Mathematical Notes.

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Introductory Lessons in Solid Geometry.—The difficulty which every student beginning the study of solid geometry finds in interpreting diagrams representing solid figures may be largely met, and an interest in the subject may be aroused at the outset, by a few lessons in drawing the simplest regular solids and in the study of the drawings made, before the formal propositions of Euclid, Book XI. are attacked.

A solid may be regarded as having dimensions in three principal and mutually perpendicular directions, which for clearness we shall describe as North and South, East and West, and vertical. Lines running in the last two of these directions may be represented in a drawing by lines at right angles to each other,

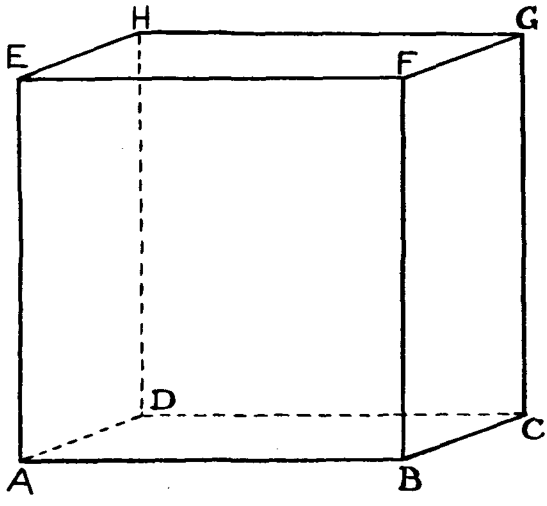

Fig. 1.

and of the actual lengths which they possess in the solid. What about the North and South lines? If we agree to represent them by lines inclined at an angle of 20° to the East and West lines and of one-third their actual length, all other lines in the solid may be at once drawn in, and a fairly convincing representation of the solid may be obtained, from the study of which the properties of the solid may be deduced.

I. The Cube.

Make a drawing of a cube, placed so that four of its edges run North and South, four East and West, and four vertical, each edge being three inches long.

Method:— Draw AB and AE (Fig. 1) at right angles to each other to represent an East and West and a vertical edge, making each three inches long. Make $\angle BAD = 20°$, with $AD =$ one inch. AD represents the North and South edge through A, and the drawing may be completed as shown in the figure.

The following exercises may be set:—

Ex. 1. The length of a diagonal of a cube is $a\sqrt{3}$, where a is the length of an edge.

Ex. 2. The distance between the middle point of the edge AE (Fig. 1) and the middle point of the diagonal $BH = \dfrac{a\sqrt{2}}{2}$.

Ex. 3. The four diagonals of a cube are concurrent and bisect one another.

Ex. 4. Make a drawing of a rectangular solid of unequal edges, and work out exercises corresponding to the foregoing.

Ex. 5. If the four diagonals of the cube are drawn in (the student should make a new drawing wherever necessary), the cube is seen to be composed of six equal pyramids with a common vertex at the point of concurrency of the diagonals, each pyramid having as base one face of the cube and as altitude a line equal to one-half the edge of the cube.

$$\text{Volume of each pyramid} = \frac{a^3}{6} = \tfrac{1}{3} \times a^2 \times \frac{a}{2}.$$

This may be mentioned as an example of the general rule that the volume of a pyramid $= \frac{1}{3} \times$ area of base $\times$ altitude.

II. The Regular Octahedron.

Make a drawing of a regular octahedron of edge three inches long.

Method:—(i) Place the octahedron so that its principal axes run North and South, East and West, and vertically. The last two will be shown full size in the drawing.

(ii) To obtain the lengths of the axes draw a triangle A′E′B′ having $\angle E' = 90°$ and $E'A' = E'B' = 3''$. Then each axis $= A'B'$.

(iii) Draw the vertical axis $AB = A'B'$, and bisect it at O. Through O draw CD at right angles to AB, making $OC = OD = OA$.

Draw EOF inclined at 20° to COD, making $OE = OF = \frac{1}{3}OA$. Then CD and EF represent the East and West axis and the North and South axis respectively.

(iv) Complete the drawing by joining the extremities of the axes as in Fig. 2.

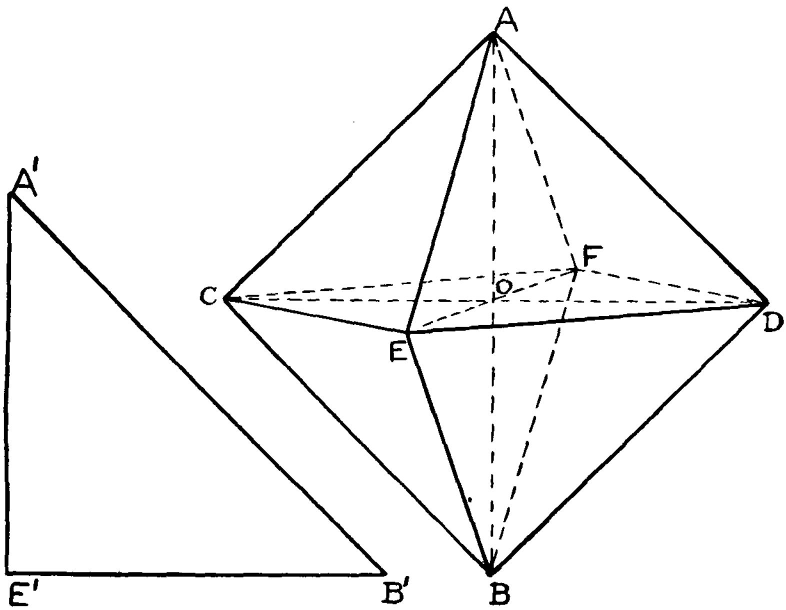

Fig. 2.

Ex. 1. The length of each axis is $a\sqrt{2}$, where a is the length of an edge.

Ex. 2. If CE is bisected at G, $AG = \dfrac{a\sqrt{3}}{2}$.

Ex. 3. Area of face $ACE = \frac{1}{2}AG \cdot CE = \dfrac{a^2\sqrt{3}}{4}$.

Total surface of the octahedron $= 2\sqrt{3} \cdot a^2$.

Ex. 4. Volume of pyramid $ACEDF = \frac{1}{3}a^2 \times \dfrac{a\sqrt{2}}{2} = \dfrac{a^3\sqrt{2}}{6}$.

Volume of octahedron $= \dfrac{a^3\sqrt{2}}{3}$.

Ex. 5. AG and BG are lines in the faces ACE and BCE respectively, both of which are perpendicular to CE, the line of intersection of these faces. Therefore $\angle AGB$ is the angle between these faces. (The formal definition of the angle between two planes might be introduced at this stage).

From the triangle AGB, $\cos AGB = \frac{AG^2 + GB^2 - AB^2}{2AG \cdot GB} = -\frac{1}{3}$.

III. The Regular Tetrahedron.

Make a drawing of a regular tetrahedron, each edge of which is three inches long.

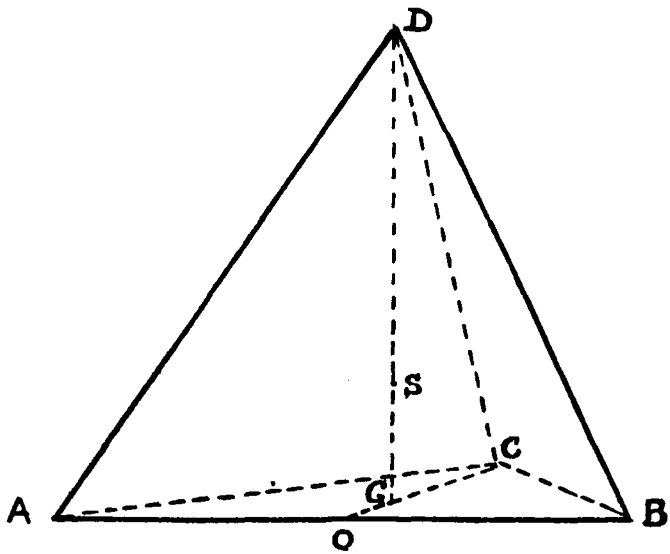

Fig. 3.

Method :—(i) Place the tetrahedron so that one face is horizontal and one edge runs East and West. Draw AB (Fig. 3), three inches long, to represent that edge, and bisect AB at O.

(ii) The median OC of the horizontal face is a North and South line. To find its length draw an equilateral triangle A′B′C′ (Fig. 4) having each side three inches long. Bisect A′B′ at O′ and join O′C′. OC is the actual length of a median of a face of the tetrahedron.

(iii) Draw OC (Fig. 3) inclined at 20° to OB and equal to one-third of O′C′. Join AC, BC. ABC represents the horizontal face of the tetrahedron.

(iv) Divide OC at G in the ratio 1 : 2. A vertical line through G is the locus of all points equidistant from A, B ,C. Hence the fourth vertex of the tetrahedron must lie on this line.

(v) To find the vertical height of the tetrahedron divide O′C′ (Fig. 4) at G′ in the ratio 1 : 2. Draw G′D′ at right angles to O′C′, and with centre C′ and radius three inches cut G′D′ at D′. G′D′ is equal to the vertical height of the tetrahedron.

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(vi) From G (Fig. 3) draw GD at right angles to AB, making GD equal to G′D′. D represents in the drawing the position of the fourth vertex of the tetrahedron, and the drawing is completed by joining DA, DB, DC.

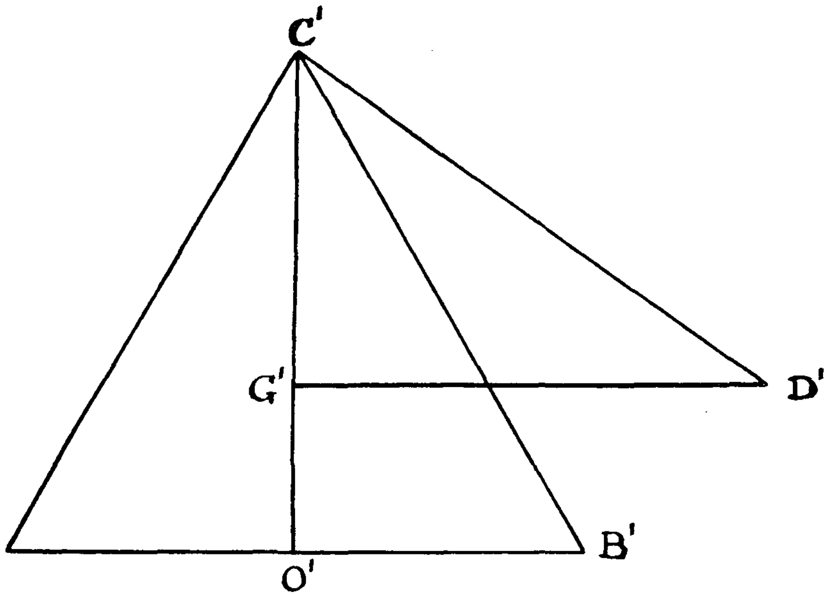

Fig. 4.

Ex. 1. The altitude of the tetrahedron is $\frac{a\sqrt{6}}{3}$, where a is the length of an edge.

$$\text{For actual length of } OC = O'C' = \frac{a\sqrt{3}}{2}.$$

$$\text{Therefore actual length of } GC = \frac{a\sqrt{3}}{3}.$$

$$\text{Now } GD^2 = CD^2 - CG^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3}.$$

$$\therefore\ GD = \frac{a\sqrt{6}}{3}.$$

Ex. 2. The surface of the tetrahedron $= a^2\sqrt{3}$.

$$\text{For area of face } ABC = \tfrac{1}{2}a \times \frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{4}.$$

Ex. 3. The volume of the tetrahedron $= \frac{a^3\sqrt{2}}{12}$.

$$\text{For volume} = \tfrac{1}{3}\triangle ABC \times GD.$$

Ex. 4. A sphere can be described which will pass through the four vertices of the tetrahedron. For GD must intersect the plane which bisects AD at right angles, and the point of intersection is equidistant from all four vertices.

Ex. 5. Calculate the radius of the circumscribing sphere, and find the position of its centre. Let S be the centre, and let $SG = x$. Then $SC^2 = SD^2$, *i.e.* $x^2 + \left(\frac{a\sqrt{3}}{3}\right)^2 = \left(\frac{a\sqrt{6}}{3} - x\right)^2$, which gives $x = \frac{a\sqrt{6}}{12}$

i.e. $GS = \frac{1}{4}GD$, and the radius of the sphere $\frac{a\sqrt{6}}{4}$.

Ex. 6. The straight lines joining the vertices of the tetrahedron to the centroids of the opposite faces are concurrent, and divide one another in the ratio 1 : 3.

Ex. 7. The straight lines joining the middle points of opposite edges of the tetrahedron pass through the centre of the circumscribing sphere and are themselves bisected there.

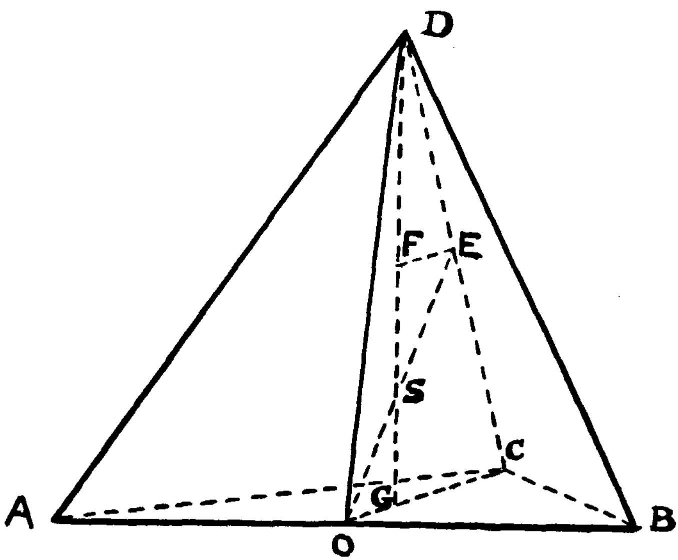

Fig. 5.

Bisect CD at E (Fig. 5) and join OE. OE and GD must intersect, since both are in the plane of $\triangle DOC$. Let them intersect at S. Bisect GD at F and join FE. Then FE is parallel to GC, and $FE = \frac{1}{2}GC = OG$. Therefore triangles OGS, EFS are congruent. Hence $GS = SF = \frac{1}{4}GD$. Therefore S is the centre of the circumscribing sphere; also $OS = SE$.

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Ex. 8. OE is the common perpendicular to AB and CD. For O is the middle point of the base AB of the isosceles triangle EAB, hence EO is perpendicular to AB. Again, E is the middle point of the base CD of the isosceles triangle OCD, hence OE is perpendicular to CD.

Ex. 9. The angle between any two faces of the tetrahedron is $\cos^{-1}(\frac{1}{3})$.

$$\text{For} \quad \cos COD = \frac{OG}{OD} = \frac{OG}{OC} = \tfrac{1}{3}.$$

The student will probably now have acquired considerable confidence in the use of his drawings, which may be further tested by applying certain of the above exercises, modified as required, to any tetrahedron. And if the principle on which these drawings have been made is borne in mind when the usual propositions are taken up, the advantage of having a definite way of constructing diagrams and a definite way of thinking and speaking of the lines there represented will be found to lessen considerably the difficulty of the subject.

PETER RAMSAY

Geometrical Construction for Refracted Ray.—

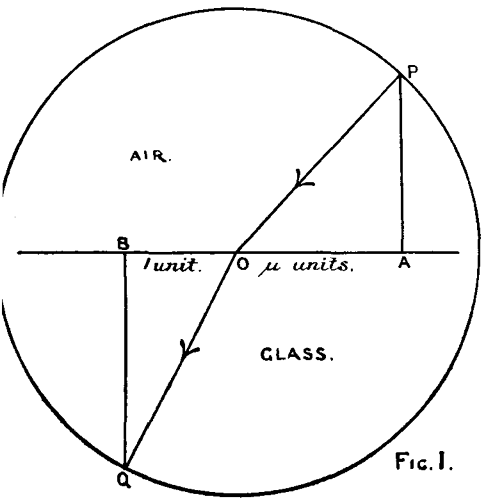

FIG. I.

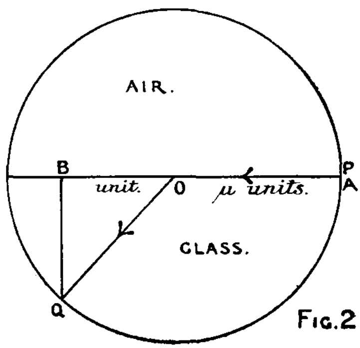

FIG. 2

The following variation seems an improvement on the usual method of tracing a refracted ray. In the usual method when an incident ray OP is given, *any* radius OP is taken and a circle described. PA is drawn perpendicular to the surface AB. OA is then sub-divided into the number of units expressed in the numerator of the fraction μ, and OB is measured backwards, the