; the second term tends to zero by (21). Now let us prove that $(\varphi_k - h_k) \searrow 0$ as $k \longrightarrow \infty$ uniformly on $\bar\Delta$. Let $\tilde\Delta_k = \{z \in \mathbb{C} : |z - \frac{\alpha_k}{1-\alpha_k}| < \frac{1}{(1-\alpha_k)^2}\}$ be the image of the unit disk under the linear transformation

$$C_k : k \longmapsto \frac{1}{(1-\alpha_k)^2}z + \frac{\alpha_k}{(1-\alpha_k)}. \tag{25}$$

Then $\varphi_k(z) = g\big((1-\alpha_k)^2 z + \alpha_k(1-\alpha_k)\big)$ is holomorphic and bounded in $\tilde\Delta_k$. The transformation (25) maps the disk $\Delta'_k = \{z \in \mathbb{C} : |z = \alpha_k(1-\alpha_k)| \le (1-\alpha_k)^2\}$ onto the unit

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disk. We consider $\varphi_k(z)$ as an element of $H_2(\tilde{\Delta}_k)$. It is clear that $\|\varphi_k\|_{H_2(\tilde{\Delta}_k)} = \|g\|_{H_2(\Delta_k)}$. Further,

$$\tilde{K}_k(z,w) = \frac{1}{1-(1-\alpha_k)^2\big((1-\alpha_k)z-\alpha_k\big)\big((1-\alpha)\bar{w}-\alpha_k\big)}$$

is the reproducing kernel for the Hardy spaces in $\tilde{\Delta}_k$. It is easy to check that (23) is the linear combination of $\tilde{K}_k(z,a''_\ell)$, $\ell = 1,\ldots,H_k$, that interpolates φ_k at the points $a''_\ell = C_k(a'_\ell)$, $\ell = 1,\ldots,N_k$. By Proposition 1

$$\|h_k\|_{H_2(\tilde{\Delta}_k)} \leq \|\varphi_k\|_{H_2(\tilde{\Delta}_k)} = \|g\|_{H_2(\Delta_k)}$$

and therefore by (13), (14) we obtain

$$\sup_{|z|\leq 1} |\varphi(z) - h_k(z)| \leq 2\|g\|_{H_2(\Delta_k)} \cdot \frac{1}{(1-\tau)^{1/2}} \exp\Big\{-\frac{1-\tau}{2}\sum_{\ell=1}^{N_k}(1-|a'_\ell|)\Big\}$$

where $\tau = \sup\{|z| : z \in \Delta'_k\} = 1-\alpha_k$. Finally, by (22)

$$\begin{aligned}\sup_{|z|\leq 1} |\varphi(z) - h_k(z)| &\leq 2\|g\|_{H_2(\Delta_k)}\frac{1}{\alpha_k} \exp\Big\{-\frac{\alpha_k}{2}\sum_{\ell=1}^{N_k}(1-|a'_\ell|)\Big\}\\ &\leq 2\|g\|_{H_2(\Delta_k)} \cdot \alpha_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.\end{aligned}$$

∎

As a final piece of evidence to support the veracity of our general result in H_1 consider the following

PROPOSITION 3. *Assume $\{a_n\}_{n=1}^\infty$ is a uniqueness set for H_1 accumulating at a point ξ_0, $|\xi_0| = 1$. If f is in H_1 and there is a neighborhood of ξ_0 in which f does not take the value b in the neighborhood then we can construct a sequence $\{f_n\}_{n=1}^\infty$ in H_1 using only the values $\{f(a_n)\}_{n=1}^\infty$ which converges to f in H_1.*

PROOF. Without loss of generality assume $\{a_n\}$ is determining, $a_n \longrightarrow 1$ and $f(w) \neq b$ in the unit disc. The function

$$F(w) = f(w) - b$$

has a holomorphic square root in the disc. Hence, if we know $f(a_n)$ we can determine the values of H_2-function

$$G(w) = \sqrt{F(w)}$$

on $\{a_n\}$. It is now possible by our earlier result to determine $\{g_n\}$ in H_2 with

$$\|g_n - G\|_{H_2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Since

$$\|g_n^2 - F\| \leq 2(\sup_n \|g_n\|_{H_2})\|g_n - G\|_{H_2}.$$

Thus we can find a sequence in H_1 converging in norm to f.

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