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NON-AVERAGING SETS, DIMENSION AND POROSITY

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ABSTRACT. A subset of the line is called non-averaging if, whenever two points belong to the set, their average does not. This paper provides an example of a closed set which is small in the sense that it is non-averaging and has porosity 1 at each of its points and yet large in the sense that its Hausdorff dimension is 1.

The Cantor ternary set C, the set of all real numbers $x = \Sigma a_i/3^i$, $a_i = 0$ or 2, has the following curious property: if x, y, and z are three distinct points of the Cantor set but are not end points of any contiguous intervals, then $y \neq (x + z)/2$. This can be seen by letting $x = \Sigma b_i/3^i$, $z = \Sigma c_i/3^i$ and n be the least natural number for which $b_n \neq c_n$. Then $(x + z)/2$ has a 1 in the nth place of its ternary expansion. Sets which do not contain the average of any two of their points will be called non-averaging. Thus, by selecting a closed subset of C which does not contain the endpoints of contiguous intervals, it is possible to obtain a non-averaging set of dimension $s = \log 2/\log 3$ and with s-measure close to 1 because $s - m(C) = 1$. It is not easy to see how to obtain a bigger non-averaging set in the sense of Hausdorff dimension.

In fact, an analogue to this arose originally in number theory. The problem was to give an estimate of the maximum number $\nu(n)$ of elements in a non-averaging subset of $\{1, 2, \ldots, n\}$. In [1] it was shown that

$$\nu((3^k + 1)/2) = 2^k \quad \text{for} \quad k < 5.$$

However, in [2], a non-averaging sequence of integers R was constructed in which the number of elements less than or equal to n was larger than $n^{1-c/\sqrt{\log n}}$ where c is an absolute constant. By using an analogous construction, a closed, non-averaging subset of $[0, 1]$ which has Hausdorff dimension 1 will be constructed. (It follows from the Lebesgue Density Theorem that a measurable, non-averaging subset cannot have positive measure.) The set will be constructed so that it has porosity 1 at each of its points, where the porosity of a set E at a point $x \in E$ is

$$\limsup_{h,k \to 0} \{|I|/|h - k| : I \subset [x - k, x + h]\backslash E\}.$$

The ingenious idea behind the non-averaging set R along with a theorem on Hausdorff dimension will facilitate the construction. R is defined to be the set of those

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natural numbers x which have

(i) $n(n+1)$ digits (for some n) with the first digit being 1.

(ii) when the digits in x are divided into blocks of size $2n-1, n-2, n-3, \ldots, 3, 2, 1$ each of the blocks ends in a zero.

(iii) if x_i is the number in the ith block $i = 1, 2, \ldots n-2$ and $\bar{x}$ is the number in the block of length $2n-1$ after the initial digit 1 is removed, then $\bar{x} = \Sigma x_i^2$.

By (ii) it is possible to average two elements of R block by block. If two elements of R have the same number of digits, then their average z has $(x_i + y_i)/2$ in each block and $(\bar{x} + \bar{y})/2$ in the first block preceded by a 1 and 0's to make up $2n-1$ digits. If $z \in R$ then $\bar{z} = \Sigma z_i^2 = \Sigma((x_i + y_i)/2)^2$ but $\bar{z} = (\bar{x} + \bar{y})/2 = \Sigma(x_i^2 + y_i^2)/2$. By subtracting these two equations for $\bar{z}$ it follows that $\Sigma((x_i - y_i)/2)^2 = 0$; that is, $x_i = y_i$ for $i = 1, 2, \ldots n-2$ and hence $\bar{x} = \bar{y}$ and $x = y$.

Only this information about R (that is, the possibility of averaging numbers block by block and producing a non-averaging set by restricting the elements in some of the blocks) will be needed in the construction of the set E.

THEOREM. *Let $\{n_k\}$ be a subsequence of the natural numbers and let $n(m)$ be the number of $j \leq m$ which are not in the subsequence. Let f be a function which takes each finite sequence of digits into a digit. Suppose E is the set of x such that if $x = \Sigma a_i/10^i$, then for each k $a_{n_k} = f(a_1, a_2 \ldots a_{n_{k-1}})$. Suppose there is a number α and a natural number N such that for each $m \geq N$, $n(m) \geq m\alpha - N$. Then* $\dim(E) \geq \alpha$ *and* $\alpha - m(E) > 0$.

PROOF. To prove that $\alpha - m(E) > 0$ it is sufficient to use the intervals from the net $\{[i/10^n, (i+1)/10^n]\}$ (cf. [3, p. 101–104]). Moreover, to show that the α-measure of a compact set is greater than zero, it suffices to use only finite covers with intervals from the net and show that the value $\Sigma|I_j|^\alpha$ is bounded away from 0. Let $H(I) = 0$ if the interior of I does not meet E and let $H(I) = M^{-1}$ if M is the minimum number of intervals of length I in the net needed to cover E. Note that if $I' \subset I$, the interior of I' meets E and $|I| = 10^{-n} = 10|I'|$, then if $n+1 \in \{n_k\}$, $H(I) = H(I')$, otherwise $H(I)/10 = H(I')$. Now suppose that A is a finite cover of E with intervals from the net in which no one interval contains any other and each interval has points of E in its interior. Select one of the smallest intervals in A. Associated with this interval is one of the next larger size from the net which contains the selected one. All the intervals which are the same size as the selected one from the net which contain points of E in their interior and are contained in the associated inverval must be in A. Thus if A' is the cover obtained by replacing these intervals with the associated larger one and retaining all the rest of A, it follows that

$$\sum_{I \in A'} H(I) = \sum_{I \in A} H(I).$$

Continuing this process reduces the cover to the single interval $[0, 1]$ and shows that $\Sigma_{I \in A} H(I) = 1$. Now assume all the intervals in A are no larger than 10^{-N}. The number of intervals of size 10^{-K} needed to cover E equals $10^{n(K)}$. Thus if $I \in A$ and $|I| = 10^{-K}$, then

$$|I|^\alpha = 10^{-K\alpha} > 10^{-n(K)-N} = 10^{-N}H(I).$$

Thus $\Sigma_{I \in A} |I|^\alpha > \Sigma_{I \in A} 10^{-N}H(I) = 10^{-1}$ and hence $\alpha - m(I) > 0$.

EXAMPLE. *There is a non-averaging set* $E \subset [0, 1]$ *such that each point of* E *has porosity* 1 *and* $\dim(E) = 1$.

CONSTRUCTION. Let $S = \{N_p\}$ be an increasing sequence of natural numbers with $N_{p+1} > N_p + 2$. Let $E(S)$ be the set of all $x \in [0, 1]$ such that if

$$x = \Sigma a_i/10^i = \Sigma b_j/10^{s(j)}$$

where $a_i = 0, 1, \ldots, 9$ and $s(j) = j(j+1)/2$ and $b_j = 0, 1, \ldots, s(j) - 1$ and the b_j and a_i satisfy for each $x \in E(S)$,

(i) $b_n + b_{n+1} \cdot 10^{s(n)} = \sum_1^{n-1} b_j^2$ when $n = N_p$

(ii) $a_{s(n)} = 0$ when $n \neq N_p$

(iii) $b_{n+2} = 0$ when $n = N_p$.

Then $E(S)$ is a non-averaging set. This follows from the same argument that was applied to R. By (ii), the elements of $E(S)$ can be averaged by averaging the blocks b_n of their decimals. By (i), if $x, y, z \in E(s)$ and $z = (x+y)/2$, then $x = y = z$ follows from the fact that each of their blocks are equal. By (iii), $E(S)$ is a porous set. For if $x = \Sigma b_j 10^{-s(j)}$, $n = N_p$, $a = \Sigma_1^{n+1} b_j 10^{-s(j)}$, then $x \in [a, a + 10^{-s(n+2)}]$ and $(a + 10^{-s(n+2)}, a + 10^{-s(n+1)})$ does not meet E. Hence, the porosity of E at x is at least $\lim(10^{-s(n+1)} - 10^{-s(n+2)})/10^{-s(n+1)} = 1$. To produce E, choose S so that the N_p increase rapidly. For example, let $N_p = 10^p$. Consider the set $\{n_k\}$ for which points $x \in E$ have their n_k-th decimal determined by (i), (ii) or (iii). Fix m and suppose n, K are such that $n(n+1)/2 > m \geq (n-1)n/2$ and $S(10^K) < m \leq S(10^{K+1})$. Then $n(m) > m - n - 3(10^{K+1} - 1)/9 - 3K$ since at most n a_i's are fixed by (ii) and at most $3(10^{K+1} - 1)/9 + 3K$ a_i's are fixed by (i) and (iii). Thus $n(m) \geq m - \sqrt{2m} - 3(10\sqrt{2m}/9 + 1/2 \log(2m))$. Let $\alpha \in (0, 1)$; then $n(m) - \alpha m \geq m(1-\alpha) - \sqrt{2m} - 3(10\sqrt{2m}/9 + 1/2 \log(2m))$. Since the right-hand side of this inequality approaches ∞, there is an N such that for $m > N$, $n(m) - \alpha m \geq -N$. Thus the set E satisfies the hypotheses of the theorem for each $\alpha \in (0, 1)$ and hence $\dim(E) = 1$.

The referee has suggested the following question: if a closed set E contains no solution of $x_1 + x_2 + \ldots + x_k = kx_{k+1}$ for $x_i \in E$ must the set have Hausdorff dimension less than 1?

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