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The fundamental gap of a kind of sub-elliptic operator

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In this paper the minimum fundamental gap of a kind of sub-elliptic operator is concerned, we deal with the existence and uniqueness of weak solution for that. We verify that the minimization fundamental gap problem can be achieved by some function, and characterize the optimal function by adopting the differential of eigenvalues.

Keywords: Fundamental gap; sub-elliptic operator; weak solution; optimal function

1. Introduction

Numerous works on the fundamental gap of the first two eigenvalues have been developed in the past few decades, in particular, the fundamental gap is defined as $\lambda_2 - \lambda_1$, where λ_1 and λ_2 represent the first and second eigenvalues of the given differential equation, respectively.

Most of these investigations are focused on the Schrödinger equation, which has been discussed in a variety of situations. In quantum mechanics, the size for fundamental gap is extremely crucial, if it is small enough, it will produce the well-known tunnelling effect, so the research on this problem is of great significance. Ashbaugh *et al.* [**4**] proposed the existence and characterization for minimal and maximal fundamental gaps when the potential is constrained by various L^p norm, moreover, Karaa extended this work on [**19**]. Chern and Shen determine [**12**] the minimum fundamental gap with certain restrictions on the class of potential functions. The estimation of the upper and lower bounds of the fundamental gap has also attracted more attention of scholars. In particular, note that the fundamental gap bound related to single-well potential with transition point as midpoint is considered in [**16**], and the symmetric potential is imposed by various constraints in [**17**]. Yu [**36**] introduced the bound for the fundamental gap under Dirichlet and Neumann boundary condition, where the single-well potential with transition point to be not midpoint. Andrews and Clutterbuck [**3**] solved the gap conjecture $\lambda_2 - \lambda_1 \geqslant \frac{3\pi^2}{d^2}$ via an ingenious way, where d is the diameter of domain and the potential is weakly convex. For more such topics, we may refer to [**10**, **15**, **18**, **22**, **23**].

The eigenvalue problem for degenerate elliptic equations has also attracted extensive attention. The author for [**24**] considered a general class of eigenvalue problems where the leading elliptic term corresponds to a convex homogeneous energy function that is not necessarily differentiable, and they derived uniqueness of the first eigenfunction. Stuart [**31**] developed bifurcation from the line of trivial solutions for a nonlinear eigenvalue problem on a bounded open subset $\Omega \in R^n$ with $n \geqslant 3$, and introduced a critically degenerate elliptic Dirichlet problem in [**32**] for $n \geqslant 2$. For more aspects relating to degenerate elliptic equations, we can learn from other references [**7**–**9**, **26**, **34**]. Meanwhile, degenerate parabolic equation theory has flourished over the past years, interested readers can refer [**1**, **6**, **27**], in particular, [**6**] established a Lebeau–Robbiano spectral inequality for a degenerate one-dimensional elliptic operator.

With regard to sub-elliptic operator, Xu [**35**] considered the existence and the regularity for the minimum points of a certain variational problem, the properties of weak solutions are presented in detail for some sub-elliptic operator in [**28**, **29**]. In terms of eigenvalues, Chen and Luo [**11**] studied the lower bound estimates for the j-th eigenvalue for some degenerate elliptic operators for $n \geqslant 2$.

However, the fundamental gap of sub-elliptic operator had been a largely under explored domain, too little work has been devoted to it, especially for the degenerate sub-elliptic operator even in the one-dimensional case. Inspired by the above researches, the goal of this paper is to study the minimization problem $\inf\limits_{V\in\mathcal{V}}(\lambda_2(V) - \lambda_1(V))$ about the following sub-elliptic equation:

$$\begin{cases} -\left(h^2(x)u'\right)' + V(x)u = \lambda u, & x \in (0,1), \\ u = 0, & x \in \{0,1\}, \end{cases} \tag{1.1}$$

where $h \in C^1[0,\,1]$ with $h(0) = 0$ and $h(x) > 0$, $x \in (0,\,1]$, and

$$V \in \mathcal{V} = \{V \in L^\infty(0,1) \mid m \leqslant V \leqslant M \text{ a.e. in } (0,1)\}, \tag{1.2}$$

where $0 < m < M$ are two given constants.

And most remarkably, in some of the latest literature [**2**, **20**, **21**], Kerner [**20**, **21**] introduced the explicit lower bound on the fundamental gap of one-dimensional Schrödinger operators with non-negative bounded potentials with Neumann boundary conditions, which operator is a uniform elliptic operator, and the class of potential function $V(x)$ is different from ours. Allali and Harrell [**2**] provided lower bounds for the gap of Sturm–Liouville problem with general single-well potential $V(x)$ with a transition point a, without any restriction on a, and also for the case where the potential is convex, while $V(x)$ demands certain monotonicity or convexity, which is also different from our work.

The remaining part of the paper proceeds as follows: § 2 is devoted to the study the weak solution space of sub-elliptic equation. Under certain assumptions, we propose a more profound characterization of the weak solution space in § 3. Section 4 begins by laying out the characteristic of the eigenvalue and the corresponding eigenfunction of (1.1), and looks at the differential of eigenvalue. Section 5 presents the existence of the minimum fundamental gap problem and the performance of the optimal function.

2. Existence of weak solution

Prior to commencing the study of eigenvalue problem for (1.1), we first shall look for the proper weak solution space of the equation:

$$\begin{cases} -\left(h^2(x)u'\right)' + V(x)u = f(x), & x \in (0,1), \\ u = 0, & x \in \{0,1\}, \end{cases} \tag{2.1}$$

where $f \in L^2(0, 1)$ are given functions. On the basis of [**1**, **6**, **25**, **29**, **30**], we provide a more comprehensive analysis of the weak solution space for this sub-elliptic operator. We remark that in the next work, the existence and uniqueness of a solution for problem (2.1) follows from the Lax–Milgram Theorem.

Throughout this paper, we let $\|\cdot\|$ denote the norm, $(\cdot, \cdot)$ denote the inner product and define

$$H^1(0,1;h) = \left\{u \in \mathcal{D}'(\Omega) \mid u \in L^2(0,1), hu' \in L^2(0,1)\right\}, \tag{2.2}$$

where

$$(u,v) = \int_0^1 uv\mathrm{d}x + \int_0^1 h^2u'v'\mathrm{d}x, \tag{2.3}$$

and

$$\|u\| = \left(\int_0^1 |u|^2\mathrm{d}x + \int_0^1 |hu'|^2\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.4}$$

Theorem 2.1. *Let $H^1(0, 1; h)$, $(\cdot, \cdot)$, $\|\cdot\|$ be defined as (2.2), (2.3) and (2.4), respectively. Then $(H^1(0, 1; h), (\cdot, \cdot))$ is a Hilbert space with norm $\|\cdot\|$. Moreover,*

$$H^1(0,1;h) = \left\{u \in \mathcal{D}'(\Omega) \mid u \in L^2(0,1), (hu)' \in L^2(0,1)\right\}, \tag{2.5}$$

and hence $hu \in H^1(0, 1)$ for all $u \in H^1(0, 1; h)$.

Proof. For the first point, it is evidently that $(H^1(0, 1; h), (\cdot, \cdot))$ is an inner product space and $\|\cdot\|$ is a norm on $H^1(0, 1; h)$. Besides, the expression (2.5) is deduced from

$$u \in L^2(0,1), hu' \in L^2(0,1) \;\Leftrightarrow\; u \in L^2(0,1), (hu)' = h'u + hu' \in L^2(0,1).$$

For the second point, let $\{u_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $H^1(0, 1; h)$. i.e.,

$$\|u_k - u_n\|_{L^2}^2 + \|hu_k' - hu_n'\|_{L^2}^2 = \|u_k - u_n\|_{H_1(0,1;h)}^2 \to 0 \text{ as } k, n \to \infty,$$

which implies that there exists u, $v \in L^2(0, 1)$ such that

$$u_k \to u \text{ and } hu_k' \to v \text{ in } L^2(0,1).$$

In addition,

$$(hu_k)' = h'u_k + hu_k' \to h'u + v \text{ in } L^2(0,1),$$

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we observe that for each $\phi \in C_c^\infty(0,\,1)$:

$$\int_0^1 (hu)\phi'\mathrm{d}x = \lim_{k\to\infty}\int_0^1 (hu_k)\phi'\mathrm{d}x = -\lim_{k\to\infty}\int_0^1 (hu_k)'\phi\mathrm{d}x = -\int_0^1 (h'u+v)\phi\mathrm{d}x,$$

one can find that

$$h'u + hu' = (hu)' = h'u + v \text{ in } \mathcal{D}'(0,1),$$

that is $hu' = v$ in $\mathcal{D}'(0,\,1)$, hence $u_k \to u$ in $H^1(0,\,1;h)$ is obtained.

For the third point, as $u \in H^1(0,\,1;h)$, we have $(hu)' \in L^2(0,\,1)$ by invoking (2.5), and thus

$$\|hu\|_{H^1(0,1)}^2 = \|hu\|_{L^2}^2 + \|(hu)'\|_{L^2}^2 \leqslant \sup_{x\in[0,1]} h^2(x)\|u\|_{L^2}^2 + \|(hu)'\|_{L^2}^2 < \infty$$

in line with the definition of $h(x)$. Overall, these results indicate that $hu \in H^1(0,\,1)$. □

Remark 2.2. Let $h = x$, then $\sqrt{x} \in H^1(0,\,1;h)$, but $\sqrt{x} \notin H^1(0,\,1)$. So, $H^1(0,\,1) \subsetneq H^1(0,\,1;h)$.

Proposition 2.3. *Let $u \in H^1(0,\,1;h)$, then $u \in C^{0,\frac{1}{2}}(0,\,1]$ and $hu \in C^{0,\frac{1}{2}}[0,\,1]$.*

Proof. Take $\delta \in (0,\,1)$, thanks to

$$\left(\inf_{x\in(\delta,1)} h(x)\right)\|u'\|_{L^2(\delta,1)} \leqslant \|hu'\|_{L^2(\delta,1)} < \infty,$$

and employing the fact $\inf_{x\in[\delta,1]} h(x) > 0$, we readily check $\|u'\|_{L^2(\delta,1)} < \infty$, consequently, $u \in H^1(\delta,\,1)$. And applying the Sobolev Embedding Theorem to find that $u \in C^{0,\frac{1}{2}}[\delta,\,1]$, due to the arbitrary of $\delta > 0$, we deduce that $u \in C^{0,\frac{1}{2}}(0,\,1]$.

Finally, we have $hu \in H^1(0,\,1)$ by theorem 2.1, then $hu \in C^{0,\frac{1}{2}}[0,\,1]$ by utilizing the Sobolev Embedding Theorem again. □

For $u \in H^1(0,\,1;h)$, we easily check that $u(0+) = \lim_{x\downarrow 0} u(x)$ exists and it is an extended real numbers, $(hu)(0) = \lim_{x\downarrow 0} h(x)u(x)$ by proposition 2.3. Take into account the boundary condition, we shall have to gain further results.

Definition 2.4. *Define*

$$H_0^1(0,1;h) = \{u \in H^1(0,1;h) \mid (hu)(0) = u(1) = 0\} \tag{2.6}$$

in the sense of

$$(hu)(0) = \lim_{x\downarrow 0} h(x)u(x), \quad u(1) = \lim_{x\uparrow 1} u(x).$$

Remark 2.5. Indeed, definition 2.4 is well-defined owing to proposition 2.3. It is obvious that $H_0^1(0,\,1;h)$ is a subspace of $H^1(0,\,1;h)$ and $C_c^\infty(0,\,1) \subset H_0^1(0,\,1;h)$.

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Moreover, from remark 2.2 we know that $H_0^1(0,\,1) \subsetneq H_0^1(0,\,1;h)$. Note that if $u \in H_0^1(0,\,1;h)$, then $hu \in H_0^1(0,\,1)$.

LEMMA 2.6. *The space* $H_0^1(0,\,1;h)$ *is a Hilbert space with inner product* $(\cdot,\,\cdot)$ *in* $H^1(0,\,1;h)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(0,\,1;h)$ be a Cauchy sequence, then $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(0,\,1;h)$, there exists $u \in H^1(0,\,1;h)$ such that

$$u_n \to u \text{ in } H^1(0,1;h) \tag{2.7}$$

from theorem 2.1. On the other side, since $\{hu_n\}_{n\in\mathbb{N}} \subset H_0^1(0,\,1)$ and

$$\|hu_n - hu_m\|^2_{H_0^1(0,1)} \leqslant C\|u_n - u_m\|^2_{L^2} + \|hu_n' - hu_m'\|^2_{L^2} \to 0 \text{ as } n, m \to \infty,$$

there exists $v \in H_0^1(0,\,1)$ such that

$$hu_n \to v \text{ in } H_0^1(0,1) \text{ as } n \to \infty.$$

This together with (2.7), that yields $v = hu$ on $(0,\,1)$, and along with the fact that $(hu)(0) = u(1) = 0$, so $u \in H_0^1(0,\,1;h)$ is derived. □

LEMMA 2.7. *Generally, the space* $H_0^1(0,\,1;h)$ *cannot be embedded compactly in* $L^2(0,\,1)$.

Proof. Assume $h = x^2$, for all $n \in \mathbb{N}$, let

$$u_n(x) = \begin{cases} -n^{\frac{3}{2}}\left(x - \dfrac{1}{n}\right), & x \in \left[0, \frac{1}{n}\right], \\ 0, & x \in \left(\frac{1}{n}, 1\right]. \end{cases}$$

Then

$$\int_0^1 |u_n(x)|^2 \mathrm{d}x = \int_0^{\frac{1}{n}} n^3 \left(x - \frac{1}{n}\right)^2 \mathrm{d}x = \frac{1}{3},$$
$$\int_0^1 |hu_n'(x)|^2 \mathrm{d}x = \int_0^{\frac{1}{n}} x^2 n^3 \left(x - \frac{1}{n}\right)^2 \mathrm{d}x = \frac{1}{5n^2},$$

and

$$\lim_{x\to 0^+} h(x)u_n(x) = \lim_{x\to 0^+} x^2 \left[-n^{\frac{3}{2}}\left(x - \frac{1}{n}\right)\right] = 0, \quad \lim_{x\to 1^-} u_n(x) = 0,$$

which implies that $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(0,\,1;h)$.

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Suppose that there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$, and $u \in L^2(0,\, 1)$, such that

$$u_{n_k} \to u \text{ strongly in } L^2(0,1),$$

then $\|u\|^2_{L^2(0,1)} = \frac{1}{3}$, and

$$\|hu'\|_{L^2(0,1)} = \lim_{n\to\infty} \|hu'_n\|_{L^2(0,1)} = 0.$$

Furthermore, there exists a subsequence of $\{u_{n_k}\}_{k\in\mathbb{N}}$, still denoted by itself, such that

$$u_{n_k} \to u \text{ a.e. in } (0,1).$$

In view of the definition of $u_{n_k}(k \in \mathbb{N})$, the above conclusion implies that $u = 0$ a.e. in (0, 1), but it contradicts to the fact $\|u\|^2_{L^2(0,1)} = \frac{1}{3}$. □

Definition 2.8. *We call $u \in H^1_0(0,\, 1; h)$ is a weak solution of* (2.1)*, provided*

$$B(u, v) = (f, v)_{L^2(0,1)}, \forall\ v \in H^1_0(0, 1; h),$$

where

$$B(u, v) = \int_0^1 h^2 u'v' \mathrm{d}x + \int_0^1 V uv \mathrm{d}x,\ \ u, v \in H^1_0(0, 1; h). \tag{2.8}$$

Now we rely on the specific bilinear form $B(u,\, v)$ for $u,\, v \in H^1_0(0,\, 1; h)$ defined in (2.8) to test the hypothesis of Lax–Milgram Theorem.

Theorem 2.9. *Let $f \in L^2(0,\, 1)$ be arbitrarily given function, then equation* (2.1) *has a unique solution in $H^1_0(0,\, 1; h)$.*

Proof. For all $v \in H^1_0(0,\, 1; h)$, we have

$$|(f, v)_{L^2(0,1)}| \leqslant \|f\|_{L^2(0,1)} \|v\|_{L^2(0,1)} \leqslant \|f\|_{L^2(0,1)} \|v\|_{H^1_0(0,1;h)},$$

and hence $f : H^1_0(0,\, 1; h) \to \mathbb{R},\ v \mapsto (f,\, v)_{L^2(0,1)}$ is a bounded linear function on $H^1_0(0,\, 1; h)$.

Next, we shall explain that $B(\cdot,\, \cdot)$ satisfies the Lax–Milgram Theorem. Indeed, through applying Cauchy inequality we obtain

$$\begin{aligned} B(u, v) &= \int_0^1 h^2 u'v' \mathrm{d}x + \int_0^1 V uv \mathrm{d}x \leqslant \|hu'\|_{L^2(0,1)} \|hv'\|_{L^2(0,1)} + M \|u\|_{L^2(0,1)} \|v\|_{L^2(0,1)} \\ &\leqslant \max\{M, 1\} \left(\|hu'\|^2_{L^2(0,1)} + \|u\|^2_{L^2(0,1)} \right)^{\frac{1}{2}} \left(\|hv'\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)} \right)^{\frac{1}{2}} \\ &= \max\{M, 1\} \|u\|_{H^1_0(0,1;h)} \|v\|_{H^1_0(0,1;h)}, \end{aligned}$$

and

$$\begin{aligned} B(u, u) = \int_0^1 h^2 |u'|^2 \mathrm{d}x + \int_0^1 V u^2 \mathrm{d}x &\geqslant \|hu'\|^2_{L^2(0,1)} + m\|u\|^2_{L^2(0,1)} \\ &\geqslant \frac{1}{2} \min\{m, 1\} \|u\|^2_{H^1_0(0,1;h)}. \end{aligned}$$

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All of these show that $B(\cdot, \cdot)$ satisfies the Lax–Milgram Theorem. On the basis of the mentioned analysis and Lax–Milgram Theorem, we see that there exists a unique $u \in H_0^1(0,\, 1; h)$ such that

$$B(u, v) = (f, v)$$

for all $v \in H_0^1(0,\, 1; h)$. □

LEMMA 2.10. *Consider*

$$L : L^2(0, 1) \to L^2(0, 1), \quad f \mapsto u,$$

where $u \in H_0^1(0,\, 1; h)$ *is the unique solution of* (2.1), *then* L *is a self-adjoint bounded linear operator.*

Proof. Employing (2.8) and

$$B(u, u) = (f, u)_{L^2(0,1)} \leqslant \|f\|_{L^2(0,1)} \|u\|_{L^2(0,1)} \leqslant \|f\|_{L^2(0,1)} \|u\|_{H_0^1(0,1;h)},$$

for which we deduce that $\|u\|_{H_0^1(0,1;h)} \leqslant C\|f\|_{L^2(0,1)}$, and thereby

$$\|Lf\|_{L^2(0,1)} = \|u\|_{L^2(0,1)} \leqslant \|u\|_{H_0^1(0,1;h)} \leqslant C\|f\|_{L^2(0,1)}.$$

Evidently, the above formula provides evidence that $L : L^2(0,\, 1) \to L^2(0,\, 1)$ is a bounded linear operator. On other side,

$$(Lf, g)_{L^2(0,1)} = (u, g)_{L^2(0,1)} = B(u, v) = (f, v)_{L^2(0,1)} = (f, Lg)_{L^2(0,1)}$$

for all $f,\, g \in L^2(0,\, 1)$, where $u,\, v$ are the solutions of (2.1) with respect to f and g respectively. Finally, we explicitly note that $L^* = L$, i.e., L is a self-adjoint operator. □

3. Further results

In this section and later sections, we further assume that

$$\|h^{-1}\|_{L^2(0,1)} \leqslant C, \tag{3.1}$$

actually, there are many functions that satisfy (3.1).

Let us consider first the boundary-value problem (2.1) with the assumption (3.1), our overall plan is first to define and then construct an appropriate weak solution u of (3.1) and only later to investigate the eigenvalue problem and other properties of u.

DEFINITION 3.1. *Denote*

$$\mathrm{Lip}_c(0,1) = \left\{ u : (0,1) \to \mathbb{R} \;\middle|\; \begin{array}{l} u \textit{ is a Lipschitz continuous function on } (0,1), \\ \textit{and } \mathrm{supp}\, u \textit{ is compact in } (0,1), \end{array} \right\},$$

and we define

$$H_c^1(0, 1; h) = \overline{\mathrm{Lip}_c(0, 1)}^{H^1(0,1;h)}.$$

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LEMMA 3.2. *The space*

$$H_c^1(0,1;h) = \overline{C_c^\infty(0,1)}^{H^1(0,1;h)}.$$

Proof. As we can see the definition above has states that $\overline{C_c^\infty(0,\,1)}^{H^1(0,1;h)} \subset H_c^1(0,\,1;h)$, it suffices to prove that, $H_c^1(0,\,1;h) \subset \overline{C_c^\infty(0,\,1)}^{H^1(0,1;h)}$. Indeed, for all $u \in H_c^1(0,\,1;h)$, there exists a sequence $u_n \in Lip_c(0,\,1)$, where u_n is a Lipschitz continuous function on $(0,\,1)$ and suppu_n is compact in $(0,\,1)$, then u_n' exists a.e. on $(0,\,1)$ and $|u_n'| \leqslant C$ a.e. on $(0,\,1)$, where C is the Lipschitz constant of u_n, these facts illustrate that $u_n \in H_0^1(0,\,1)$ with compact support on $(0,\,1)$. It is well known that there exists $\{v_n\}_{n\in\mathbb{N}} \subset C_c^\infty(0,\,1)$ such that

$$\|v_n - u_n\|_{H_0^1(0,1)} \to 0,\ n \to \infty,$$

and we have

$$\|v_n - u_n\|_{H^1(0,1;h)} \leqslant C\|v_n - u_n\|_{H_0^1(0,1)} \to 0,\ n \to \infty$$

by the definition of $h(x)$, finally,

$$\|v_n - u\|_{H^1(0,1;h)} \leqslant \|v_n - u_n\|_{H^1(0,1;h)} + \|u_n - u\|_{H^1(0,1;h)} \to 0,\ n \to \infty,$$

so the desired result is verified. □

LEMMA 3.3. *The space $H_c^1(0,\,1;h)$ is a complete subspace of $H^1(0,\,1;h)$, and $u(0) = u(1) = 0$ for any $u \in H_c^1(0,\,1;h)$. Moreover, $H_c^1(0,\,1;h)$ can be embedded compactly in $L^2(0,\,1)$.*

Proof. 1. Let $\{u_n\}_{n\in\mathbb{N}} \subset H_c^1(0,\,1;h)$ be a Cauchy sequence, then there exists $u \in H^1(0,\,1;h)$ such that

$$u_n \to u \text{ in } H^1(0,1;h) \text{ as } n \to \infty.$$

Hence, for arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for all $n \geqslant N$ we have $\frac{1}{2^n} < \frac{\epsilon}{2}$ and

$$\|u_n - u\|_{H^1(0,1;h)} < \frac{\epsilon}{2}.$$

Note that for each $n \in \mathbb{N}$, there exists $\phi_n \in C_c^\infty(0,\,1)$, such that

$$\|u_n - \phi_n\|_{H^1(0,1;h)} < \frac{1}{2^n} < \frac{\epsilon}{2},$$

these evidences suggest that

$$\|u - \phi_n\|_{H^1(0,1;h)} \leqslant \|u - u_n\|_{H^1(0,1;h)} + \|u_n - \phi_n\|_{H^1(0,1;h)} < \epsilon$$

for all $n \geqslant N$. i.e., $u \in H_c^1(0,\,1;h)$.

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2. For any $u \in H_c^1(0,\,1;h)$, there exists $\phi_n \in C_c^\infty(0,\,1)$ such that

$$\|u-\phi_n\|_{H^1(0,1;h)} \to 0, \quad n\to\infty,$$

and consider that

$$\|u-\phi_n\|_{H^1(\delta,1)} \leqslant \left(\min\left\{\inf_{x\in[\delta,1]} h(x), 1\right\}\right)^{-1} \|u-\phi_n\|_{H^1(0,1;h)},$$

hence using Sobolev Embedding Theorem, for any $\delta\in(0,\,1)$,

$$C_1\|u-\phi_n\|_{C^{0,\frac{1}{2}}[\delta,1]} \leqslant \|u-\phi_n\|_{H^1(\delta,1)} \leqslant C_2\|u-\phi_n\|_{H^1(0,1;h)}. \tag{3.2}$$

Clearly, $\lim\limits_{n\to\infty}\phi_n(1)=u(1)=0$. Besides, it should be noted here that according to the integration absolutely continuous of h^{-1}, for arbitrary $\epsilon>0$, there exists $\xi>0$, for all $E\subset(0,\,1)$, $|E|<\xi$, we have

$$\int_E |h^{-1}(t)|^2 \mathrm{d}t < \epsilon^2. \tag{3.3}$$

Then for all $\delta\in(0,\,1)$, since

$$\begin{aligned} |\phi_n(\delta)-\phi_m(\delta)| &= |\int_\delta^1 \phi_n'(x)-\phi_m'(x)\mathrm{d}x| \leqslant \int_\delta^1 |h^{-1}h(\phi_n'(x)-\phi_m'(x)|\mathrm{d}x \\ &\leqslant \left(\int_\delta^1 |h^{-1}|^2\mathrm{d}x\right)^{\frac{1}{2}} \|h(\phi_n'-\phi_m')\|_{L^2(\delta,1)} \\ &\leqslant \epsilon\|h(\phi_n'-\phi_m')\|_{L^2(0,1)} \to 0, n, m\to\infty, \end{aligned}$$

which shows that ϕ_n is uniformly convergent, we have $\lim\limits_{n\to\infty}\lim\limits_{\delta\to 0}\phi_n(\delta) = \lim\limits_{\delta\to 0}\lim\limits_{n\to\infty}\phi_n(\delta)=0$, furthermore, we have $u(0)=\lim\limits_{\delta\to 0}u(\delta)=0$ from (3.2) and proposition 2.3.

3. Let $\{u_n\}_{n\in\mathbb{N}} \subset H_c^1(0,\,1;h)$ be a bounded set. On one hand, note that

$$\|u_n\|_{H^1(\delta,1)} \leqslant \left(\min\left\{\inf_{x\in[\delta,1]} h(x), 1\right\}\right)^{-1} \|u_n\|_{H^1(0,1;h)}$$

for all $\delta\in(0,\,1)$, as a consequence,

$$C_1\|u_n\|_{C^{0,\frac{1}{2}}[\delta,1]} \leqslant \|u_n\|_{H^1(\delta,1)} \leqslant C_2\|u_n\|_{H^1(0,1;h)} \tag{3.4}$$

by Sobolev Embedding Theorem.

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Due to (3.3), for any $0 < x < y < 1$, we have

$$|u_n(x) - u_n(y)| = \left|\int_x^y u_n'(t)\mathrm{d}t\right| \leqslant \int_x^y |u_n'(t)|\mathrm{d}t \leqslant \int_x^y h^{-1}h|u_n'(t)|\mathrm{d}t$$
$$\leqslant \left(\int_x^y |h^{-1}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \|hu_n'\|_{L^2(0,1)}$$
$$< \epsilon\|hu_n'\|_{L^2(0,1)} \leqslant C\epsilon,$$

for all $|x - y| < \xi$, x, $y \in (0,\, 1)$. This implies that $\{u_n\}_{n\in\mathbb{N}} \subset C[0,\, 1]$ is a bounded and equicontinuous sequence according to $u \in C^{0,\frac{1}{2}}(0,\, 1]$ and $u(0) = u(1) = 0$ for all $u \in H_c^1(0,\, 1; h)$. The Arzela–Asoli Theorem tells us that there exists a convergent subsequence of $\{u_n\}_{n\in\mathbb{N}}$, still denoted by itself, and $u_0 \in C[0,\, 1]$, such that

$$u_n \to u_0 \text{ in } C[0,1] \text{ as } n \to \infty,$$

that is a sure sign that

$$u_n \to u_0 \text{ strongly in } L^2(0,1) \text{ as } n \to \infty.$$

□

Lemma 3.4. *There is a relationship among the space* $H_0^1(0,\, 1)$, $H_c^1(0,\, 1; h)$, $H_0^1(0,\, 1; h)$ *and* $H^1(0,\, 1; h)$ *by*

$$H_c^1(0,1;h) \subsetneqq H_0^1(0,1;h) \subsetneqq H^1(0,1;h),$$
$$H_0^1(0,1) \subsetneqq H_c^1(0,1;h), h(x) \in C^1(0,1].$$

Proof. That is easily verifiable for $H_c^1(0,\, 1; h) \subsetneqq H_0^1(0,\, 1; h) \subsetneqq H^1(0,\, 1; h)$ based on lemma 3.3 and the definition of $H_0^1(0,\, 1; h)$.

For $H_0^1(0,\, 1) \subsetneqq H_c^1(0,\, 1; h)$, we give an example. Let $h(x) = x^{\frac{1}{4}}$, $g_1(x) = \sqrt{x}$, $g_2(x) = (x-1)^2$, let $f(x)$ be infinitely differentiable satisfying $f^{(k)}(\frac{1}{2}) = g_1^{(k)}(\frac{1}{2})$ and $f^{(k)}(\frac{3}{4}) = g_2^{(k)}(\frac{3}{4})$, where $(\cdot)^{(k)}$ is the k-th derivative of $(\cdot)$, $k = 0,\, 1,\, 2\cdots$. Consider

$$u(x) = \begin{cases} g_1(x), & x \in \left[0, \frac{1}{2}\right], \\ f(x), & x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ g_2(x), & x \in \left[\frac{3}{4}, 1\right], \end{cases}$$

then

$$\|u'(x)\|_{L^2(0,1)}^2 = \int_0^{\frac{1}{2}} |\frac{1}{2\sqrt{x}}|^2\mathrm{d}x + \int_{\frac{1}{2}}^{\frac{3}{4}} |f'(x)|^2\mathrm{d}x + \int_{\frac{3}{4}}^1 4|x-1|^2\mathrm{d}x \to \infty,$$

clearly, $u \notin H_0^1(0,\, 1)$. Now let $\xi \in C_c^\infty(R)$ satisfy

$$0 \leqslant \xi \leqslant 1, \xi|_{[0,1]} \equiv 1, \quad \xi|_{R-[-1,2]} \equiv 0,$$

and let

$$u_m(x) = \begin{cases} u(x)[1 - \xi(mx)], & x \in [0, \frac{1}{2}], \\ u(x)[1 - \xi(mx + 1 - m)], & x \in [\frac{1}{2}, 1], \end{cases}$$

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where $m > 4$. We observe that $mx > 2$ if $x \in (\frac{2}{m}, \frac{1}{2}]$ and $mx + 1 - m < -1$ if $x \in [\frac{1}{2}, 1 - \frac{2}{m})$. According to the definition of ξ, we have $\xi = 0$ on $(\frac{2}{m}, 1 - \frac{2}{m})$, so that $u_m(x) = u(x)$ on $(\frac{2}{m}, 1 - \frac{2}{m})$. See that

$$\begin{aligned}\int_0^1 |u_m - u|^2 \mathrm{d}x &= \int_0^{\frac{2}{m}} |u(x)[1 - \xi(mx)] - u(x)|^2 \mathrm{d}x \\ &\quad + \int_{1-\frac{2}{m}}^1 |u(x)[1 - \xi(mx + 1 - m)] - u(x)|^2 \mathrm{d}x \\ &= \int_0^{\frac{2}{m}} |u(x)\xi(mx)|^2 \mathrm{d}x + \int_{1-\frac{2}{m}}^1 |u(x)\xi(mx + 1 - m)|^2 \mathrm{d}x \\ &\leqslant \int_0^{\frac{2}{m}} x dx + \int_{1-\frac{2}{m}}^1 |x - 1|^4 \mathrm{d}x \leqslant \frac{2}{m^2} + \left(\frac{2}{m}\right)^5 \to 0, m \to \infty,\end{aligned}$$

since $\xi(mx)$, $\xi(mx + 1 - m) \leqslant 1$. Moreover, since $u'_m = u'(1 - \xi) - mu\xi'$ and $\xi = \xi' = 0$ on $(\frac{2}{m}, 1 - \frac{2}{m})$, we have $u'_m(x) = u'(x)$ on $(\frac{2}{m}, 1 - \frac{2}{m})$. Then

$$\begin{aligned}\int_0^1 |hu'_m - hu'|^2 \mathrm{d}x &= \int_0^1 |hu'(1 - \xi) - hmu\xi' - hu'|^2 \mathrm{d}x = \int_0^1 | - hu'\xi - hmu\xi'|^2 \mathrm{d}x \\ &\leqslant 2\int_0^1 |hu'\xi|^2 \mathrm{d}x + 2\int_0^1 |hmu\xi'|^2 \mathrm{d}x \\ &\leqslant 2\int_0^{\frac{2}{m}} |\frac{1}{2} x^{\frac{1}{4}} x^{-\frac{1}{2}}|^2 \mathrm{d}x + 2\int_{1-\frac{2}{m}}^1 |2x^{\frac{1}{4}}(x - 1)|^2 \mathrm{d}x \\ &\quad + C\int_0^{\frac{2}{m}} |mx^{\frac{1}{4}}\sqrt{x}|^2 \mathrm{d}x + C\int_{1-\frac{2}{m}}^1 |mx^{\frac{1}{4}}(x - 1)|^2 \mathrm{d}x \\ &\leqslant \sqrt{\frac{2}{m}} + C\left(\frac{2}{m}\right)^3 + \frac{C}{\sqrt{m}} + \frac{C}{m^3} \to 0, m \to \infty,\end{aligned}$$

therefore $u_m(x) \to u(x)$ in $H^1(0,\, 1; h)$. Note that $u_m = 0$ on $[0, \frac{1}{m}] \cup [1 - \frac{1}{m}, 1]$, we can mollify the u_m to produce functions $\omega_m \in C_c^\infty(0,\, 1)$ such that $\omega_m \to u$ in $H_c^1(0,\, 1; h)$, thus $u \in H_c^1(0,\, 1; h)$. □

Lemma 3.5. *The space $H_c^1(0,\, 1; h)$ is separable and reflexive.*

Proof. Define $L_2^2(0,\, 1) = L^2(0,\, 1) \times L^2(0,\, 1)$, where

$$\|u\|_{L_2^2(0,1)} = \left(\int_0^1 |u_1|^2 \mathrm{d}x + \int_0^1 |u_2|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

for $u = (u_1,\, u_2) \in L_2^2(0,\, 1)$ as the norm of space $L_2^2(0,\, 1)$. It is no doubt that $L_2^2(0,\, 1)$ is a separable space in the light of that $L^2(0,\, 1)$ is a separable. Set

$$Pu = (u, hu'), u \in H^1(0, 1; h),$$

evidently, $W = \{Pu | u \in H^1(0,\, 1; h)\}$ is a subspace of $L_2^2(0,\, 1)$. From $\|Pu\|_{L_2^2(0,1)} = \|u\|_{H^1(0,1;h)}$, we know that P is an isometric isomorphism of mapping $H^1(0,\, 1; h)$ to W. In view of the fact that $H^1(0,\, 1; h)$ is complete, W is a closed subspace of $L_2^2(0,\, 1)$, furthermore, W is separable. Note that P is an isometric isomorphism, then $H^1(0,\, 1; h)$ has the same properties. Lemma 3.4 implies that the space $H_c^1(0,\, 1; h)$ is a separable Hilbert space. Similarly, reflexivity is also obtained. □

DEFINITION 3.6. *Define*

$$B(u, v) = \int_0^1 h^2 u' v' \mathrm{d}x + \int_0^1 V uv \mathrm{d}x, \;\; u, v \in H_c^1(0, 1; h), \tag{3.5}$$

we call $u \in H_c^1(0,\, 1; h)$ *is a solution to equation* (2.1) *if*

$$B(u, v) = (f, v)_{L^2(0,1)}$$

for all $v \in H_c^1(0,\, 1; h)$.

THEOREM 3.7. *Let* $f \in L^2(0,\, 1)$ *be arbitrarily given function, then equation* (2.1) *has a unique solution in* $H_c^1(0,\, 1; h)$.

Proof. Same to the proof of theorem 2.9. □

So far, we have constructed a suitable weak solution space $H_c^1(0,\, 1; h)$, especially this space can be compactly embedded into $L^2(0,\, 1)$ space, this feature provides an vital theoretical support for the study of eigenvalues in § 4.

4. Characterization of eigenvalues

For the fundamental gap optimization problem of Schrödinger operator [**4**, **16**, **19**], the differential of eigenvalues is a valuable tool and an important link in the process of exploration, the optimality conditions for extremum problems will be given with the help of formulae for derivatives of eigenvalues.

At the beginning this section mainly investigates the characteristics of eigenvalue and eigenfunction, so as to better serve the later exploration of the first derivative of eigenvalues. We will exploit the distinct techniques, it primarily depends on the properties of compact operators, Harnack inequality and so on [**5**, **13**, **14**, **33**]. Among them, the proof for Harnack inequality is placed in the appendix.

LEMMA 4.1. *Define*

$$L : L^2(0, 1) \to L^2(0, 1), \quad f \mapsto u,$$

where $u \in H_c^1(0,\, 1; h)$ *is the unique solution of* (2.1). *Then* L *is a self-adjoint compact linear operator.*

Proof. By the same argument of lemma 2.10 we obtain that L is a self-adjoint bounded linear operator. This together with lemma 3.3 we obtain L is a compact linear operator. □

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Remark 4.2. It is obvious that $\dim H_c^1(0,\,1;h)=\infty$, and hence $0\in\sigma(L)$, $\sigma(L)-\{0\}=\sigma_p(L)-\{0\}$, and $\sigma(L)-\{0\}$ is a sequence tending to 0, where $\sigma(L)$ is the spectrum of L and $\sigma_p(L)$ represents the point spectrum of L.

Theorem 4.3. *Denote*

$$Su=-(h^2u')'+Vu,\quad u\in H_c^1(0,1;h).$$

Then (i) *Each eigenvalue of S is real.*

(ii) *Furthermore, if we repeat eigenvalue according to its (finite) multiplicity, we have*

$$\sigma(S)=\{\lambda_k\}_{k\in\mathbb{N}},$$

where

$$0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\cdots$$

and

$$\lambda_k\to\infty \text{ as } k\to\infty.$$

(iii) *Finally, there exists an orthonormal basis $\{w_k\}_{k\in\mathbb{N}}\subset L^2(0,\,1)$, where $w_k\in H_c^1(0,\,1;h)$ is an eigenfunction corresponding to λ_k:*

$$Sw_k=\lambda_k w_k \text{ in } (0,1),\quad w_k\in H_c^1(0,1;h),k\in\mathbb{N}.$$

Proof. Set $L=S^{-1}$, then L is a self-adjoint compact linear operator owing to lemma 4.1. Also noteworthy,

$$(Lf,f)=(u,f)=B(u,u)\geqslant 0$$

for any given $f\in L^2(0,\,1)$.

Consider $L^2(0,1)$ is separable, by invoking theorem D7 (pp. 728) in [**13**], there exists a countable orthonormal basis of $L^2(0,1)$ consisting of eigenvectors of L. It should be pointed out that for $\eta\neq 0$, $Lw=\eta w$ if and only if $Sw=\lambda w$ for $\lambda=\frac{1}{\eta}$. Consequently, we prove theorem 4.3. □

Theorem 4.4. (i) *We have*

$$\lambda_1=\min\{B(u,u)\mid u\in H_c^1(0,1;h),\|u\|_{L^2(0,1)}=1\}.$$

(ii) *Furthermore, the above minimum is attained for a function $w_1\in H_c^1(0,\,1;h)$, positive within $(0,\,1)$, which solves*

$$Sw_1=\lambda_1 w_1 \text{ in } (0,1).$$

If $u\in H_c^1(0,\,1;h)$ is any weak solution of

$$Su=\lambda_1 u \text{ in } (0,1).$$

then u is a multiple of w_1.

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(iii) *Let u_k denote the k-th normalized eigenfunction for operator S, then $\lambda_k = \min\{B(u,\, u) \mid u \in H_c^1(0,\, 1; h),\, u \perp V_{k-1},\, \|u\|_{L^2(0,1)} = 1\}$, where $V_{k-1} =$ span$\{u_1,\, u_2 \cdots,\, u_{k-1}\}$, the equality holds if and only if $u = w_k$.*

(iv) *The connected components of the open sets $\Omega_+ = \{x \in (0,\, 1),\, u_k(x) > 0\}$ and $\Omega_- = \{x \in (0,\, 1),\, u_k(x) < 0\}$ are called the nodal domains of u_k, suppose the eigenvalue λ_k of the operator S is simple, then u_k has at most k nodal domains, $k \geqslant 2$.*

Proof. We carry out this proof by several steps.

(i) From theorem 4.3, we have

$$\begin{cases} B(w_k, w_k) = \lambda_k (w_k, w_k) = \lambda_k, \\ B(w_k, w_l) = \lambda_k (w_k, w_l) = 0, k, l = 1, 2 \cdots, k \neq l. \end{cases} \tag{4.1}$$

Since $\{w_k\}_{k=1}^{\infty}$ is the orthogonal basis of $L^2(0,\, 1)$, if $u \in H_c^1(0,\, 1; h)$ and $\|u\|_{L^2(0,1)} = 1$, then we write

$$u = \sum_{k=1}^{\infty} d_k w_k, \quad d_k = (u, w_k), \quad \sum_{k=1}^{\infty} d_k^2 = \|u\|^2_{L^2(0,1)} = 1, \tag{4.2}$$

the series converging in $L^2(0,\, 1)$. By (4.1), $\frac{w_k}{\sqrt{\lambda_k}}$ is an orthonormal subset of $H_c^1(0,\, 1; h)$, endowed with the new inner product $B(\cdot,\, \cdot)$. Indeed,

$$\begin{aligned} c_1 \|u\|^2_{H_c^1(0,1;h)} &\leqslant \int_0^1 h^2 |u'|^2 \mathrm{d}x + m \int_0^1 |u|^2 \mathrm{d}x \leqslant B(u, u) \\ &\leqslant \int_0^1 h^2 |u'|^2 \mathrm{d}x + M \int_0^1 |u|^2 \mathrm{d}x \leqslant c_2 \|u\|^2_{H_c^1(0,1;h)}. \end{aligned} \tag{4.3}$$

Moreover, $B(w_k,\, u) = \lambda_k (w_k,\, u) = 0$ for $k = 1,\, 2 \cdots$, which implies that $u \equiv 0$ due to $\lambda_k > 0$ and $\{w_k\}_{k=1}^{\infty}$ is the orthonormal basis of $L^2(0,\, 1)$. Hence $u = \sum\limits_{k=1}^{\infty} \mu_k \frac{w_k}{\sqrt{\lambda_k}}$ for $\mu_k = B(u,\, \frac{w_k}{\sqrt{\lambda_k}})$, the series converging in $H_c^1(0,\, 1; h)$. Together with the equality (4.2), we know that $\mu_k = d_k \sqrt{\lambda_k}$, thus $u = \sum\limits_{k=1}^{\infty} d_k w_k$ is convergent in $H_c^1(0,\, 1; h)$.

By employing the equalities (4.1) and (4.2), then

$$B(u, u) = \sum_{k=1}^{\infty} d_k^2 \lambda_k \geqslant \lambda_1,$$

and the equality holds for $u = w_1$, clearly, the desired result (i) is obtained.

(ii) We next claim that if $u \in H_c^1(0,\, 1; h)$ and $\|u\|_{L^2(0,1)} = 1$, then u is a weak solution of

$$\begin{cases} Su = \lambda_1 u, x \in (0, 1), \\ u(0) = u(1) = 0 \end{cases} \tag{4.4}$$

if and only if

$$B(u, u) = \lambda_1. \tag{4.5}$$

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Naturally, (4.4) implies (4.5). On the other side, suppose that (4.5) is established, writing $d_k = (u,\, w_k)$, then

$$\sum_{k=1}^{\infty} d_k^2 \lambda_1 = \lambda_1 = B(u,u) = \sum_{k=1}^{\infty} d_k^2 B(w_k, w_k) = \sum_{k=1}^{\infty} d_k^2 \lambda_k.$$

Therefore

$$\sum_{k=1}^{\infty} d_k^2 (\lambda_k - \lambda_1) = 0.$$

Evidently, we have $d_k = (u,\, w_k) = 0$ provided that $\lambda_k > \lambda_1$, then $u = \sum_{k=1}^{n} (u,\, w_k) w_k$ for some n according to the multiplicity finiteness of λ_1, where $Sw_k = \lambda_1 w_k$. Hence

$$Su = \sum_{k=1}^{n} (u, w_k) S w_k = \sum_{k=1}^{n} \lambda_1 (u, w_k) w_k = \lambda_1 u, \tag{4.6}$$

the claim is confirmed.

From (i), we see that if u is a eigenfunction of λ_1, then $|u|$ is one also. By lemma A.1, we must have $|u|$ is positive in $(0,\, 1)$ and hence λ_1 has a positive eigenfunction. This argument indicates that the eigenfunctions of λ_1 are either positive or negative and thereby it is impossible that two of them are orthogonal, i.e. λ_1 is simple.

(iii) We will apply the mathematical induction to prove it. Suppose the conclusion is established for $k = 2, \cdots, N-1$. For $u \perp V_{N-1}$, then equality (4.1) must be satisfied, and note that

$$u = \sum_{k=1}^{\infty} (u, w_k) w_k = \sum_{k=N}^{\infty} (u, w_k) w_k = \sum_{k=N}^{\infty} d_k w_k,$$

therefore

$$\sum_{k=1}^{\infty} d_k^2 B(w_k, w_k) = B(u,u) = \sum_{k=N}^{\infty} d_k^2 B(w_k, w_k),$$

thus we must have $d_1 = 0, \cdots, d_{N-1} = 0$ by (4.1), this leads to $\sum_{k=N}^{\infty} d_k^2 = 1$. Moreover,

$$B(u,u) = \sum_{k=N}^{\infty} d_k^2 B(w_k, w_k) = \sum_{k=N}^{\infty} d_k^2 \lambda_k \geqslant \lambda_N \sum_{k=N}^{\infty} d_k^2 = \lambda_N, \tag{4.7}$$

this inequality holds if and only if $u = w_N$.

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(iv) Suppose u_k has more than k nodal domains, let $G_1, \cdots, G_k, G_{k+1}$ be the nodal domains of u_k and consider

$$v = \sum_{n=1}^{k+1} a_n u_k \chi_{G_n}$$

where $a_1, \cdots, a_{k+1}$ are constants to be chosen later on. It is not hard to find that $u_k \chi_{G_n} \in H_c^1(0, 1; h)$ for every n, and thus $v \in H_c^1(0, 1; h)$. We may choose a non-trivial $(N+1)$-tuple $(a_1, \cdots, a_{k+1})$ such that v is orthogonal to the eigenfunctions $u_1, \cdots, u_k$, this is allowed since there are N equations and $N+1$ coefficients. Next, multiplying by an appropriate coefficient such that $\int_0^1 v^2 \mathrm{d}x = 1$, we record the new coefficient as $(c_1, c_2, \cdots, c_{k+1})$. According to the assumption that the eigenvalue of S is simple, by the same way as (4.7), we have

$$B(v, v) = \int_0^1 h^2 |v'|^2 \mathrm{d}x + \int_0^1 V|v'|^2 \mathrm{d}x \geqslant \lambda_{k+1} > \lambda_k. \tag{4.8}$$

Additionally, since $-(h^2(u_k\chi_{G_n})')' + V u_k \chi_{G_n} = \lambda_k u_k \chi_{G_n}$ for $n = 1, 2 \cdots, k+1$, the equality

$$\int_{G_n} h^2 |(u_k\chi_{G_n})'|^2 \mathrm{d}x + \int_{G_n} V|u_k\chi_{G_n}|^2 \mathrm{d}x = \lambda_k \int_{G_n} |u_k\chi_{G_n}|^2 \mathrm{d}x,$$

leads to

$$\begin{aligned} B(v, v) &= \int_0^1 h^2 |v'|^2 \mathrm{d}x + \int_0^1 V|v|^2 \mathrm{d}x \\ &= \sum_{n=1}^{k+1} \int_{G_n} h^2 |c_n (u_k\chi_{G_n})'|^2 \mathrm{d}x + \sum_{n=1}^{k+1} \int_{G_n} V|c_n u_k \chi_{G_n}|^2 \mathrm{d}x \\ &= \lambda_k \sum_{n=1}^{k+1} \int_{G_n} c_n^2 |u_k\chi_{G_n}|^2 \mathrm{d}x = \lambda_k, \end{aligned} \tag{4.9}$$

this contradicts inequality (4.8). □

The above results provide the definition of eigenvalue and prove the characteristics of the first eigenfunction by utilizing Harnack inequality. In order to fully understand the behavior of eigenvalues and eigenfunctions of equation (1.1), we require more detailed exploration content.

THEOREM 4.5. *The eigenvalue* λ_k *of equation* (1.1) *is simple,* $k > 2$.

Proof. Suppose u and v are two nontrivial linear independent solutions corresponding to λ_k, then

$$\begin{aligned} (h^2u')'v - (V - \lambda_k)uv = 0, \\ (h^2v')'u - (V - \lambda_k)uv = 0. \end{aligned} \tag{4.10}$$

Let $W_h = h^2u'v - h^2v'u$, then $W_h' = (h^2u')'v - (h^2v')'u$. By invoking [**13**] (chapter 6), the interior regularity for $u, v \in H_{loc}^2(\delta, 1)$ is obtained for any

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$\delta \in (0, 1)$. Then we discover that $u, v \in C^{1,\frac{1}{2}}[\delta, 1]$ take advantage of Sobolev Embedding Theorem for any $\delta \in (0, 1)$ and $u(1) = v(1) = 0$. For all $\phi \in C_c^\infty(0, 1)$, we have $\int_0^1 W_h' \phi \mathrm{d}x = \int_0^1 ((h^2u')'v - (h^2v')'u)\phi \mathrm{d}x = \int_0^1 [(V - \lambda_k)uv - (V - \lambda_k)uv]\phi \mathrm{d}x = 0$ by (4.10), therefore $W_h' = 0$ a.e. in the sense of distribution for $x \in [0, 1]$, which implies that W_h is a constant C a.e., combined with the fact that $W_h(1) = 0$ and W_h is continuous on $[\delta, 1]$ for any $\delta \in (0, 1)$, we immediately obtain that $C = 0$ a.e. on $[0, 1]$.

If $W_h = 0$ a.e. on $[0, 1]$, that is, $u'v - v'u = 0$ a.e. on $(0, 1)$, i.e. $(\frac{u}{v})'v^2 = 0$ a.e. on $(0, 1)$, also given that there is no positive measure subset in $(0, 1)$ such that $v = 0$ on it by lemma 4.1.3 on [**5**], then $(\frac{u}{v})' = 0$ a.e. on $(0, 1)$. This argument shows that $\frac{u}{v}$ is a constant on $(0, 1)$ a.e., which contradicts that u and v are linear independent, so that there is one and only one linearly independent solution to each λ_k. □

LEMMA 4.6. *Let the functions u and v that do not vanish identically on $(0, 1)$ and satisfy the equations*

$$-(h^2(x)u'(x))' + V_1(x)u(x) = 0, \tag{4.11}$$

$$-(h^2(x)v'(x))' + V_2(x)v(x) = 0, \tag{4.12}$$

where $h \in C^1[0, 1]$ with $h(0) = 0$ and $h(x) > 0$ for $x \in (0, 1]$, $V_1(x) > V_2(x)$ in $(0, 1)$. Suppose x_1 and x_2 be two consecutive zeros of u, then there is at least one zero x_0 of v and satisfy $x_1 < x_0 < x_2$, where $x_0 \in (0, 1)$.

Proof. Let $u(x) > 0$ on (x_1, x_2), suppose $v(x)$ has no zero point on (x_1, x_2), without loss of generality, let $v(x) > 0$ on (x_1, x_2). Multiplying equation (4.11) by $u(x)$ and equation (4.12) by $\frac{u^2(x)}{v(x)}$ and subtracting, that yields the equality

$$-(h^2(x)u'(x))'u(x) + (h^2(x)v'(x))'\frac{u^2(x)}{v(x)} + (V_1 - V_2)u^2(x) = 0. \tag{4.13}$$

Integrating this equality over (x_1, x_2), then yields

$$\int_{x_1}^{x_2}(V_1 - V_2)u^2(x)\mathrm{d}x + \int_{x_1}^{x_2}(h^2(x)v'(x))'\frac{u^2(x)}{v(x)} - (h^2(x)u'(x))'u(x)\mathrm{d}x = 0. \tag{4.14}$$

Integrating by parts and utilizing the fact that $u(x_1) = u(x_2) = 0$

$$\begin{aligned}&\int_{x_1}^{x_2}(V_1 - V_2)u^2(x)\mathrm{d}x - \int_{x_1}^{x_2} h^2(x)v'(x)\left(\frac{u^2(x)}{v(x)}\right)'\mathrm{d}x + \int_{x_1}^{x_2} h^2(x)|u'(x)|^2\mathrm{d}x\\ &= \int_{x_1}^{x_2}(V_1 - V_2)u^2(x)\mathrm{d}x + \int_{x_1}^{x_2} h^2(x)\left(\frac{u'(x)v(x) - v'(x)u(x)}{v(x)}\right)^2\mathrm{d}x\\ &= 0,\end{aligned} \tag{4.15}$$

however, the left side of the equality is positive, this is a contradiction. □

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THEOREM 4.7. *The eigenfunction u_k of equation (1.1) has exactly $k-1$ zeroes in $(0,\,1)$, $k = 1,\,2\cdots$.*

Proof. For $k=1$, theorem 4.4 implies that u_1 has no zero point on $(0,\,1)$, so it be verified. For $k=2$, we know that u_2 and u_1 are orthogonal and u_1 is positive on $(0,\,1)$, then u_2 has at least one zero. Meanwhile, u_2 has at most two node domains according to (iv) of theorem 4.4 and 4.5, hence u_2 has exactly one node in $(0,\,1)$.

For $k \geqslant 3$, consider

$$\begin{cases} -(h^2(x)u_i'(x))' + (V(x)-\lambda_i)u_i(x) = 0, i = 2,3,\cdots, \\ -(h^2(x)u_{i+1}'(x))' + (V(x)-\lambda_{i+1})u_{i+1}(x) = 0, i = 2,3,\cdots, \end{cases}$$

we observe that $V(x)-\lambda_2 > V(x)-\lambda_3$, by invoking lemma 4.6, it can be obtained that u_3 has at least two zeros. Combined with (iv) of theorem 4.4, it is apparent that u_3 has at most three node domains, therefore u_3 must have two zeros in $(0,\,1)$. The rest can be deduced by this method, so that we get u_k of equation (1.1) has exactly $k-1$ zeroes in $(0,\,1)$, $k=1,\,2\cdots$. □

The above results have introduced the explicit feature of eigenvalues and corresponding eigenfunctions, then we will calculate the differential of eigenvalues from the perspective of partial differential equation, which is the key step to derive the nature of the optimal function.

THEOREM 4.8. *Consider the following problem:*

$$\begin{cases} -(h^2(x)u'(t,x))' + V(t,x)u(t,x) = \lambda(t)u(t,x), & x \in (0,1), \\ u(t,x) = 0, & x \in \{0,1\} \end{cases} \tag{4.16}$$

with a parameter $t \in (a,\,b)$, $h \in C^1[0,\,1]$ with $h(0)=0$ and $h(x)>0$ for $x \in (0,\,1]$. And $V(t,\,x) : (a,\,b)\times[0,\,1] \to \mathbb{R}$ and $m \leqslant V(t,\,x) \leqslant M$ a.e., suppose that $\frac{\partial V}{\partial t}(x,\,t)$ exists for a.e. $x \in [0,\,1]$ and for all $t \in (a,\,b)$.

Let $u_i(t,\,x)$ be the i-th normalized eigenfunction concerned with $V(t,\,x)$, then the derivative of the i-th eigenvalue $\lambda_i(t)$ in relation to t is

$$\dot{\lambda}_i = \int_0^1 \dot{V}|u_i(t,x)|^2 \mathrm{d}x, \quad i = 1,2.$$

Proof. For the sake of simplicity, let us divide the discussion into several parts.

1°. Let $u_i(t) := u_i(t,\,x)$ $(i=1,\,2)$ denote the i-th normalized eigenfunction for problem (4.16), and u_i^M $(i=1,\,2)$ denote the i-th normalized eigenfunction associated with $V(t,\,x)=M$ in (4.16). Taking $t \in (a,\,b)$, by definition in theorem 4.4

$$\begin{aligned} \lambda_1(t) &= \inf_{u \in H_c^1(0,1;h)} \frac{\int_0^1 h^2|u'|^2\mathrm{d}x + \int_0^1 V(t)|u|^2\mathrm{d}x}{\|u\|^2_{L^2(0,1)}} \leqslant \inf_{u \in H_c^1(0,1;h)} \frac{\int_0^1 h^2|u'|^2\mathrm{d}x + \int_0^1 M|u|^2\mathrm{d}x}{\|u\|^2_{L^2(0,1)}} \\ &= \frac{\int_0^1 h^2|(u_1^M)'|^2\mathrm{d}x + \int_0^1 M|u_1^M|^2\mathrm{d}x}{\|u_1^M\|^2_{L^2(0,1)}} \leqslant C, \end{aligned}$$

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likewise, we easily check that λ_2 has the same properties, that is $\lambda_i(t) \leqslant C$, $\forall t \in (a,\, b)$, $i = 1,\, 2$. Furthermore, we obtain

$$\int_0^1 h^2|u_i'(t)|^2\mathrm{d}x + \int_0^1 V(t)|u_i(t)|^2\mathrm{d}x = \lambda_i(t)\|u_i(t)\|^2_{L^2(0,1)} = \lambda_i(t) \leqslant C, i = 1, 2,$$

so that

$$\|u_i(t)\|_{H_c^1(0,1;h)} \leqslant C, \quad \forall t \in (a, b),\ i = 1, 2. \tag{4.17}$$

2°. Taking $\{s_n\}_{n\in\mathbb{N}}$ with $s_n \to t$, there exists a subsequence of $\{s_n\}$ by 1°, still denoted by itself, and $\lambda_i^* = \liminf_{n\to\infty} \lambda_i(s_n) \in \mathbb{R}$ such that $\lambda_i(s_n) \to \lambda_i^*$, $i = 1,\, 2$. In addition, we can further extract a subsequence of $\{s_n\}_{n\in\mathbb{N}}$ from (4.17), still denoted by itself, and $u_i^* \in H_c^1(0,\, 1; h)$ such that

$$u_i(s_n) \to u_i^* \text{ weakly in } H_c^1(0, 1; h), i = 1, 2, \tag{4.18}$$

by employing lemma 3.3

$$u_i(s_n) \to u_i^* \text{ strongly in } L^2(0, 1), i = 1, 2, \tag{4.19}$$

at the limit, the constraint

$$\|u_i^*\|_{L^2(0,1)} = 1, i = 1, 2 \tag{4.20}$$

is preserved. Thanks to (4.19), there exists a subsequence of $\{u_i(s_n)\}_{n\in\mathbb{N}} \subset L^2(0,\, 1)$, still denoted by itself, such that

$$u_i(s_n) \to u_i^* \text{ a.e.}, i = 1, 2. \tag{4.21}$$

Note that $u_i(s_n)$ is the solution of (4.16) with respect to $V(s_n)$, owing to the fact that $V(s_n) \to V(t)$ strongly in $L^\infty(0,\, 1)$ as $n \to \infty$, by applying (4.18) and (4.19), elementary computations lead to

$$\begin{cases} -(h^2 u_i^{*\prime})' + V(t)u_i^* = \lambda_i^* u_i^*, & \text{in } (0, 1), \\ u_i^* = 0, & \text{on } \{0, 1\}. \end{cases} \tag{4.22}$$

Clearly, u_i^* is a nonzero eigenfunction related to λ_i^* whence (4.20) and (4.22), $i = 1,\, 2$. Given that $u_1(s_n)$ is the first positive eigenfunction for $V(s_n)$, so $u_1^* > 0$ a.e. by (4.21), it is no doubt that u_1^* must change sign except $\lambda_1^* = \lambda_1(t)$, for this reason we must have

$$\lambda_1^* = \lambda_1(t) \text{ and } u_1^* = u_1(t). \tag{4.23}$$

Besides, since $u_2(s_n)$ is the second eigenfunction so it must change sign once by theorem 4.7, and consider that (4.21) and $(u_2^*,\, u_1^*) = 0$, the eigenfunction u_2^* exactly has one node on $(0,\, 1)$, which shows that u_2^* is the second eigenfunction for V^* according to theorem 4.7, so that $\lambda_2^* = \lambda_2(t)$ and $u_2^* = u_2(t)$. In view of this, we conclude that $\lambda_i(s) \to \lambda_i(t)$ as $s \to t$, $i = 1,\, 2$.

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Now, we shall show $u_i(s) \to u_i(t)$ in $L^2(0,\,1)$, $i = 1,\,2$. Indeed, if not, there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subset (a,\,b)$ and $\epsilon_0 > 0$ such that $s_n \to t$ and

$$\|u_1(s_n) - u_1(t)\|_{L^2(0,1)} \geqslant \epsilon_0. \tag{4.24}$$

Follow the inequality (4.17), there exists a subsequence of $\{s_n\}_{n\in\mathbb{N}}$, still denoted by itself, and $\hat{u}_1 \in H^1_c(0,\,1;h)$ such that $u_1(s_n)$ weakly converges to $\hat{u}_1$ in $H^1_c(0,\,1;h)$ and $u_1(s_n)$ strongly converges to $\hat{u}_1$ in $L^2(0,\,1)$ as $\|\hat{u}_1\|_{L^2(0,1)} = 1$. Hence, we can extract a subsequence of $\{u_1(s_n)\}_{n\in\mathbb{N}} \subset L^2(0,\,1)$, still denoted by itself, such that $u_1(s_n) \to \hat{u}_1$ a.e., we deduce that $\|\hat{u}_1 - u_1(t)\|_{L^2(0,1)} \geqslant \epsilon_0$ whence (4.24). However, in the same manner as above (see (4.23)), the result $\hat{u}_1 = u_1(t)$ is obtained, which is absurd. The same argument is applied to $u_2(t)$ again, the desired result is then received.

3°. We observe that

$$\begin{cases} -(h^2 u_i'(s))' + V(s)u_i(s) = \lambda_i(s)u_i(s), \\ -(h^2 u_i'(t))' + V(t)u_i(t) = \lambda_i(t)u_i(t), \end{cases}$$

thereby

$$-(h^2u_i'(s) - h^2u_i'(t))' + V(s)u_i(s) - V(t)u_i(t) = \lambda_i(s)u_i(s) - \lambda_i(t)u_i(t), \quad i = 1,2,$$

i.e.,

$$\begin{aligned} &-(h^2u_i'(s) - h^2u_i'(t))' + (V(s) - V(t))u_i(s) + V(t)(u_i(s) - u_i(t)) \\ &= (\lambda_i(s) - \lambda_i(t))u_i(s) + \lambda_i(t)(u_i(s) - u_i(t)), i = 1,2. \end{aligned}$$

Multiplying both sides by $u_i(s) - u_i(t)$ and then integrating on $(0,\,1)$, we see that

$$\begin{aligned} &\int_0^1 h^2|u_i'(s) - u_i'(t)|^2 \mathrm{d}x + \int_0^1 (V(s) - V(t))u_i(s)(u_i(s) - u_i(t))\mathrm{d}x \\ &\quad + \int_0^1 V(t)|u_i(s) - u_i(t)|^2 \mathrm{d}x \\ &= (\lambda_i(s) - \lambda_i(t)) \int_0^1 u_i(s)(u_i(s) - u_i(t))\mathrm{d}x + \lambda_i(t) \int_0^1 |u_i(s) - u_i(t)|^2 \mathrm{d}x, i = 1,2, \end{aligned}$$

utilizing the Hölder inequality, we attain further results

$$\begin{aligned} &\int_0^1 h^2|u_i'(s) - u_i'(t)|^2 \mathrm{d}x + \int_0^1 V(t)|u_i(s) - u_i(t)|^2 \mathrm{d}x \\ &= \int_0^1 (V(t) - V(s))u_i(s)(u_i(s) - u_i(t))\mathrm{d}x + (\lambda_i(s) \\ &\quad - \lambda_i(t)) \int_0^1 u_i(s)(u_i(s) - u_i(t))\mathrm{d}x + \lambda_i(t) \int_0^1 |u_i(s) - u_i(t)|^2 \mathrm{d}x, \\ &\leqslant C\|u_i(s)\|_{L^2(0,1)}\|u_i(s) - u_i(t)\|_{L^2(0,1)} + C\|u_i(s) - u_i(t)\|^2_{L^2(0,1)} \\ &\leqslant C\left(\|u_i(s) - u_i(t)\|_{L^2(0,1)} + \|u_i(s) - u_i(t)\|^2_{L^2(0,1)}\right),\ i = 1,2. \end{aligned}$$

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This suggests that if $u_i(s) \to u_i(t)$ in $L^2(0,\,1)$, then $u_i(s) \to u_i(t)$ in $H^1_c(0,\,1;h)$, $i = 1,\,2$.

4°. Since

$$\begin{aligned}
&\int_0^1 h^2|u_i'(s) - u_i'(t)|^2\mathrm{d}x + \int_0^1 V(t)|u_i(s) - u_i(t)|^2\mathrm{d}x \\
&\quad = \int_0^1 h^2|u_i'(s)|^2\mathrm{d}x - 2\int_0^1 h^2u_i'(s)u_i'(t)\mathrm{d}x + \int_0^1 h^2|u_i'(t)|^2\mathrm{d}x \\
&\qquad + \int_0^1 V(t)|u_i(s)|^2\mathrm{d}x - 2\int_0^1 V(t)u_i(s)u_i(t)\mathrm{d}x + \int_0^1 V(t)|u_i(t)|^2\mathrm{d}x \\
&\quad = \lambda_i(s) + \int_0^1 V(t)|u_i(s)|^2\mathrm{d}x - \int_0^1 V(s)|u_i(s)|^2\mathrm{d}x - 2\lambda_i(t)\int_0^1 u_i(s)u_i(t)\mathrm{d}x + \lambda_i(t) \\
&\quad = (\lambda_i(s) - \lambda_i(t)) + \int_0^1 (V(t) - V(s))|u_i(s)|^2\mathrm{d}x + \lambda_i(t)\int_0^1 |u_i(s) - u_i(t)|^2\mathrm{d}x,
\end{aligned}$$

naturally,

$$\begin{aligned}
|\lambda_i(s) - \lambda_i(t)| \leqslant &\int_0^1 h^2|u_i'(s) - u_i'(t)|^2\mathrm{d}x + C\|u_i(s) - u_i(t)\|^2_{L^2(0,1)} \\
&+ \int_0^1 |V(s) - V(t)||u_i(s)|^2\mathrm{d}x,
\end{aligned}$$

for $i = 1,\,2$. Combine this result with 4°, in addition $V(s) \to V(t)$ as $s \to t$, this suggests that when $u_i(s) \to u_i(t)$ in $L^2(0,\,1)$, we have $\lambda_i(s) \to \lambda_i(t)$, $i = 1,\,2$.

5°. We claim that $\lambda_i(s) \to \lambda_i(t)$ as $s \to t$, then $u_i(s) \to u_i(t)$ in $L^2(0,\,1)$ as $s \to t$, $i = 1,\,2$. If not, suppose that there exist $\epsilon > 0$ and a sequence $\{u_i(s_n)\}_{n\in\mathbb{N}}$ such that $s_n \to t$ but $\|u_i(s_n) - u_i(t)\|_{L^2(0,1)} \geqslant \epsilon$. In the same way as 3°, there exists $\hat{u}_i \in H^1_c(0,\,1;h)$ such that $u_i(s_n)$ weakly converges to $\hat{u}_i$ in $H^1_c(0,\,1;h)$, $u_i(s_n)$ strongly converges to $\hat{u}_i$ in $L^2(0,\,1)$ with $\|\hat{u}_i\|_{L^2(0,1)} = 1$ and $u_i(s_n) \to \hat{u}_i$ a.e., $i = 1,\,2$. Then we obtain $\|\hat{u}_i - u_i(t)\|_{L^2(0,1)} \geqslant \epsilon$.

We see that $u_1(s_n) - u_1(t) \in H^1_c(0,\,1;h)$ and

$$\begin{aligned}
\lambda_1(t) &= \inf_{u\in H^1_c(0,1;h),u\neq 0} \frac{\int_0^1 h^2|u'|^2\mathrm{d}x + \int_0^1 V(t)|u|^2\mathrm{d}x}{\int_0^1 |u|^2\mathrm{d}x} \\
&\leqslant \frac{\int_0^1 h^2|\hat{u}_1' - u_1'(t)|^2\mathrm{d}x + \int_0^1 V(t)|\hat{u}_1 - u_1(t)|^2\mathrm{d}x}{\int_0^1 |\hat{u}_1 - u_1(t)|^2\mathrm{d}x}.
\end{aligned} \tag{4.25}$$

Similarly, $u_2(s_n) - u_2(t) \in H^1_c(0,\,1;h)$ and

$$\lambda_2(t) = \inf_{\substack{u\in H^1_c(0,1;h),u\neq 0,\\ (u,u_1(t))=0}} \frac{\int_0^1 h^2|u'|^2\mathrm{d}x + \int_0^1 V(t)|u|^2\mathrm{d}x}{\int_0^1 |u|^2\mathrm{d}x}.$$

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If $(\hat{u}_2 - u_2(t),\, u_1(t)) \neq 0$, we must have

$$\lambda_2(t) \neq \frac{\int_0^1 h^2|\hat{u}_2' - u_2'(t)|^2\mathrm{d}x + \int_0^1 V(t)|\hat{u}_2 - u_2(t)|^2\mathrm{d}x}{\int_0^1 |\hat{u}_2 - u_2(t)|^2\mathrm{d}x}, \tag{4.26}$$

and if $(\hat{u}_2 - u_2(t),\, u_1(t)) = 0$, then

$$\lambda_2(t) \leqslant \frac{\int_0^1 h^2|\hat{u}_2' - u_2'(t)|^2\mathrm{d}x + \int_0^1 V(t)|\hat{u}_2 - u_2(t)|^2\mathrm{d}x}{\int_0^1 |\hat{u}_2 - u_2(t)|^2\mathrm{d}x}. \tag{4.27}$$

For $i = 1, 2$, due to $\lambda_i(t)$ is simple, the equality of (4.25) and (4.27) holds if and only if $\hat{u}_i - u_i(t) = cu_i(t)$, thus $\hat{u}_1 = (c+1)u_1(t)$, the normalization condition guarantees $c = 0$ or $c = -2$, since $1 = \|\hat{u}_i\|_{L^2(0,1)} = |c+1|\|u_i(t)\|_{L^2(0,1)} = |c+1|$. Under the condition $\|\hat{u}_i - u_i(t)\|_{L^2(0,1)} \geqslant \epsilon$, the possibility of $c = 0$ is excluded, $i = 1, 2$. And simultaneously we also analyse that $c \neq -2$ for $i = 1$ due to $\hat{u}_1 \geqslant 0$ a.e. Without loss of generality, we may choose the sign of the second eigenfunction to be positive to the left of the node and to be negative in the other side. It is easily found that $\hat{u_2} = -u_2(t)$ for $c = -2$, it means that $\hat{u_2}$ and $u_2(t)$ have the same node, so this case can not happen. All these facts have demonstrated that the inequality of (4.25) and (4.27) holds strictly.

Formula from 4°,

$$\begin{aligned}
&\int_0^1 h^2|(u_i'(s_n) - u_i'(t))|^2\mathrm{d}x + \int_0^1 V(t)|u_i(s_n) - u_i(t)|^2\mathrm{d}x - \lambda_i(t)\int_0^1 |u_i(s_n) - u_i(t)|^2\mathrm{d}x \\
&\quad = (\lambda_i(s_n) - \lambda_i(t)) + \int_0^1 (V(t) - V(s_n))|u_i(s_n)|^2\mathrm{d}x, i = 1, 2,
\end{aligned}$$

let $s_n \to t$, we check that the right side of above equation converges to 0, but the limit on the left is nonzero by virtue of the above analysis, which causes a contradiction. So far, the desired result is proved.

6°. By 3°,

$$\begin{aligned}
&-(h^2(u_i'(s) - u_i'(t)))' + (V(s) - V(t))u_i(s) + V(t)(u_i(s) - u_i(t)) \\
&= (\lambda_i(s) - \lambda_i(t))u_i(s) + \lambda_i(t)(u_i(s) - u_i(t)), i = 1, 2,
\end{aligned}$$

multiplying $u_i(t)$ on the both sides of this equality and integrating on (0, 1), then yields

$$\begin{aligned}
&\int_0^1 h^2(u_i'(s) - u_i'(t))u_i'(t)\mathrm{d}x \\
&\quad + \int_0^1 (V(s) - V(t))u_i(s)u_i(t)\mathrm{d}x + \int_0^1 V(t)u_i(t)(u_i(s) - u_i(t))\mathrm{d}x \\
&= (\lambda_i(s) - \lambda_i(t))\int_0^1 u_i(s)u_i(t)\mathrm{d}x + \lambda_i(t)\int_0^1 u_i(t)(u_i(s) - u_i(t))\mathrm{d}x, \quad i = 1, 2,
\end{aligned}$$

i.e.

$$\int_0^1 h^2 \frac{u_i'(s) - u_i'(t)}{s-t} u_i'(t) \mathrm{d}x + \int_0^1 \frac{V(s) - V(t)}{s-t} u_i(s) u_i(t) \mathrm{d}x + \int_0^1 V(t) u_i(t) \frac{u_i(s) - u_i(t)}{s-t} \mathrm{d}x = \frac{\lambda_i(s) - \lambda_i(t)}{s-t} \int_0^1 u_i(s) u_i(t) \mathrm{d}x + \lambda_i(t) \int_0^1 u_i(t) \frac{u_i(s) - u_i(t)}{s-t} \mathrm{d}x, \quad i = 1, 2. \tag{4.28}$$

Also, given that

$$-(h^2 u_i'(t))' + V(t) u_i(t) = \lambda_i(t) u_i(t),$$

multiplying $\frac{u_i(s)-u_i(t)}{s-t}$ on both sides, after integrating on (0, 1) yields

$$\int_0^1 h^2 u_i'(t) \frac{u_i'(s) - u_i'(t)}{s-t} \mathrm{d}x + \int_0^1 V(t) u_i(t) \frac{u_i(s) - u_i(t)}{s-t} \mathrm{d}x = \lambda_i(t) \int_0^1 u_i(t) \frac{u_i(s) - u_i(t)}{s-t} \mathrm{d}x, \tag{4.29}$$

further,

$$\frac{\lambda_i(s) - \lambda_i(t)}{s-t} \int_0^1 u_i(s) u_i(t) \mathrm{d}x = \int_0^1 \frac{V(s) - V(t)}{s-t} u_i(s) u_i(t) \mathrm{d}x, \quad i = 1, 2.$$

whence (4.28) and (4.29). Concerning that $u_i(s) \to u_i(t)$ in $L^2(0, 1)$ as $s \to t$, from the dominated convergence theorem we have

$$\dot{\lambda}_i(t) = \lim_{s \to t} \frac{\lambda_i(s) - \lambda_i(t)}{s-t} \int_0^1 u_i(s) u_i(t) \mathrm{d}x = \lim_{s \to t} \int_0^1 \frac{V(s) - V(t)}{s-t} u_i(s) u_i(t) \mathrm{d}x = \int_0^1 \dot{V}(t) |u_i(t)|^2 \mathrm{d}x, \quad i = 1, 2.$$

□

Remark 4.9. The results of the above theorem 4.8 can be extended to the case of $i > 2$, the approach used is similar, which we will not repeat here.

The classical path to explore the minimum problem is to obtain the optimality condition, denote $\Gamma(V) = \lambda_2(V) - \lambda_1(V)$. Consider the set $\mathcal{V}$, recall that $P(x)$ is called admissible perturbation function associated to $V(x)$, provided $V(x) + tP(x) \in \mathcal{V}$ for any small $|t|$, where $P(x)$ is measurable bounded and real valued function. Sometimes we will say that only nonpositive t or non-negative t is allowed, but if there is no explicit limit, t can have any sign. Now and then we consider $V(t, x) = V(x) + tP(x)$, theorem 4.8 guarantees that

$$\left. \frac{\mathrm{d}\Gamma(V(x,t))}{\mathrm{d}t} \right|_{t=0} = \int_0^1 P(x) \left(|u_2(x)|^2 - |u_1(x)|^2 \right) \mathrm{d}x, \tag{4.30}$$

where $u_i(x)$ be the i-th normalized eigenfunction for $V(x)$, $i = 1, 2$.

5. Fundamental gap

In this section we commence by exploring the existence of the minimum fundamental gap when $V(x)$ is limited to the set $\mathcal{V}$, and then we further characterize the optimal function $V^*(x)$ and describe its behavior.

THEOREM 5.1. *The fundamental gap $\lambda_2(V) - \lambda_1(V)$ reaches its minimum on the set $\mathcal{V}$.*

Proof. We shall prove the existence of the minimizer V^* such that $\Gamma^* = \Gamma(V^*) = \inf\limits_{V\in\mathcal{V}}(\lambda_2(V) - \lambda_1(V))$.

Let $\{V^k\}_{k\in\mathbb{N}}$ such that

$$\Gamma(V^k) \downarrow \inf_{V\in\mathcal{V}} \Gamma(V) = \Gamma^*. \tag{5.1}$$

In view of the compactness of the class $\mathcal{V}$, then there exists a subsequence $\{V^k\}_{k\in\mathbb{N}} \subset L^\infty(0,\, 1)$ such that

$$V^k \to V^* \text{ weakly star in } L^\infty(0,1), \tag{5.2}$$

and

$$\lambda_2(V^k) \to \lambda_2^*, \lambda_1(V^k) \to \lambda_1^*, \Gamma^* = \lambda_2^* - \lambda_1^*. \tag{5.3}$$

Let $\{(\lambda_j^k,\, u_j^k)\}_{k\in\mathbb{N}}$ be a sequence concerned with V^k, λ_j^k denotes the j-th eigenvalue of (1.1), u_j^k be the normalized eigenfunction in $H_c^1(0,\, 1; h)$, $j = 1,\, 2$. By the definition of eigenvalue, we realize that

$$\begin{aligned}\lambda_1^k &= \inf_{u\in H_c^1(0,1;h),u\neq 0} \frac{\int_0^1 h^2|u'|^2\mathrm{d}x + \int_0^1 V^k|u|^2\mathrm{d}x}{\int_0^1 |u|^2\mathrm{d}x}\\ &\leqslant \inf_{u\in H_c^1(0,1;h),u\neq 0} \frac{\int_0^1 h^2|u'|^2\mathrm{d}x + \int_0^1 M|u|^2\mathrm{d}x}{\int_0^1 |u|^2\mathrm{d}x} \leqslant C,\end{aligned}$$

so, in like manner, we obtain that $\lambda_2^k \leqslant C$. Further,

$$\int_0^1 h^2|(u_j^k)'|^2\mathrm{d}x + \int_0^1 V^k|u_j^k|^2\mathrm{d}x = \lambda_j^k\|u_j^k\|^2_{L^2(0,1)} = \lambda_j^k \leqslant C, i = 1, 2, \tag{5.4}$$

we have that u_j^k is bounded in $H_c^1(0,\, 1; h)$, and hence by the compactness of the embedding in lemma 3.3, a subsequence, which we take as $\{u_j^k\}_{k\in\mathbb{N}}$ itself, such that

$$u_j^k \to u_j^* \text{ weakly in } H_c^1(0,1;h), \quad j = 1, 2, \tag{5.5}$$

and

$$u_j^k \to u_j^* \text{ strongly in } L^2(0,1), j = 1, 2. \tag{5.6}$$

Hence, there exists a subsequence, still take as $\{u_j^k\}_{k\in\mathbb{N}}$ itself, such that

$$u_j^k \to u_j^* \text{ a.e.}, j = 1, 2. \tag{5.7}$$

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For all $v \in H_c^1(0,\,1;h)$, seeing that

$$\int_0^1 h^2(u_j^k)'v'\mathrm{d}x + \int_0^1 V^k u_j^k v dx = \lambda_j^k \int_0^1 u_j^k v dx, j = 1, 2, \forall v \in H_c^1(0, 1; h),$$

let $k \to \infty$, then by (5.2) (5.3) and (5.5) (5.6)

$$\int_0^1 h^2(u_j^*)'v'\mathrm{d}x + \int_0^1 V^* u_j^* v dx = \lambda_j^* \int_0^1 u_j^* v dx, j = 1, 2, \forall v \in H_c^1(0, 1; h). \quad (5.8)$$

This implies that λ_j^* be the element of the spectrum for V^*, $j = 1,\,2$. Consider that u_1^k is the first eigenfunction involved with λ_1^k, and the ground states are characterized as the positive eigenfunctions on $(0,\,1)$, this fact determines that $u_1^* > 0$ on $(0,\,1)$ by (5.7), so that we must have $\lambda_1^* = \lambda_1(V^*)$. In a similar fashion, we see that u_2^k change once sign in $(0,\,1)$, so that u_2^* change once sign by (5.7) and $(u_2^*,\,u_1^*) = 0$, utilizing theorem 4.7, we have u_2^* is the second eigenfunction for V^* and $\lambda_2^* = \lambda_2(V^*)$. All this suggests that $\Gamma(V^*) = \lambda_2(V^*) - \lambda_1(V^*)$. □

THEOREM 5.2. *Consider the differential equation* (1.1) *for* $V(x) \in \mathcal{V}$, *then the minimum fundamental gap* $\lambda_2 - \lambda_1$ *is attained by*

$$V^*(x) = m\chi_\omega + M\chi_{\omega^c} \ a.e., \quad (5.9)$$

where $\omega = \{x \in (0,\,1) \mid |u_2^*(x)|^2 - |u_1^*(x)|^2 \geqslant 0\}$, $u_i^*(x)$ *be the* i*-th normalized eigenfunction associated to* V^*, $i = 1,\,2$. *Furthermore,* $|\omega| > 0$ *and* $|\omega^c| > 0$.

Proof. In order to prove (5.9), we demand display that the set $T = \{x \in (0,\,1) \mid m < V^*(x) < M\}$ is zero measure. If not, let $T^k = \{x \in [0,\,1] \mid m + \frac{1}{k} < V^*(x) < M - \frac{1}{k}\}$, then we may write $T = \bigcup_{k=1}^{\infty} T^k$, we assume that at least one of them is a positive measure for T^k. For any $x^* \in T^k$ and any measurable sequence of subsets $G_{k,j} \subset T^k$ including x^*, the optimal condition of the fundamental gap extremum problem determines

$$\left.\frac{\mathrm{d}\Gamma(V(x,t))}{\mathrm{d}t}\right|_{t=0} = \int_{G_{k,j}} \left(|u_2^*(x)|^2 - |u_1^*(x)|^2\right)\mathrm{d}x = 0,$$

where $V(x,\,t) = V(x) + tP(x)$ and $P(x) = \chi_{G_{k,j}}$ is an admissible perturbation for any small $t \in (-\frac{1}{k},\,\frac{1}{k})$. We claim that $|u_2^*(x)|^2 = |u_1^*(x)|^2$ on T^k for any $k > 1$, so that $|u_2^*(x)|^2 = |u_1^*(x)|^2$ on the set T. Indeed, if there is a subset $G \subset T^k$ of positive measure such that $|u_2^*(x)|^2 - |u_1^*(x)|^2 > 0$, $P(x) = \chi_G$ is an admissible perturbation, then we have

$$\left.\frac{\mathrm{d}\Gamma(V(x,t))}{\mathrm{d}t}\right|_{t=0} = \int_G |u_2^*(x)|^2 - |u_1^*(x)|^2\mathrm{d}x > 0,$$

however, this is in contradiction with the fact that $V^*(x)$ is the optimal function. Likewise same procedure is carried out, we can obtain that there is no positive measure subset on T with $|u_2^*(x)|^2 - |u_1^*(x)|^2 < 0$.

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Actually, the case $|u_2^*(x)|^2 = |u_1^*(x)|^2$ only works on the set with zero measure. Without loss of generality, suppose $T^+ = \{x \in T \mid u_2^*(x) > 0\}$ with positive measure, concerning that u_1^* has no zero point on $(0, 1)$, as a result, $T = T^+ \cup T^-$, where $T^- = \{x \in T \mid u_2^*(x) < 0\}$. On T^+, seeing that $u_1^* - u_2^* = 0$ a.e., thus $(u_2^* - u_1^*)' = 0$ a.e., from equation (1.1) we can infer that

$$(h^2(u_2^* - u_1^*)')' + V(u_1^* - u_2^*) = \lambda_1 u_1^* - \lambda_2 u_2^*. \tag{5.10}$$

From the proof of theorem 4.5, we know that $u_1^*,\, u_2^* \in H^2_{loc}(0,\,1)$ and $u_1^*,\, u_2^* \in C^{1,\frac{1}{2}}(\delta,\,1)$ for any $\delta \in (0,\,1)$. Let $F(x) = (h^2(u_2^* - u_1^*)')' + V(u_1^* - u_2^*) - (\lambda_1 u_1^* - \lambda_2 u_2^*)$, we obtain $F(x) = 0$ a.e. on T^+ in the sense of distribution. The left side of equation (5.10) vanishes on T^+ a.e., it directly caused $\lambda_1 = \lambda_2$, a contradiction, so that T is a set of zero measure. This ensures that the optimal function is $V^* = m\chi_\omega + M\chi_{\omega^c}$ for some ω.

In this regard, we have to study further, there is no doubt that χ_ω is an admissible perturbation for any small $t \geqslant 0$ when $V^*(x) = m$, theorem 4.8 guarantees

$$\left.\frac{\mathrm{d}\Gamma(V(x,t))}{\mathrm{d}t}\right|_{t=0} = \int_\omega |u_2^*(x)|^2 - |u_1^*(x)|^2 \mathrm{d}x \geqslant 0,$$

for this we may come to a conclusion $|u_2^*(x)|^2 - |u_1^*(x)|^2 \geqslant 0$. If this is not true, there must be a set $S \subset \omega_0$ of positive measure such that $|u_2^*(x)|^2 - |u_1^*(x)|^2 < 0$ on S and

$$\left.\frac{\mathrm{d}\Gamma(V(x,t))}{\mathrm{d}t}\right|_{t=0} = \int_S |u_2^*(x)|^2 - |u_1^*(x)|^2 \mathrm{d}x < 0$$

for admissible perturbation χ_S, $t \geqslant 0$, this obviously fail to meet the optimality of $V^*(x)$. By applying the same argument, the result $|u_2^*(x)|^2 - |u_1^*(x)|^2 \leqslant 0$ as $V^* = M$ is straightforward.

Concerned with the normalization condition $\int_0^1 |u_2^*(x)|^2 - |u_1^*(x)|^2 \mathrm{d}x = 0$, which indicates that both the sets ω and ω^c are nonempty. □

THEOREM 5.3. *The set ω^c mentioned in theorem 5.2 only contains one interval.*

Proof. For any given function $V(x)$, the set S_1 and S_1^c are nonempty in the light of normalization condition, here $S_1 = \{x \in (0,\,1) \mid u_2(x)|^2 - |u_1(x)|^2 \geqslant 0\}$, thereby $|u_2(x)|^2 = |u_1(x)|^2$ has at least one solution. In reality, the number of solutions for this equation shall not exceed two, the following will be explained in detail.

Consider

$$\left(\frac{u_2(x)}{u_1(x)}\right)' = \frac{u_2'(x)u_1(x) - u_1'(x)u_2(x)}{|u_1(x)|^2} = \frac{g(x)}{|u_1(x)|^2}, \tag{5.11}$$

and note that

$$\begin{aligned}(h^2(x)g(x))' &= (h^2(x)u_2'(x))'u_1(x) - (h^2(x)u_1'(x))'u_2(x)\\ &= (V(x) - \lambda_2)u_1(x)u_2(x) - (V(x) - \lambda_1)u_1(x)u_2(x)\\ &= (\lambda_1 - \lambda_2)u_1(x)u_2(x),\end{aligned}$$

so that

$$h^2(x)g(x) = \int_0^x (h^2(t)g(t))'\mathrm{d}t = \int_0^x (\lambda_1 - \lambda_2)u_1(t)u_2(t)\mathrm{d}t, \quad x \in (0, \alpha), \tag{5.12}$$

and

$$h^2(x)g(x) = -\int_x^1 (h^2(t)g(t))'\mathrm{d}t = \int_x^1 (\lambda_2 - \lambda_1)u_1(t)u_2(t)\mathrm{d}t, \quad x \in (\alpha, 1), \tag{5.13}$$

where α denotes the node of the eigenfunction $u_2(x)$. Without loss of generality, suppose $u_2(x) > 0$ in $(0,\ \alpha)$ and $u_2(x) < 0$ in $(\alpha,\ 1)$, hence $h^2(x)g(x) < 0$ on $(0,\ 1)$ by (5.12) and (5.13), especially, $g(x) < 0$ on $(0,\ 1)$. Moreover, thanks to (5.11) we have

$$\left(\frac{|u_2(x)|^2}{|u_1(x)|^2}\right)' = 2\left(\frac{u_2(x)}{u_1(x)}\right)\left(\frac{u_2(x)}{u_1(x)}\right)' < 0, \quad x \in (0, \alpha),$$

and

$$\left(\frac{|u_2(x)|^2}{|u_1(x)|^2}\right)' = 2\left(\frac{u_2(x)}{u_1(x)}\right)\left(\frac{u_2(x)}{u_1(x)}\right)' > 0, \quad x \in (\alpha, 1),$$

that is, $\frac{|u_2(x)|^2}{|u_1(x)|^2}$ is strictly monotonic decreasing on $(0,\ \alpha)$ and strictly monotonic increasing on $(\alpha,\ 1)$.

Suppose there are two different points x^1, $x^2 \in (0,\ \alpha)$ such that $|u_2(x)|^2 = |u_1(x)|^2$, which lead to $\frac{|u_2(x^2)|^2}{|u_1(x^2)|^2} = \frac{|u_2(x^1)|^2}{|u_1(x^1)|^2}$, this is opposite to the condition of strictly monotone for $\frac{|u_2(x)|^2}{|u_1(x)|^2}$ on $(0,\ \alpha)$, hence the equation $|u_1(x)|^2 = |u_2(x)|^2$ has at most one solution on $(0,\ \alpha)$. Likewise, the same conclusion is achieved on $(\alpha,\ 1)$. Consequently, there exist two points x_0, x_1, $0 \leqslant x_0 < \alpha < x_1 \leqslant 1$ with

$$|u_2(x)|^2 - |u_1(x)|^2 \begin{cases} > 0, x \in (0, x_0) \cup (x_1, 1), \\ < 0, x \in (x_0, x_1), \end{cases} \tag{5.14}$$

and both sets are nonempty. Take into consideration that $u_1(x), u_2(x)$ are continuous on $(0,\ 1)$, we must have $|u_2(x_0)|^2 = |u_1(x_0)|^2$ and $|u_2(x_1)|^2 = |u_1(x_1)|^2$.

Recognizing the facts $|u_2^*(x)|^2 - |u_1^*(x)|^2 \leqslant 0$ when $V^*(x) = M$ according to theorem 5.2 and the inequality 5.14, we point out that the set ω^c only contains one interval. □

Appendix A. Harnack's inequality

LEMMA A.1 Harnack's inequality. *Let* $0 \leqslant f \in L^2(0,\ 1)$, *and* $0 \leqslant u \in H_c^1(0,\ 1; h)$ *be the solution of* (2.1) *with respect to* f. *Then there exists a constant* $C > 0$ *such that*

$$\sup_{x\in\Omega'} u(x) \leqslant C \inf_{x\in\Omega'} u(x)$$

for any $\Omega' \subset\subset \overline{\Omega} \subset (0,\ 1)$.

Proof. The proof method is similar to [**14**], but for clarity, we show specific details.

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Let $x_0 \in (0,\,1)$, $\overline{B(x_0,\,R)} \subset (0,\,1)$. Assume $0 < r_1 < r_2 < R$, and $\zeta \in C_c^\infty(\mathbb{R})$, $0 \leqslant \zeta \leqslant 1$, $\zeta|_{B(x_0, r_1)} = 1$, $\zeta|_{\mathbb{R}-B(x_0,\,r_2)} = 0$, and $|\zeta'| \leqslant \frac{2}{r_2 - r_1}$.

For each $\beta \neq 0$, taking $v = \zeta^2 \overline{u}^\beta$ with $\overline{u} = u + k$, $k > 0$, then

$$v' = 2\zeta\zeta'\overline{u}^\beta + \beta\zeta^2\overline{u}^{\beta-1}u'.$$

From which we obtain that

$$\begin{aligned}\int_0^1 fv\mathrm{d}x &= \int_0^1 h^2u'v'\mathrm{d}x + \int_0^1 Vuv\mathrm{d}x\\ &= \int_0^1 h^2u'\left(2\zeta\zeta'\overline{u}^\beta + \beta\zeta^2\overline{u}^{\beta-1}u'\right)\mathrm{d}x + \int_0^1 Vuv\mathrm{d}x,\end{aligned}$$

i.e.

$$\beta\int_0^1 h^2|u'|^2\zeta^2\overline{u}^{\beta-1}\mathrm{d}x = \int_0^1 fv\mathrm{d}x - 2\int_0^1 h^2u'\zeta\zeta'\overline{u}^\beta\mathrm{d}x - \int_0^1 Vuv\mathrm{d}x,$$

thus by Young inequality, we have

$$\begin{aligned}\int_0^1 h^2|u'|^2\zeta^2\overline{u}^{\beta-1}\mathrm{d}x &= \frac{1}{\beta}\int_0^1 fv\mathrm{d}x - \frac{2}{\beta}\int_0^1 h^2u'\zeta\zeta'\overline{u}^\beta\mathrm{d}x - \frac{1}{\beta}\int_0^1 Vuv\mathrm{d}x\\ &\leqslant \frac{1}{|\beta|}\left(k^{-1}\int_0^1 f\zeta^2\overline{u}^{\beta+1}\mathrm{d}x\right) + \frac{1}{|\beta|}\left(\epsilon\int_0^1 h^2|u'|^2|\zeta|^2\overline{u}^{\beta-1}\mathrm{d}x\right)\\ &\quad + \frac{1}{|\beta|}\left(\frac{1}{\epsilon}\int_0^1 h^2|\zeta'|^2\overline{u}^{\beta+1}\mathrm{d}x\right) + \frac{C}{|\beta|}\int_0^1 |\zeta|^2\overline{u}^{\beta+1}\mathrm{d}x\end{aligned}$$

for any $\epsilon > 0$. Choosing $\epsilon \leqslant \frac{|\beta|}{2}$, thereby obtaining the inequality

$$\int_0^1 h^2\zeta^2|u'|^2\overline{u}^{\beta-1}\mathrm{d}x \leqslant C(|\beta|)\left(\int_0^1\left(\zeta^2 + |\zeta'|^2\right)\overline{u}^{\beta+1}\mathrm{d}x + k^{-1}\int_0^1 f\zeta^2\overline{u}^{\beta+1}\mathrm{d}x\right).$$

Set

$$\theta = \begin{cases}\overline{u}^{\frac{\beta+1}{2}}, & \beta \neq -1,\\ \log\overline{u}, & \beta = -1,\end{cases}$$

then $\theta' = \frac{\beta+1}{2}\overline{u}^{\frac{\beta-1}{2}}u'$ for $\beta \neq -1$, and

$$\int_0^1 |\zeta\theta'|^2\mathrm{d}x \leqslant \begin{cases}C(|\beta|)(\beta+1)^2\left(\int_0^1\left(\zeta^2 + |\zeta'|^2\right)\theta^2\mathrm{d}x + k^{-1}\int_0^1 f\zeta^2\theta^2\mathrm{d}x\right), & \beta \neq -1,\\ C(|\beta|)\int_0^1\left(|\zeta|^2 + |\zeta'|^2 + k^{-1}f\zeta^2\right)\mathrm{d}x, & \beta = -1.\end{cases} \tag{A.1}$$

Consequently, we have

$$\|\zeta\theta\|^2_{L^6(0,1)} \leqslant C\int_0^1\left(|\zeta\theta'|^2 + |\theta\zeta'|^2\right)\mathrm{d}x \tag{A.2}$$

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by the Sobolev inequality, where C is independent of $|\beta|$. The interpolation inequality implies

$$\int_0^1 f\zeta^2\theta^2 \mathrm{d}x \leqslant \|f\|_{L^2(0,1)}\|\zeta\theta\|^2_{L^4(0,1)} \leqslant \|f\|_{L^2(0,1)}(\epsilon\|\zeta\theta\|_{L^6(0,1)} + \epsilon^{-3}\|\zeta\theta\|_{L^2(0,1)})^2, \tag{A.3}$$

hence, combine with (A.1),(A.2) and (A.3) and select the appropriate ϵ, we see that

$$\|\zeta\theta\|^2_{L^6(0,1)} \leqslant C\left(1 + |\beta+1|\right)^8 \|(\zeta + |\zeta'|)\theta\|^2_{L^2(0,1)}.$$

Let $\gamma = \beta + 1$, note that $\zeta + |\zeta'| \leqslant 1 + \frac{2}{r_2 - r_1} \leqslant \frac{C}{r_2 - r_1}$, then

$$\|\theta\|_{L^6(B(x_0,r_1))} \leqslant C\frac{(1+|\gamma|)^4}{r_2 - r_1}\|\theta\|_{L^2(B(x_0,r_2))}. \tag{A.4}$$

For $\overline{B(x_0,\, R)} \subset (0,\, 1)$ and $p \neq 0,\, r < R$, we introduce

$$\Phi(p, r) = \left(\int_{B(x_0,r)} |\overline{u}|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

then we have $\Phi(\infty,\, r) = \mathop{lim}\limits_{p\to\infty} \Phi(p,\, r) = \mathop{sup}\limits_{B(x_0,r)} \bar{u}$, $\Phi(-\infty,\, r) = \mathop{lim}\limits_{p\to-\infty} \Phi(p,\, r) = \mathop{inf}\limits_{B(x_0,r)} \bar{u}$ from [**14**]. From (A.4), we easily check that

$$\begin{cases} \Phi(3\gamma, r_1) \leqslant \left(C\frac{(1+|\gamma|)^4}{r_2-r_1}\right)^{\frac{2}{|\gamma|}} \Phi(\gamma, r_2), & \gamma > 0, \\ \Phi(\gamma, r_2) \leqslant \left(C\frac{(1+|\gamma|)^4}{r_2-r_1}\right)^{\frac{2}{|\gamma|}} \Phi(3\gamma, r_1), & \gamma < 0. \end{cases} \tag{A.5}$$

For $\beta > 0$, then $\gamma > 1$, taking $p > 1$, set $\gamma = \gamma_m = 3^{m-1}p$, $R_m = \rho_0 + (\rho_1 - \rho_0)^m$, where $\rho_0 = r_1$, $\rho_1 = \frac{r_2+r_1}{2}$, $m = 1,\, 2\cdots$, so that, thanks to (A.5),

$$\Phi(3^m p, R_{m+1}) \leqslant C\Phi(p, R_1),$$

letting m tend to ∞, then

$$\sup_{B(x_0,\rho_0)} \overline{u} \leqslant C\|\overline{u}\|_{L^p(B(x_0,\rho_1))}. \tag{A.6}$$

For $\beta < 0$ and $\gamma < 1$, we may employ a similar method to prove it, for any p_0, p such that $0 < p_0 < p < 3$, $p > 1$, we have

$$\Phi(p, \rho_1) \leqslant C\Phi(p_0, \rho_2), \quad 0 < \gamma < 1, \tag{A.7}$$

and

$$\Phi(-p_0, \rho_2) \leqslant C\Phi(-\infty, \rho_0), \quad \gamma < 0, \tag{A.8}$$

where $\rho_2 = r_2$. Indeed, when $0 < \gamma < 1$ and $p_0 < 1$, we may take $\gamma = p_0$, there exists s such that $3^s\gamma \geqslant p > 1$, by (A.5)

$$\begin{aligned} \Phi(p, \rho_1) = \|\overline{u}\|_{L^p(B(x_0,\rho_1))} &\leqslant C\|\overline{u}\|_{L^{3^s\gamma}(B(x_0,\rho_1))} \leqslant C\|\overline{u}\|_{L^\gamma(B(x_0,\rho_2))} \\ &= C\|\overline{u}\|_{L^{p_0}(B(x_0,\rho_2))} = C\Phi(p_0, \rho_2). \end{aligned} \tag{A.9}$$

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When $p_0 \geqslant 1$, we may take $3\gamma = p_0$, there exists t such that $3^t\gamma \geqslant p > 1$, (A.7) is obtained by the same procedure. Consider the case of $\gamma < 0$, let $\gamma_m = -3^{m-1}p_0$, $R_m = \rho_0 + (\rho_2 - \rho_0)^m$, $m = 1, 2, \cdots$, (A.8) can be obtained by utilizing equation (A.5) again.

To further explore this content, we shall further verify that

$$\Phi(p_0, \rho_2) \leqslant C\Phi(-p_0, \rho_2). \tag{A.10}$$

From the second of the estimates (A.1), with the aid of the Hölder inequality, the result

$$\int_{B(x_0,r_1)} |\theta'|\mathrm{d}x \leqslant |B(x_0, r_1)|^{\frac{1}{2}} \left(\int_{B(x_0,r_1)} |\theta'|^2\mathrm{d}x\right)^{\frac{1}{2}} \leqslant C$$

is obtained for any $r_1 \in (0, r_2)$. Consequently, according to theorem 7.21 [**14**], there exists a positive p_0 such that, for

$$\theta_0 = \frac{1}{|B(x_0, r_2)|} \int_{B(x_0,r_2)} \theta\mathrm{d}x,$$

we know

$$\int_{B(x_0,r_2)} e^{p_0|\theta-\theta_0|}\mathrm{d}x \leqslant C,$$

and hence

$$\int_{B(x_0,r_2)} e^{p_0\theta}\mathrm{d}x \int_{B(x_0,r_2)} e^{-p_0\theta}\mathrm{d}x \leqslant C.$$

Then combined with the definition $\theta = \log \overline{u}$, the result (A.10) can be easily obtained. These results (A.7),(A.8) and (A.10) indicate that

$$\Phi(p, \rho_1) \leqslant C\Phi(-\infty, \rho_0),$$

i.e.

$$\|\overline{u}\|_{L^p(B(x_0,\rho_1))} \leqslant C \inf_{B(x_0,\rho_0)} \overline{u},$$

coupled with (A.6), we have $\sup\limits_{B(x_0,\rho_0)} \overline{u} \leqslant C \inf\limits_{B(x_0,\rho_0)} \overline{u}$. Now let $\Omega' \subset\subset \Omega$ and choose $x_1, x_2 \in \overline{\Omega'}$ such that $u(x_1) = \sup\limits_{\Omega'} u$ and $u(x_2) = \inf\limits_{\Omega'}$u. Take $\Gamma \subset \overline{\Omega'}$ be the line joining x_1 and x_2 and choose R such that $4R < dist(\Gamma, \partial\Omega)$. Since Γ can be covered by a finite number N of balls of radius R, utilizing the above estimation on each ball and combining all inequalities, the desired result is obtained. □

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