Triangles Triply in Perspsctive.

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1. The following way of viewing triangles triply in perspective is of interest.

Two triangles ABC and PQR are said to be "in perspective" if the joins of corresponding vertices be concurrent.

Thus if U be the point of concurrency we have the following collineations :—

ARU
BQU
CPU

2. *Triangles doubly in perspective are also triply in perspective.*

Let the two triangles ABC and PQR be doubly in perspective, the two centres of perspective being U and V. We then have the following collineations :—

ARU ; AQV
BQU ; BPV
CPU ; CRV.

Now all cubic curves through the eight points ABCPQRUV will have a ninth point W in common. But one cubic through the above eight points will be the degenerate cubic formed of the three lines

ARU
BPV
CQ

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Hence W must lie on this cubic, and must therefore lie on CQ. Similarly W must lie on BR and AP.

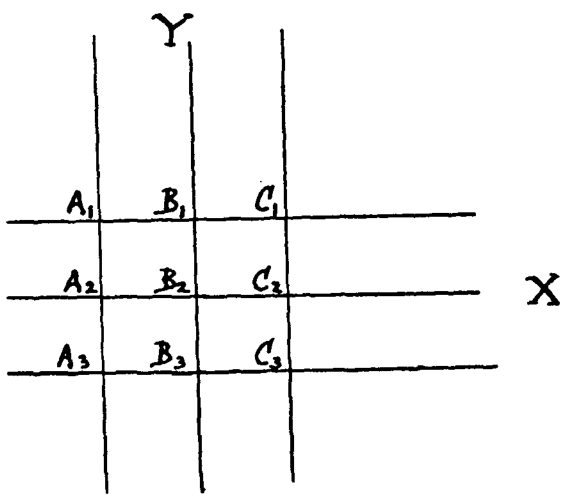
Hence the given two triangles ABC and PQR will be triply in perspective according to the following scheme of collineation :—

$$\begin{array}{lll} ARU; & AQV; & APW \\ BQU; & BPV; & BRW \\ CPU; & CRV; & CQW. \end{array} \qquad (1)$$

3. *ABC, PQR, UVW form a system of three triangles, each pair of which is triply in perspective, the vertices of the third triangle being the centres of perspective of the other two.*

This is plain at once if we read the scheme (1) in rows instead of columns, whence we see that A is the centre of the collineations ARU, AQV and APW, and similarly for B and C.

4. The well-known concurrencies in connexion with three parallel lines meeting three other parallel lines will now be obvious from the above systems.



For let the two systems of three parallel lines intersect at infinity in X and Y respectively. Then to prove that (say) B_3C_2, C_1A_3, A_2B_1

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are concurrent, we have only to note the following two systems of perspective :—

$$\begin{array}{ll} B_1C_1X\,; & B_1B_3Y \\ C_2A_2X\,: & C_2C_1Y \\ A_3B_3X\,; & A_3A_2Y, \end{array}$$

whence by rotating the middle column once more we get

$$\begin{array}{l} B_1A_2Z \\ C_2B_3Z \\ A_3C_1Z. \end{array}$$

5. *To find the connexion between the system of triangles triply in perspective of article (2) and a cubic curve through their nine centres of perspective.*

Consider such a cubic parametrically in the form

$$\begin{aligned} x &= \wp u \\ y &= \wp' u \end{aligned}$$

the periods of the elliptic functions being $2\pi_1$ and $2\pi_2$.

Let the parameters of the points A, B, etc., be the corresponding small letters a, b, etc.

Then in virtue of the collineations of article 2 we get

$$\left.\begin{aligned} a+r+u &= \text{period.} \\ b+q+u &= \quad,, \\ c+p+u &= \quad,, \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} a+q+v &= \text{period.} \\ b+p+v &= \quad,, \\ c+r+v &= \quad,, \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned} a+p+w &= \text{period.} \\ b+r+w &= \quad,, \\ c+q+w &= \quad,, \end{aligned}\right\} \qquad (3)$$

Adding severally the numbers of (1), (2), (3) we see that $3u$, $3v$, $3w$ differ from each other only by periods.

Rearranging and reasoning as in article 3 we see that $3a$, $3b$, $3c$ differ only by periods, and so for $3p$, $3q$, $3r$.

Also if we subtract the first two rows of (1) we get

$$a-b=q-r+\text{period}.$$

Hence the period involved is the same in all cases.

We may therefore take

$$\begin{aligned} a &= \lambda \\ b &= \lambda+\frac{2\pi_1}{3} \qquad (4) \\ c &= \lambda+\frac{4\pi_1}{3} \end{aligned}$$

$$\begin{aligned} p &= \mu \\ q &= \mu+\frac{2\pi_1'}{3} \qquad (5) \\ r &= \mu+\frac{4\pi_1}{3} \end{aligned}$$

$$\begin{aligned} u &= \nu+\frac{2\pi_1}{3} \\ v &= \nu+\frac{4\pi_1}{3} \qquad (6) \\ w &= \nu \end{aligned}$$

whence the collineations of article 2 will be satisfied if

$$\lambda+\mu+\nu=\text{period}.$$

Hence if the parameters λ, μ, ν represent the points L, M, N, we see that L, M, N will be collinear.

We thus get from (4), (5), (6) of the present article the following theorem :—

If L, M, N be three collinear points on a cubic, and if we project them through any set of three collinear flexes on to the cubic again, we get three triangles triply in perspective, each triangle corresponding to the points got on projection from L, M, N respectively, and conversely, and three triangles triply in perspective can be dealt with by this method in an infinity of ways in virtue of the infinity of cubics that can be drawn through their vertices.