Proceedings of the Edinburgh Mathematical Society (1976-77), **20**, 287-291 ©

NEW SETS OF EQUI-ISOCLINIC n-PLANES FROM OLD

by S. G. HOGGAR

(Received 10th August 1975)

Introduction

Two n-planes Γ and Δ in real Euclidean r-space $\boldsymbol{R}^r$ are called *isoclinic* with parameter λ if the angle θ between any x in Γ and its orthogonal projection Px on Δ is unique, with $\cos^2\theta = \lambda$. Let $v_\lambda(n, r)$ denote the maximum number of equi-isoclinic (i.e. pairwise isoclinic) n-planes in $\boldsymbol{R}^r$ with parameter λ.

Problem 1. *Compute $v_\lambda(n, r)$ for given n, r, λ.*

Let k be any positive integer. Lemmens and Seidel (4) derived the upper bound $(r - r\lambda)/(n - r\lambda)$, for $n - r\lambda > 0$, which is the same for (kn, kr) as for (n, r). They deduced that if the bound is attained for (n, r) then it is attained for each (kn, kr); for every set of v equi-isoclinic n-planes in $\boldsymbol{R}^r$ yields a set of v equi-isoclinic kn-planes in $\boldsymbol{R}^{kr}$ with the same parameter, via the tensor product construction. In fact all the above applies to vector spaces over $\boldsymbol{F}$, where $\boldsymbol{F} = \boldsymbol{R}$, $\boldsymbol{C}$, or the quaternions $\boldsymbol{H}$ (3). We adopt here the aim of deducing from a set of equi-isoclinic planes as many inequalities as possible which are relevant to Problem 1. Hence:

Problem 2. *What new sets of equi-isoclinic planes can be constructed from a given set?*

We give constructions which yield, in an obvious notation

$$v_\lambda(n, r, \boldsymbol{H}) \leq v_\lambda(2n, 2r, \boldsymbol{C}), \qquad v_\lambda(n, r, \boldsymbol{C}) \leq v_\lambda(2n, 2r, \boldsymbol{R}),$$

and for $\boldsymbol{F} = \boldsymbol{R}$, $\boldsymbol{C}$: $v_\lambda(n, r, \boldsymbol{F}) \leq v_{\lambda^n}(1, \binom{r}{n}, \boldsymbol{F})$, for example.

A short preview of some material in this paper is found in (2), which contains the definitions of Construction 2 and of *isoclinic functor*, and a special case of the exterior powers used in Section 2.

1. New n-planes by embedding matrices

We have inclusions $\boldsymbol{R} \subset \boldsymbol{C} \subset \boldsymbol{H}$, leading to:

Proposition 1.1. $v_\lambda(n, r, \boldsymbol{R}) \leq v_\lambda(n, r, \boldsymbol{C}) \leq v_\lambda(n, r, \boldsymbol{H})$. *If the first attains the upper bound $(r - r\lambda)/(n - r\lambda)$, so do the rest; if the second attains it, so does the third.*

However, we can confirm that C and H do add new possibilities by the result of (3): $v_{2/5}(1,2)=2$, 3, 6 respectively for $F=R, C, H$.

Notation. Let $F(m, n)$ denote the set of m by n matrices over F and let $F(n)=F(n, n)$. Throughout this section we identify linear transformations with matrices via the standard basis of F^r, $(1,0,\ldots,0)$, $(0,1,0,\ldots,0)$ etc. The *adjoint*, or transpose conjugate of a matrix M is written M^*. Extending a result of (4), we have

Proposition 1.2. *Let $F=R$, C, or H, then*

(a) *$M\in F(r)$ is the orthogonal projection onto its range iff $M^2=M=M^*$,*

(b) *two n-planes in F^r with projections P, Q are isoclinic with parameter λ iff $PQP=\lambda P$.*

Construction 1.

We can view H as a right vector space over C of dimension two by writing quaternions in the form $z+jw$ $(z, w\in C)$, where $j^2=-1$ and $ja=\bar{a}j$ for every $a\in C$. Applying this to linear transformations yields an injective function ϕ from quaternionic to complex matrices (5) which replaces each entry $z+jw$ by the block $\begin{pmatrix} z & -\bar{w} \\ w & \bar{z}\end{pmatrix}$. Thus if $M\in H(m, n)$ then $\phi M\in C(2m, 2n)$. Now ϕ is easily seen to preserve

(1) composition of homomorphisms,
(2) identity homomorphisms,
(3) adjoints,
(4) multiplication by real scalars.

If Γ is a plane in H^r with projection P, let $\phi\Gamma=(\phi P)C^{2r}$, that is, the column space of ϕP.

Theorem 1.3. *If Γ, Δ are isoclinic n-planes in H^r with parameter λ then $\phi\Gamma$, $\phi\Delta$ are isoclinic $2n$-planes in C^{2r} with parameter λ.*

Proof. Firstly, Properties (1), (2) imply that ϕ sends isomorphisms to isomorphisms. Thus, as Γ, Δ have the same dimension, so do $\phi\Gamma$, $\phi\Delta$. In fact the latter have dimension $2n$ (exercise for the reader).

Secondly, (1), (3) show the property $M^2=M=M^*$ is preserved by ϕ (that is, $(\phi M)^2=\phi M=(\phi M)^*$), which therefore sends projections to projections, by 1.2.

Finally, $\phi\Gamma$ and $\phi\Delta$ are isoclinic with parameter λ because the condition $PQP=\lambda P$ of 1.2 is preserved by ϕ, from Properties (1), (4).

Corollary 1.4. $v_\lambda(n, r, H)\leq v_\lambda(2n, 2r, C)$. *If the first attains the upper bound $(r-r\lambda)/(n-r\lambda)$, so does the second.*

Example 1.5. A set of 4 equiangular lines in $\boldsymbol{H}^2$ with parameter 1/3 is given by vectors $(1, j)$, $(1-\sqrt{2}i,\ j-\sqrt{2}k)$, $(\sqrt{2}+\sqrt{3}+i,\ \sqrt{2}j-\sqrt{3}j+k)$, $(\sqrt{2}-\sqrt{3}+i,\ \sqrt{2}j+\sqrt{3}j+k)$. This set attains the upper bound $(r-r\lambda)/(n-r\lambda)$, so is maximal.

Example 1.6. Since the isoclinic condition is trivial for lines, it is interesting that Example 1.5 gives, via construction 1, a set of equi-isoclinic 2-planes in $\boldsymbol{C}^4$ with the same parameter, 1/3. The second set is maximal by 1.4, and is spanned by the pairs of vectors

$$\begin{aligned}&(1,0,0,1),(0,1,-1,0);(1-\sqrt{2}i,0,0,1+\sqrt{2}i),(0,1+\sqrt{2}i,-1+\sqrt{2}i,0);\\ &\quad(\sqrt{2}+\sqrt{3}+i,0,0,\sqrt{2}-\sqrt{3}-i),(0,\sqrt{2}+\sqrt{3}-i,-\sqrt{2}+\sqrt{3}-i,0);\\ &\quad(\sqrt{2}-\sqrt{3}+i,0,0,\sqrt{2}+\sqrt{3}-i),(0,\sqrt{2}-\sqrt{3}-i,-\sqrt{2}-\sqrt{3}-i,0).\end{aligned}$$

It is natural to conjecture $v_\lambda(n, r, \boldsymbol{H}) = v_\lambda(2n, 2r, \boldsymbol{C})$, but this is false in general since $v_{3/7}(2, 4, \boldsymbol{H}) = 6$, while $v_{3/7}(4, 8, \boldsymbol{C}) = 8$, (3).

Construction 2. Described in (2), this is the analogy of construction 1 which makes complex n-planes Γ into real $2n$-planes $\psi\Gamma$. From it, we make here the following deductions.

Corollary 1.7. $v_\lambda(n, r, \boldsymbol{C}) \leqslant v_\lambda(2n, 2r, \boldsymbol{R})$. *If the first attains the bound* $(r-r\lambda)/(n-r\lambda)$, *so does the second.*

Remark. Just as for ϕ, the inequality may be strict. For instance $v_{4/9}(4, 8, \boldsymbol{C}) = 7$ but $v_{4/9}(8, 16, \boldsymbol{R}) = 10$ (3). On the other hand, from $\boldsymbol{R} \subset \boldsymbol{C}$, we have

$$v_\lambda(n, r, \boldsymbol{C}) \leqslant v_\lambda(2n, 2r, \boldsymbol{R}) \leqslant v_\lambda(2n, 2r, \boldsymbol{C}),$$

so that if the first equals the third then equality does hold in 1.7.

Similarly for

$$v_\lambda(n, r, \boldsymbol{H}) \leqslant v_\lambda(2n, 2r, \boldsymbol{C}) \leqslant v_\lambda(2n, 2r, \boldsymbol{H}).$$

We note that in the fully solved case $r = 2n$ with $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ (3) *there exists no value of n* for which the above (four) inequalities are simultaneously strict.

2. New n-planes by exterior powers

Let Λ^k denote the usual exterior power of a vector space, over $F = \boldsymbol{R}$ or $\boldsymbol{C}$. It can be shown that Λ^k preserves

(1) composition of homomorphisms,
(2) identity homomorphisms,
(3) adjoints,

and that if $T\colon V \to W$ is a homomorphism and λ a scalar then $\Lambda^k(\lambda T) = \lambda^k(\Lambda^k T)$.

Theorem 2.1. *Let $F=R$ or C. If Γ, Δ are isoclinic n-planes in F^r with parameter λ then for $1 \leq k \leq n$, $\Lambda^k\Gamma$ and $\Lambda^k\Delta$ are isoclinic $\binom{n}{k}$-planes in $F^{\binom{r}{k}}$ with parameter λ^k.*

Proof. The dimensions of $\Lambda^k\Gamma$ and $\Lambda^k\Delta$ are as stated because the definition of induced homomorphism is actually independent of the choice of basis. As Λ^k preserves both adjoints and composition of homomorphisms, it also preserves orthogonal projections, by 1.2(a). (Compare ϕ and ψ). For the parameter we observe that $PQP=\lambda P$ implies $(\Lambda^kP)(\Lambda^kQ)(\Lambda^kP)=\lambda^k(\Lambda^kP)$. The proof is completed by applying 1.2(b) to the latter equality.

Corollary 2.2. *For $F=R$, C,* $v_\lambda(n,r) \leq v_{\lambda^k}(\binom{n}{k},\binom{r}{k})$, $1 \leq k \leq n$.

Corollary 2.3. *For $F=R$, C,* $v_\lambda(n,r) \leq v_{\lambda^n}(1,\binom{r}{n})$.

Examples 2.4. Our first example (1.5) was of 4 equiangular lines in H^2 with parameter 1/3. ϕ makes these into equi-isoclinic 2-planes in C^4 (1.6) and ψ then produces 4-planes in R^8, both sets with parameter 1/3. Using Λ^2, Λ^3, Λ^4, etc we can get infinitely many new examples (not in general of maximal size). The reader may care to consider how, using the examples in hand, we could derive lines in C^6 with parameter 1/9, 6-planes in R^{28} with parameter 1/9, lines in R^{70} with parameter 1/81.

3. Isoclinic functors

The constructions ϕ, ψ, Λ^k each associate to every vector space V of a certain kind, a new space $\mathcal{F}(V)$, and to each homomorphism $T: V \to W$ a new homomorphism $\mathcal{F}(T): \mathcal{F}(V) \to \mathcal{F}(W)$ in such a way that Properties (1), (2) hold. They are thus examples of covariant functors between the three categories V_F $(F=R, C, H)$ whose objects are the spaces F^m and whose morphisms are the F-linear homorphisms between them. Specifically, $\phi: V_H \to V_C$, $\psi: V_C \to V_R$, $\Lambda^k: V_C \to V_C$ and $\Lambda^k: V_R \to V_R$ (Λ^k does not work for H because of the noncommutativity of H).

However, these all have extra properties which ensure they send equi-isoclinic planes to equi-isoclinic planes, under the standard inner-product $((x_1,\ldots,x_m),(y_1,\ldots,y_m))=\bar{x}_1y_1+\cdots+\bar{x}_my_m$ on F^m. This motivates the following.

Definition. A (covariant) functor $\mathcal{F}$ between any two of the categories V_F is *isoclinic* if it preserves adjoints and, for each homomorphism $T: V \to W$ and $\lambda \in [0,1]$ we have $\mathcal{F}(\lambda T)=f(\lambda)\mathcal{F}(T)$, where $f:[0,1] \to [0,1]$. If $\mathcal{F}$ satisfies this definition then it sends a set of equi-isoclinic n-planes with parameter λ to a set of equi-isoclinic d-planes (say) with

https://doi.org/10.1017/S0013091500026511 Published online by Cambridge University Press

parameter $f(\lambda)$. We may call $n \to d$ the *dimension function* of $\mathcal{F}$ and f its *parameter function*.

Problem 3. *Determine all isoclinic functors.*

REFERENCES

(1) F. HIRZEBRUCH, *Topological Methods in Algebraic Geometry*, 3rd edn, (Springer, Berlin, 1966).

(2) S. G. HOGGAR, New equi-isoclinic n-planes from old, *Proc. 5th British* Combinatorial Conference, Aberdeen (1975), *Congressus Numerantium* **15** (1976), 319–322.

(3) S. G. HOGGAR, Quaternionic Equi-isoclinic n-planes, *Ars Combinatoria* **2** (1976), 11–13.

(4) P. W. H. LEMMENS and J. J. SEIDEL, Equi-isoclinic Subspaces of Euclidean Spaces, *Proc. Kon. Ned. Akad. Wet. Ser.* A, **76** (1973), 98–107.

(5) I. R. PORTEOUS, *Topological Geometry* (New University Mathematics Series, Van Nostrand, London, 1969).

UNIVERSITY OF GLASGOW