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PF-RINGS OF GENERALISED POWER SERIES

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Let R be a commutative ring and $(S, \leqslant)$ a strictly ordered monoid which satisfies the condition that $0 \leqslant s$ for every $s \in S$. We show that the generalised power series ring $[[R^{S,\leqslant}]]$ is a PF-ring if and only if R is a PF-ring.

1. INTRODUCTION AND PRELIMINARIES

Let R be a commutative ring. Recall that R is a PF-ring if every projective R-module is free. A famous result of Quillen and Suslin independently states that for a field F, every finitely generated projective $F[x_1, \ldots, x_n]$-module is free. In [**1**], it was proved that $R[[x_1, \ldots, x_n]]$ is a PF-ring if and only if R is a PF-ring. In this paper, we shall prove that the generalised power series ring $[[R^{S,\leqslant}]]$ is a PF-ring if and only if R is a PF-ring, where $(S, \leqslant)$ is a strictly ordered monoid which satisfies the condition that $0 \leqslant s$ for every $s \in S$. As an application, we obtain some new examples of PF-rings.

All rings considered here are commutative with identity. Any concept and notation not defined here can be found in [**5, 6, 7**]. For a ring R, we denote by $U(R)$ and $J(R)$ the multiplicative group of units, and the Jacobson radical of R, respectively.

Let $(S, \leqslant)$ be an ordered set. Recall that $(S, \leqslant)$ is *Artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leqslant)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [**5, 6, 7**].

Let $(S, \leqslant)$ be a *strictly ordered* monoid (that is, $(S, \leqslant)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a commutative ring. Let $A = [[R^{S,\leqslant}]]$ be the set of all maps $f : S \longrightarrow R$ such that $supp(f) = \{s \in S \mid f(s) \neq 0\}$ is Artinian and narrow. With pointwise addition, A is an Abelian additive group. For every $s \in S$ and $f_1, \ldots, f_m \in A$, let $X_s(f_1, \ldots, f_m) = \{(u_1, \ldots, u_m) \in S^m \mid s = u_1 + \cdots + u_m,\ f_1(u_1) \neq 0, \ldots, f_m(u_m) \neq 0\}$. It follows from [**6**, 1.16] that $X_s(f_1, \ldots, f_m)$ is finite. This fact allows us to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v).$$

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With this operation, and pointwise addition, A becomes a commutative ring, which is called *the ring of generalised power series.* The elements of A are called *generalised power series* with coefficients in R and exponents in S.

For example, if $S = \mathbb{N}$ and $\leqslant$ is the usual order, then $[[R^{\mathbb{N},\leqslant}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leqslant$ is the trivial order, then $[[R^{S,\leqslant}]] = R[S]$, the monoid-ring of S over R. Further examples are given in [**5**]. Many results on $[[R^{S,\leqslant}]]$ have been obtained in [**2, 3, 4, 5, 6, 7**].

We shall use the following notations introduced by Ribenboim in [**5**].

Let $a \in R$. Define a mapping $c_a \in [[R^{S,\leqslant}]]$ as follows:

$$c_a(0) = a, \quad c_a(s) = 0, \quad 0 \neq s \in S.$$

Let $s \in S$. Define a mapping $e_s \in [[R^{S,\leqslant}]]$ as follows:

$$e_s(s) = 1, \quad e_s(t) = 0, \quad s \neq t \in S.$$

Then R is canonically embedded as a subring of $[[R^{S,\leqslant}]]$, and S is canonically embedded as a submonoid of $\left([[R^{S,\leqslant}]] - \{0\}, \bullet\right)$. It is easy to see that e_0 is the identity of $[[R^{S,\leqslant}]]$.

2. Main Results

We shall henceforth assume that $(S, \leqslant)$ is a strictly ordered monoid which satisfies the condition:

$$\text{(SO)} \qquad 0 \leqslant s \quad \text{for every} \quad s \in S.$$

Lemma 2.1. [**6**] *Let $f \in [[R^{S,\leqslant}]]$. Then $f \in U\left([[R^{S,\leqslant}]]\right)$ if and only if $f(0) \in U(R)$.*

Corollary 2.2. *Let $f \in [[R^{S,\leqslant}]]$. Then f is in $J\left([[R^{S,\leqslant}]]\right)$ if and only if $f(0)$ is in $J(R)$.*

Proof: Suppose that $f(0) \in J(R)$. Then $1 - rf(0) \in U(R)$ for every $r \in R$. For each $g \in [[R^{S,\leqslant}]]$, we have $(gf)(0) = \sum_{(u,v)\in X_0(g,f)} g(u)f(v) = g(0)f(0)$ by the condition (S0). Thus $(e_0 - gf)(0) = e_0(0) - (gf)(0) = 1 - g(0)f(0) \in U(R)$. By Lemma 2.1, it follows that $e_0 - gf \in U\left([[R^{S,\leqslant}]]\right)$, which means that $f \in J\left([[R^{S,\leqslant}]]\right)$.

Conversely suppose that $f \in J\left([[R^{S,\leqslant}]]\right)$. For every $r \in R$, $e_0 - c_r f \in U\left([[R^{S,\leqslant}]]\right)$. Thus, by Lemma 2.1, $1 - rf(0) = (e_0 - c_r f)(0) \in U(R)$, and so $f(0) \in J(R)$. ∎

Proposition 2.3. *There exists a group isomorphism $K_0[[R^{S,\leqslant}]] \cong K_0 R$.*

PROOF: Since $(S, \leqslant)$ satisfies the condition (S0), it is easy to see that for any $f, g \in [[R^{S,\leqslant}]]$, $(fg)(0) = \sum_{(u,v)\in X_0(f,g)} f(u)g(v) = f(0)g(0)$. Thus there exist ring homomorphisms

$$\alpha : [[R^{S,\leqslant}]] \longrightarrow R$$
$$f \mapsto f(0)$$

and

$$\beta : R \longrightarrow [[R^{S,\leqslant}]]$$
$$r \mapsto c_r.$$

Clearly $\alpha\beta = 1_R$. Thus $K_0\alpha$ is a surjective homomorphism. Let $f \in \operatorname{Ker}(\alpha)$. Then $f(0) = 0 \in J(R)$. By Corollary 2.2, it follows that $f \in J\big([[R^{S,\leqslant}]]\big)$. This means that $\operatorname{Ker}(\alpha) \subseteq J\big([[R^{S,\leqslant}]]\big)$. Thus, by [**8**, Proposition 9], $K_0\alpha$ is a monomorphism. Now the result follows. ∎

We note that the group isomorphism above is also a ring isomorphism since the rings we considered are commutative (see [**1**]).

A ring R is called a Hermite ring provided for every $(r_1, \dots, r_n) \in R^n$, if there exists $(p_1, \dots, p_n) \in R^n$ such that $r_1p_1 + \dots + r_np_n = 1$, then there exists a $n \times n$ matrix M over R with first row $(r_1, \dots, r_n)$ and $\det(M)$ a unit in R.

PROPOSITION 2.4. *$[[R^{S,\leqslant}]]$ is a Hermite ring if and only if R is a Hermite ring.*

PROOF: Let $[[R^{S,\leqslant}]]$ is a Hermite ring. Suppose that $(r_1, \dots, r_n)$ and $(p_1, \dots, p_n)$ are in R^n such that $r_1p_1 + \dots + r_np_n = 1$. Since $(c_{r_1}c_{p_1} + \dots + c_{r_n}c_{p_n})(s) = \sum_{i=1}^{n} (c_{r_i}c_{p_i})(s) = \sum_{i=1}^{n} \sum_{(u,v)\in X_s(c_{r_i}, c_{p_i})} c_{r_i}(u)c_{p_i}(v) = 0 = e_0(s)$ when $s \neq 0$, and $(c_{r_1}c_{p_1} + \dots + c_{r_n}c_{p_n})(0) = (c_{r_1}c_{p_1})(0) + \dots + (c_{r_n}c_{p_n})(0) = r_1p_1 + \dots + r_np_n = 1 = e_0(0)$, we have

$$c_{r_1}c_{p_1} + \dots + c_{r_n}c_{p_n} = e_0.$$

Since $[[R^{S,\leqslant}]]$ is a Hermite ring, there exists a $n \times n$ matrix M over $[[R^{S,\leqslant}]]$ with first row $(c_{r_1}, \dots, c_{r_n})$ and $\det(M)$ a unit in $[[R^{S,\leqslant}]]$. Suppose that

$$M = \begin{pmatrix} c_{r_1} & c_{r_2} & \dots & c_{r_n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{pmatrix}.$$

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Denote

$$N = \begin{pmatrix} r_1 & r_2 & \dots & r_n \\ f_{21}(0) & f_{22}(0) & \dots & f_{2n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(0) & f_{n2}(0) & \dots & f_{nn}(0) \end{pmatrix}.$$

Since S satisfies the condition (S0), it is easy to see that

$$\begin{aligned} \det(M)(0) &= \Bigl(\sum_{i_1 \dots i_n} (-1)^{\pi(i_1 \dots i_n)} c_{r_{i_1}} f_{2i_2} \dots f_{ni_n}\Bigr)(0) \\ &= \sum_{i_1 \dots i_n} (-1)^{\pi(i_1 \dots i_n)} r_{i_1} f_{2i_2}(0) \dots f_{ni_n}(0) = \det(N). \end{aligned}$$

By Lemma 2.1, it follows that $\det(N) \in U(R)$. Thus R is a Hermite ring.

Conversely suppose that R is a Hermite ring. Assume that $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ are in $[[R^{S,\leqslant}]]^n$ such that $\sum_{i=1}^{n} f_i g_i = e_0$, the identity of ring $[[R^{S,\leqslant}]]$. Then

$$\sum_{i=1}^{n} f_i(0) g_i(0) = 1.$$

Since R is a Hermite ring, there exists a $n \times n$ matrix

$$P = \begin{pmatrix} f_1(0) & f_2(0) & \dots & f_n(0) \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{pmatrix}$$

over R with first row $\bigl(f_1(0), \dots, f_n(0)\bigr)$ and $\det(P) \in U(R)$. Let

$$Q = \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ c_{r_{21}} & c_{r_{22}} & \dots & c_{r_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r_{n1}} & c_{r_{n2}} & \dots & c_{r_{nn}} \end{pmatrix}.$$

Then, by condition (S0), it is easy to see that $\bigl(\det(Q)\bigr)(0) = \det(P) \in U(R)$. Thus, by Lemma 2.1, it follows that $\det(Q) \in U\bigl([[R^{S,\leqslant}]]\bigr)$. This means that $[[R^{S,\leqslant}]]$ is a Hermite ring. ∎

Now we have:

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THEOREM 2.5. *Let $(S, \leqslant)$ be a strictly ordered monoid which satisfies the condition that $0 \leqslant s$ for every $s \in S$. Then $[[R^{S,\leqslant}]]$ is a PF-ring if and only if R is a PF-ring.*

PROOF: It is well-known that a commutative ring A is a PF-ring if and only if A is a Hermite ring and there exists a ring isomorphism $K_0A \cong \mathbb{Z}$ (see, for example, [**9**]). Thus the result follows from Proposition 2.3 and 2.4. ∎

COROLLARY 2.6. [1] *$R[[x_1, \dots, x_n]]$ is a PF-ring if and only if R is a PF-ring.*

PROOF: Let $S = \mathbb{N} \times \cdots \times \mathbb{N}$ (n copies) with the product of the usual order. Then $[[R^{S,\leqslant}]] \cong R[[x_1, \dots, x_n]]$. Now the result follows from Theorem 2.5. ∎

The following corollaries will give other examples of PF-rings.

COROLLARY 2.7. *Let $\mathbb{Q}^+ = \{a \in \mathbb{Q} \mid a \geqslant 0\}, \mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geqslant 0\}$. Then the rings $[[\mathbb{Z}^{\mathbb{N},\leqslant}]], [[\mathbb{Z}^{\mathbb{Q}^+,\leqslant}]]$ and $[[\mathbb{Z}^{\mathbb{R}^+,\leqslant}]]$ are PF-rings, where $\leqslant$ is the usual order.*

COROLLARY 2.8. *Let $(S_1, \leqslant_1), \dots, (S_n, \leqslant_n)$ be strictly ordered monoids which satisfy the condition that $0 \leqslant_i s$ for every $s \in S_i, i = 1, \dots, n$. Denote by $(lex \leqslant)$ and $(revlex \leqslant)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_n$. Then R is a PF-ring if and only if $[[R^{S_1 \times \cdots \times S_n, (lex \leqslant)}]]$ is a PF-ring if and only if $[[R^{S_1 \times \cdots \times S_n, (revlex \leqslant)}]]$ is a PF-ring.*

PROOF: It is easy to see that $(S_1 \times \cdots \times S_n, (lex \leqslant))$ is a strictly ordered monoid which satisfies the condition that $(0, \dots, 0)(lex \leqslant)(s_1, \dots, s_n)$ for every $(s_1, \dots, s_n) \in S_1 \times \cdots \times S_n$. Thus, by Theorem 2.5, R is a PF-ring if and only if $[[R^{S_1 \times \cdots \times S_n, (lex \leqslant)}]]$ is a PF-ring.

The proof of the another assertion is similar. ∎

Let R be a ring, and consider the multiplicative monoid $\mathbb{N}_{\geqslant 1}$, endowed with the usual order $\leqslant$. Then $A = [[R^{\mathbb{N}_{\geqslant 1},\leqslant}]]$ is the ring of arithmetical functions with values in R, endowed with the Dirichlet convolution:

$$(fg)(n) = \sum_{d|n} f(d)g(n/d), \quad \text{for each} \quad n \geqslant 1.$$

COROLLARY 2.9. *$[[R^{\mathbb{N}_{\geqslant 1},\leqslant}]]$ is a PF-ring if and only if R is a PF-ring.*

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