Canad. Math. Bull. Vol. **59** (3), 2016 pp. 575–584
http://dx.doi.org/10.4153/CMB-2015-009-5

Cohomogeneity One Randers Metrics

Jifu Li, Zhiguang Hu, and Shaoqiang Deng

Abstract. An action of a Lie group G on a smooth manifold M is called *cohomogeneity one* if the orbit space M/G is of dimension 1. A Finsler metric F on M is called *invariant* if F is invariant under the action of G. In this paper, we study invariant Randers metrics on cohomogeneity one manifolds. We first give a sufficient and necessary condition for the existence of invariant Randers metrics on cohomogeneity one manifolds. Then we obtain some results on invariant Killing vector fields on the cohomogeneity one manifolds and use them to deduce some sufficient and necessary conditions for a cohomogeneity one Randers metric to be Einstein.

1 Introduction

Let M be a smooth manifold and G a Lie group. An action of G on M is called *cohomogeneity one* if the orbit space M/G is of dimension 1. This notion was first introduced by Mostert [19]. If G acts on M properly, then there exists a G-invariant complete Riemannian metric h on M. In this case the manifold with the metric h is called a cohomogeneity one Riemannian manifold. Cohomogeneity one Riemannian manifolds have been studied extensively, and many interesting results have been obtained. For example, many interesting new and significant examples, including Einstein metrics and positively curved metrics, have been constructed; see [2–5, 7, 13–15, 19, 21, 23–25]. There are also some studies on cohomogeneity one action on Alexandrov spaces [12], which is a natural synthetic generalization of Riemannian geometry.

The purpose of this paper is to initiate the study of cohomogeneity one action on Finsler spaces. Due to the complexity of the general case, we will focus on Randers spaces. Randers metrics were introduced by G. Randers in the context of general relativity. Hence they have important applications in the theory of relativity. In geometry, Randers metrics provide a rich source of explicit examples that are neither Riemannian nor locally Minkowskian. In [10, 11], Deng and Hou study homogeneous Randers metrics and invariant Einstein–Randers metrics on homogeneous manifolds (see also [9]) and obtain some interesting results. Since a cohomogeneity one Riemannian manifold is a natural generalization of a Riemannian homogeneous manifold, it is interesting to study cohomogeneity one actions on the Finsler manifolds. Hopefully this consideration will lead to serious study on general cohomogeneity one Finsler spaces.

Received by the editors September 15, 2014.
Published electronically April 18, 2016.
Supported by NSFC (no. 11271198, 11221091 11226053, 11301385, 51535008) and SRFPD of China.
The corresponding author is S. Deng.
AMS subject classification: **53C30**, 53C60.
Keywords: Cohomogeneity one actions, normal geodesics, invariant vector fields, Randers metrics.

In Section 2 we recall some definitions and fundamental results on cohomogeneity one Riemannian manifolds and Finsler geometry. Section 3 is devoted to studying invariant Randers metrics on cohomogeneity one manifolds. A complete description of invariant Randers metrics on cohomogeneity one Riemannian manifolds is given. In Section 4, we obtain a complete description of invariant Killing vector fields on cohomogeneity one Riemannian manifolds. This result is used to present some sufficient and necessary conditions for a cohomogeneity one Randers metric to be Einstein.

We remark here that the authors are listed in order based on their contribution.

2 Preliminaries

In this section we first give some fundamental facts about cohomogeneity one Riemannian manifold; for details see [2–4, 7, 27]. Then we recall some definitions and facts about Randers manifolds. We will also fix the notation to be used throughout the paper.

Let M be a manifold and G a connected compact Lie transformation group on M. If G acts effectively on M and has a codimension one orbit, or equivalently, if the orbit space M/G has dimension 1 (see [7, 19]), then M is called a cohomogeneity one G-manifold. Since G is compact, we can choose a G-invariant Riemannian metric h on M. We say that (M, h) is a cohomogeneity one Riemannian manifold. Note that in this case the orbit space $I = M/G$ becomes a one-dimensional metric space under the natural projection $\pi: M \to M/G$. Mostert [19] proved that the metric space M/G must be homomorphism to one of the following:

(i) $I = \mathbb{R}$;
(ii) $I = [0, +\infty)$;
(iii) $I = S^1 = \mathbb{R}/\mathbb{Z}$;
(iv) $I = [0, L]$.

Denote $(0, L)$ by I^0. In the following, we only consider case (iv),as the other cases can be treated similarly. Note that in the case (iv) the manifold M is a compact manifold, and its fundamental group is finite.

Definition 2.1 Let (M, h) be a Riemannian cohomogeneity one manifold. A G-orbit in M is called singular (resp. regular) if the image under the natural projection π is a boundary (resp. internal) point. A point $x \in M$ is called singular (resp. regular) if the orbit $G \cdot x$ is singular (resp. regular). The set of all regular points of M is denoted by M_r.

Definition 2.2 Let (M, h) be a cohomogeneity one Riemannian manifold. A complete geodesic γ is called normal if it is perpendicular to all orbits.

The existence of normal geodesics is proved in [7]. Let $S_1 = \pi^{-1}(0)$ and $S_2 = \pi^{-1}(L)$ be two singular orbits and let γ be a normal geodesic on M initiating at $x_1 \in S_1$, parameterized by arc length. Then γ is a normal geodesic from S_1 to S_2, and $x_2 = \gamma(L) \in S_2$. Let $H = G_{\gamma(L/2)}$ be the isotropy subgroup of G at the midpoint $\gamma(L/2)$. Then for any $0 < t < L$, we have $H = G_{\gamma(t)}$. Hence H preserves any point in the normal geodesic. Let $K_i = G_{x_i}, (i = 1, 2)$ be the singular isotropy subgroups. Then we have $H \subset \{K_1, K_2\} \subset G$, and it can be proved that the coset spaces K_i/H, $(i = 1, 2)$ are spheres [2]. Every principal orbit is homomorphic to the reductive homogeneous space G/H. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{h} = \mathrm{Lie}\, H$. Fix a bi-invariant inner product Q on $\mathfrak{g}$ and let $\mathfrak{m} = (\mathfrak{h})^{\perp}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then we have a direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, which satisfies the condition

$$\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}, \quad \forall h \in H.$$

For each $t \in I^0$, the tangent space $T_{\gamma(t)}(G \cdot \gamma)$ can be identified with $T_H(G/H)$ via the fundamental vector field $\widetilde{X}$ and $X \in \mathfrak{m}$. On the other hand, any G-invariant metric h on M_r must be of the following form:

$$h = dt^2 + h_t,$$

where h_t is a G-invariant metric on G/H. Define

$$h(\widetilde{X}, \widetilde{Y})_{\gamma(t)} = h_t(X, Y) = Q(\mathrm{P}_t X, Y), \forall\ X, Y \in \mathfrak{m},$$

where $\mathrm{P}_t\colon \mathfrak{m} \to \mathfrak{m}$ is a Q symmetric $\mathrm{Ad}(H)$-equivalent endmorphism, and $\widetilde{X}$ is a fundamental vector field on M_r generated by $X \in \mathfrak{m}$; see [16] for details.

Proposition 2.3 (see [9, 11] or [10]) *There is a bijection between the set of invariant vector fields on $(G/H, h)$ and the subspace*

$$V := \{X \in \mathfrak{m} \mid \mathrm{Ad}(h)X = X, \forall h \in H\}.$$

Furthermore, the vector field $\widetilde{X}$ on $(G/H, h)$ generated by $X \in V$ is a G-invariant Killing vector field if and only if X satisfies $h([X, Y]_{\mathfrak{m}}, Z) + h(Y, [X, Z]_{\mathfrak{m}}) = 0$, for all $Y, Z \in \mathfrak{m}$.

Lemma 2.4 ([7]) *Let (M, h) be a cohomogeneity one Riemannian manifold such that M/G is a compact space. If (M, h) is Ricci flat, then h is flat.*

We now recall some results on Randers metrics. A Randers metric is a Finsler metric of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form whose length with respect to α is everywhere less than 1. There is another way to express such metrics, namely,

$$F(x, y) = \frac{\sqrt{h(W, y)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(W, y)}{\lambda}, \tag{2.1}$$

where h is a Riemannian metric, $y \in T_x M$, W is a vector field on smooth manifold M with $h(W, W) < 1$, and $\lambda = 1 - h(W, W)$ [6]. We call the pair (h, W) the *navigation data* of the Randers metric F. If F is a G invariant Randers metric on a cohomogeneity one manifold M, then (M, F) is called a *cohomogeneity one Randers space*.

A Finsler metric F on M is called Einstein if its Ricci scalar $\mathrm{Ric}(x, y)$, where $x \in M$ and $y \in T_x(M)\backslash\{0\}$, has no dependence on the direction y [5]. The following result is a kind of Schur's Lemma in Finsler geometry.

Lemma 2.5 ([5]) *The Ricci scalar of an Einstein–Randers metric in dimension greater than 2 is necessarily constant.*

https://doi.org/10.4153/CMB-2015-009-5 Published online by Cambridge University Press

In the general case, it is still an open problem whether the above lemma is true. Obviously, if (M,F) is a homogeneous Einstein–Finsler space, then the Ricci scalar must be constant.

3 Invariant Randers Metrics on Cohomogeneity One Riemannian Manifolds

Let (M,h) be a cohomogeneity one Riemannian manifold under the action of a compact Lie group G with $M/G=[0,L]$. Fix a a normal geodesic γ on (M,h). Let $H=G_{\gamma(L/2)}$ and $K_1=G_{\gamma(0)}(K_2=G_{\gamma(L)})$ be the principal isotropy subgroups and singular isotropy subgroup, respectively. Then $H\subset\{K_1,K_2\}\subset G$ and the coset spaces $K_i/H,(i=1,2)$ are spheres with the induced Riemannian metrics.

Define a map $\varphi\colon G/H\times I^0\to M$ by $\varphi_t(gH):=\varphi(gH,t)=g\cdot\gamma(t)$. Obviously, φ_t is well defined and is a G equivariant diffeomorphism from G/H to each principal orbit. Given a vector field X on G/H, define $\widetilde{X}_t=(\varphi_t)_*X,\forall t\in I^0$. Then for each $t\in I^0$, $\widetilde{X}_t$ is a vector field on $G\cdot\gamma(t)$. Conversely, for each $t\in I^0$, a vector field $\widetilde{Y}_t\in T(G\cdot\gamma(t))$ on $G\cdot\gamma(t)$ can be given by $\widetilde{Y}_t=(\varphi_t)_*Y$, where Y is a vector field on G/H. Hence, the vector fields on a principal orbit $G\cdot\gamma(t)$ are in one to one correspondence to that on G/H. Whence we have the following lemma.

Lemma 3.1 *Let X be a G-invariant vector field on G/H. Then $\widetilde{X}_t$ defined as above is a G-invariant vector field on $G\cdot\gamma(t),\forall t\in I^0$, and vice versa.*

By Proposition 2.3 and Lemma 3.1, we see that there is a bijection between the G-invariant vector fields on a principal orbit $G\cdot\gamma(t)$ and the space

$$V=\{u\in\mathfrak{m}\mid \mathrm{Ad}(h)u=u,\forall h\in H\}.$$

Hence,

$$\widetilde{u}|_{g\gamma(t)}=\frac{d}{ds}\Big|_{s=0}\exp(su)\gamma(t),\quad \forall t\in I^0,g\in G,u\in V,$$

is a G-invariant vector field that is tangent to each principal orbit.

Now let $u\in V$. If for any $k\in K_1$ (resp. K_2), we have $\mathrm{Ad}(k)u=u$, then the induced vector field $\widetilde{u}$ generated by u on a singular orbit G/K_1 (resp. G/K_2) is a G-invariant vector field. The converse statement is obviously true.

Now we summarize the above results. We first define a vector field. Let $u\in V$ and $g\in G$.

(i) If $\mathrm{Ad}(k)u=u,\forall k\in K_i,i=1,2$, define

$$\widehat{u}|_{g\gamma(t)}=\frac{d}{ds}\Big|_{s=0}g\exp(su)\cdot\gamma(t),\quad \forall t\in I.$$

(ii) If $\mathrm{Ad}(k)u=u,\forall k\in K_1$, but $\mathrm{Ad}(k)u\neq u$, for some $k\in K_2$, define

$$\widehat{u}|_{g\gamma(t)}=\begin{cases}0, & t=L,\\ \frac{d}{ds}\big|_{s=0}g\exp(su)\cdot\gamma(t), & t\in[0,L).\end{cases}$$

(iii) If $\mathrm{Ad}(k)u \neq u$, for some $k \in K_1$, but $\mathrm{Ad}(k)u = u$, for $k \in K_2$, define

$$\widehat{u}|_{g\gamma(t)} = \begin{cases} 0, & t = 0, \\ \frac{d}{ds}\big|_{s=0} g\exp(su)\cdot\gamma(t), & t \in (0, L]. \end{cases}$$

(iv) If $\mathrm{Ad}(k)u \neq u$, $k \in K_i$, $i = 1, 2$, define

$$\widehat{u}|_{g\gamma(t)} = \begin{cases} 0, & t = 0, L, \\ \frac{d}{ds}\big|_{s=0} g\exp(su)\cdot\gamma(t), & t \in (0, L). \end{cases}$$

Then we have the following theorem.

Theorem 3.2 *The vector field $\widehat{u}|_{g\gamma(t)}$ defined above is a G-invariant vector field on the manifold M.*

Let $T = \frac{\partial}{\partial t}$ be the geodesic vector field along the normal geodesic $\gamma(t)$, $t \in I$.

(i) If $K_1 = H = K_2$, define

$$\widehat{T}_{g\gamma(t)} = (d\tau_g)_{\gamma(t)} T, \quad \forall t \in I.$$

(ii) If $K_1 = H$ and $H \neq K_2$, define

$$\widehat{T}|_{g\gamma(t)} = \begin{cases} 0, & t = L, \\ (d\tau_g)_{\gamma(t)} T, & t \in [0, L). \end{cases}$$

(iii) If $K_1 \neq H$ and $H = K_2$, define

$$\widehat{T}|_{g\gamma(t)} = \begin{cases} 0, & t = 0, \\ (d\tau_g)_{\gamma(t)} T, & t \in (0, L], \end{cases}$$

where τ_g is the transformation of G/H defined by $\tau_g\colon g'H \to gg'H$. Then $T_{g\gamma(t)}$ is a well-defined and G-invariant vector field on M. Hence any G-invariant vector field $\widehat{X}$ on M can be uniquely written as

$$\widehat{X} = c_1\widehat{u_1}|_{g\gamma(t)} + \cdots + c_s\widehat{u_s}|_{g\gamma(t)} + c_{s+1}\widehat{T}|_{g\gamma(t)},$$

where $u_1, \ldots, u_s$ is a basis of V, and $c_1, \ldots, c_{s+1}$ are G-invariant functions on M.

Since each G-invariant Randers metric on manifold M can be constructed by a navigation data (h, U), where h is a Riemannian metric and U is a G-invariant vector field with $h(U, U) < 1$, we have the following theorem.

Theorem 3.3 *Let (M, h) be a cohomogeneity one Riemannian manifold with $M/G = I = [0, L]$. Then there is a bijection between the set of G-invariant Randers metrics with underlying Riemannian manifold (M, h) and the set*

$$W := V \cup \{T\} = \{u \in \mathfrak{m} \mid \mathrm{Ad}(h)u = u, \forall h \in H\} \cup \{T\}.$$

https://doi.org/10.4153/CMB-2015-009-5 Published online by Cambridge University Press

4 Killing Vector Fields and Invariant Einstein-Randers Metrics

In the above section we have described invariant Randers metrics on a cohomogeneity one Riemannian manifold. In this section we will study some geometric properties of the invariant Randers metric. In particular, we will prove that an invariant Randers metric on a cohomogeneity one Riemannian manifold is an Einstein metric if and only if in the navigation data the Riemannian metric is an Einstein metric and the corresponding vector field is a Killing vector field with respect to the Riemannian metric. This shows that Killing vector fields of a cohomogeneity one Riemannian manifold will play an important role in our study.

We first give two ways of constructing Killing vector fields on (M, h).

The first construction. Let $X \in V$. Then $\widetilde{X}$ is an G-invariant vector field on $(G/H \times I^0, dt^2 + h_t)$, that is,

$$\widetilde{X}|_{g\gamma(t)} = \frac{d}{ds}|_{s=0} g \exp(sX) \cdot \gamma(t).$$

Then $\widetilde{X}$ is a Killing vector field if and only if the corresponding one parameter transformation group

$$\phi_s \colon M_r \to M_r, \quad g\gamma(t) \to g \exp(sX) \cdot \gamma(t), t \in I^0,$$

consisting of isometries of h. In particular, we have

$$h(\widetilde{Y}_1, \widetilde{Y}_2)|_{r(t)} = h\big(d\phi_s(\widetilde{Y}_1), d\phi_s(\widetilde{Y}_2)\big)|_{r(t)}.$$

A direct calculation (see [9] or [10]) then shows that

$$d\phi_s(\widetilde{Y}_i) = d\tau_{\exp(sX)}\big(\widetilde{e^{\mathrm{ad}(sX)}(Y_i)}\big), \quad i = 1, 2,$$

where $\tau_{\exp(sX)}$ is the transformation of G/H defined by $gH \to \exp(sX)gH$.

By the G-invariance of h_t, we have

$$h_t(\widetilde{Y}_1, \widetilde{Y}_2)|_{r(t)} = h_t\big(\widetilde{e^{\mathrm{ad}(sX)}(Y_1)}, \widetilde{e^{\mathrm{ad}(sX)}(Y_2)}\big)|_{r(t)},$$

that is,

$$Q(P_t Y_1, Y_2) = Q\big(P_t e^{\mathrm{ad}(sX)}(Y_1), e^{\mathrm{ad}(sX)}(Y_2)\big).$$

Taking the derivative with respect to s and considering the value at $s = 0$, we get

$$Q\big(P_t[X, Y_1], Y_2\big) + Q\big(P_t[X, Y_2], Y_1\big) = 0, \quad \text{for all } t \in I^0, Y_1, Y_2 \in m.$$

Conversely, if the above formula holds, then a backward argument shows that $\widetilde{X}$ is a Killing vector field on (M_r, h). Combining with Theorem 3.2, we have the following proposition.

Proposition 4.1 *Let (M, h) be a cohomogeneity one Remannian G-manifold. Suppose $X \in V$. Then the induced vector field $\widehat{X}$ defined before Theorem 3.2 is a Killing vector field on (M, h) if and only if X satisfies*

$$Q\big(P_t[X, Y_1], Y_2\big) + Q\big(P_t[X, Y_2], Y_1\big) = 0, \quad \textit{for all } t \in I^0, Y_1, Y_2 \in m.$$

Note The Killing vector field $\widehat{X}$ constructed in Proposition 4.1 may not be smooth on the manifold M.

The second construction. Since $h = dt^2 + h_t$, the Riemannian manifold $(G/H \times I^0, h)$ is a warped product (see [8]). Let π_1 and π_2 be the natural projection from M onto G/H and I^0, respectively. We call $\{gH\} \times I^0 = \pi_1^{-1}(gH)$ the fibers and $G/H \times \{t\} = \pi_2^{-1}(t)$ the leaves. Vectors tangent to the leaves are called *horizontal*, and those tangent to the fibers are called *vertical*. We identify the vector field on G/H with the π_1 related vertical vector field on $G/H \times I^0$, and the vector field on I^0 with the π_2 related horizontal vector field on $G/H \times I^0$.

Let X be a vector field on $G/H \times I^0$. Denote $X_1 := (\pi_{1*}(X), 0) = \pi_{1*}(X)$ and $X_2 := (0, \pi_{2*}(X)) = \pi_{2*}(X)$. Then X_1 is a vector field on $G/H \times \{t\}, \forall t \in I^0$, and X_2 is a vector field on $\{gH\} \times I^0, \forall g \in G$. Since $L_{X_1} dt^2 = 0$, we have the following lemma.

Lemma 4.2 *Let X be a vector field on $(G/H \times I^0, h)$ with $h = dt^2 + h_t$. Then*

$$L_X h = L_{X_2} dt^2 + L_X h_t.$$

In particular, if $h_t = f(t)g$, then $L_X h = L_{X_2} dt^2 + X_2(f)g + f L_{X_1} g$, where f is a smooth positive function on I^0 and g is a homogeneous metric on G/H.

The following proposition follows from Lemma 4.2.

Proposition 4.3 *A vector field Y is a Killing vector field on $(G/H, h_t)$ if and only if its horizontal lift $\overline{Y}$ is a Killing vector field on $G/H \times I^0$.*

In particular, we have the following proposition.

Proposition 4.4 *If $h = dt^2 + f(t)g$, where f is a smooth positive function on I^0 and g is a homogeneous metric on G/H, then we have*

(i) *a vector field Y is a Killing vector field on $(G/H, h_t)$ if and only if its horizontal lift $\overline{Y}$ is a Killing vector field on $(G/H \times I^0, h)$;*

(ii) *a vector field Z is a Killing vector field of (I^0, dt^2) and $Z(f) = 0$ if and only if its vertical lift $\overline{Z}$ is a Killing vector field on $(G/H \times I^0, h)$.*

Remark Proposition 4.4 is also true for the cases of the warped product of two semi-Riemannian manifold and the generalized Robertson-Walker spacetimes; see [20, exercise 2] and [22, Proposition 3.7].

If $Zh_t \neq 0$, then Z is not a Killing vector field on $G/H \times I^0$ by Proposition 4.4. Hence, we only need to consider G-invariant Killing vector fields on (M_r, h) generated by the natural lift of Killing vector fields on $(G/H, h_t)$.

Combining Proposition 2.3 and Proposition 4.3, we have the following corollary.

Corollary 4.5 *With the same assumptions as in Proposition 4.4, Let*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

be the decomposition of $\mathfrak{g} = \mathrm{Lie}\, G$. Define

$$V = \{X \in \mathfrak{m} \mid \mathrm{Ad}(h)X = X, \forall h \in H\}.$$

https://doi.org/10.4153/CMB-2015-009-5 Published online by Cambridge University Press

If $X \in V$ satisfies $Q(P_t[X,Y]_{\mathfrak{m}}, Z) + Q(P_t Y, [X,Z]_{\mathfrak{m}}) = 0$, $\forall Y, Z \in \mathfrak{m}$, $t \in I^0$, then the vector field $\widetilde{X}$ on $(G/H, h)$ generated by $X \in V$ is a G-invariant Killing vector field, and the horizontal lift $\overline{X}$ of $\widetilde{X}$ is a G-invariant Killing vector field on (M_r, h). Furthermore, as in the proof of Theorem 3.2, we can obtain a G-invariant Killing vector field $\widehat{X}$ on (M, h).

If $W \in V$, then $\widehat{W}$ defined in Theorem 3.2 is a G-invariant vector field on (M, h). Let

$$F = F(g\gamma(t), y), t \in I, y \in T_{g\gamma(t)}M$$

be given by (2.1). Then F is a G-invariant Randers metric on M. Let $A(t)$, $t \in I^0$ be the Cartan tensor of the Randers metric F. If F is a smooth Randers metric, then $A(t)$ is continuous on I, *i.e.*, W satisfies Theorem 3.2(i). On the other hand, if $\lim_{x\to 0} A(t)$ and $\lim_{x\to L} A(t)$ exist, then we can redefine $A(t)$ to make it continuous on I. In both cases, we call $A(t)$ complete on I. Now we can prove the main result of this paper.

Theorem 4.6 *Let M be a manifold with $\dim M \geq 2$ and G a compact Lie group that acts on M such that M is a cohomogeneity one G-manifold with $M/G = I = [0, L]$. Let H be the principal isotropy subgroup of G with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let $W \in \mathfrak{m}$ be an H fixed vector with $Q(P_t W, W) < 1$ and let $\widehat{W}$ be the induced G-invariant vector field on (M, h). If the Cartan tensor $A(t)$ of the induced Randers metric F with navigation data $(h, \widehat{W})$ is complete, then the Randers metric F is Einstein–Randers with Ricci constant K on (M_r, h) if and only if h is Einstein with Ricci constant K and W satisfies*

$$Q(P_t[W, Y_1], Y_2) + Q(P_t[W, Y_2], Y_1) = 0, \quad \forall t \in I^0, Y_1, Y_2 \in m.$$

Proof By the result of Bao and Robles [5], the Randers metric (M, F) with navigation data (h, W) is Einstein if and only if h is an Einstein Riemannian metric and W is a homothetic vector field. Moreover, W must be a Killing vector field if h has nonzero scalar curvature. Now we prove the "only if" part. There are two cases:

(a) (M, h) is not Ricci flat. Then W must be a Killing vector field, and the assertion follows from the result of Bao and Robles [5].
(b) (M, h) is Ricci flat. Then by Lemma 2.4, (M, h) must be flat. We assert that in this case the vector field $\widehat{W}$ must also be a Killing vector field. In fact, otherwise $\widehat{W}$ must be a homothetic vector field with dilation $\sigma \neq 0$. Then by Bao, Robles, and Shen's result in [6], the Randers metric F must be of constant flag curvature $-\frac{1}{16}\sigma^2$ on the regular part (M_r, h). Since the Cartan tensor is complete, Akbar-Zadeh's theorem [1] then implies that F must be Riemannian on (M_r, h). Hence, $\widehat{W} = 0$ on (M_r, h). By the definition of $\widehat{W}$, we get $\widehat{W} = 0$ on (M, h), which is a contradiction. Hence, $\widehat{W}$ must be a Killing vector field, and the assertion follows.

The "if" part can be proved by a backward argument. ∎

Remark Theorem 4.6 is true in the case where $M/G = S^1$ without the condition of the Cartan tensor being complete. The proof is similar and will be omitted.

Finally, we give an example to describe some of the results in this paper.

https://doi.org/10.4153/CMB-2015-009-5 Published online by Cambridge University Press

Example 4.7 Let (M,h) be a $(2m+2)$-dimensional compact Riemannian manifold, where $M = [a,b] \times SU(m+1)/SU(m)$, $(a < b)$. Then the action of $SU(m+1)$ on M is cohomogeneity one, with the principal orbit $SU(m+1)/SU(m)$. Let $h = dt^2 + f(t)g_0$, where f is a smooth positive function on $[a,b]$ and g_0 is the standard Riemannian metric on $SU(m+1)/SU(m)$. If there is a positive constant λ satisfying the equations

$$(2m+1)f'' + \lambda f = 0,$$
$$Ric_0 = (\lambda f^2 + ff'' + 2mf'^2)g_0,$$

where Ric_0 is the Ricci tensor of g_0, then (M,h) is an Einstein manifold with Einstein constant λ and has constant sectional curvature by [7]. From [26], we know that there is a bijection between the set of invariant Killing vector fields on $SU(m+1)/SU(m)$ and the subspace

$$V = \left\{ \begin{pmatrix} -\frac{c\sqrt{-1}}{m} I_m & \\ & c\sqrt{-1} \end{pmatrix} \Big| \, c \in \mathbb{R} \right\}.$$

Let W be a vector field on $SU(m+1)/SU(m)$ generated by $w \in V$ with $f(t)g_0(w,w) < 1$, $t \in [a,b]$. Then the Randers metric F with navigation data $(g, \overline{W})$ is an Einstein–Randers metric with Ricci constant λ and has constant flag curvature on (M,h), where $\overline{W}$ is the horizontal lift of W to M.

References

[1] H. Akbar-Zadeh, *Sur les espaces de Finsler à courbures sectionnelles constantes.* Acad. Roy. Belg. Bull. Cl. Sci. **74**(1988), no. 10, 281–322.

[2] A. V. Alekseevsky and D. V. Alekseevsky, *G-manifolds with one dimensional orbit space.* In: Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math., 8, American Mathematical Society, Providence, RI, 1992, pp. 1–31.

[3] A. V. Alekseevsky and D. V. Alekseevsky, *Riemannian G-manifold with one-dimensional orbit space.* Ann. Glob. Anal. Geom. **11**(1993), no. 3, 197–211.

[4] D. V. Alekseevsky, *Riemannian manifolds of cohomogeneity one.* In: Differential geometry and its applications (Eger, 1989), Colloq. Math. Soc. János Bolyai, 56, North-Holland, Amsterdam, 1992, pp. 9–22.

[5] D. Bao and C. Robles, *Ricci and flag curvatures in Finsler geometry.* In: A sampler of Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ., 50, Cambridge University Press, Cambridge, 2004, pp. 197–260.

[6] D. Bao, C. Robles, and Z. Shen, *Zermelo navigation on Riemannian manifolds.* J. Differential Geom. **66**(2004), 377–435.

[7] L. Bèrard Bergery, *Sur les nouvelles variétés riemanniennes d'Einstein.* Publ. Inst. E. Cartan. **4**(1982), 1–60.

[8] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature.* Trans. Amer. Math. Soc. **145**(1969), 1–49. http://dx.doi.org/10.1090/S0002-9947-1969-0251664-4

[9] S. Deng, *Homogeneous Finsler spaces.* Springer Monographs in Mathematics, Springer, New York, 2012.

[10] S. Deng and Z. Hou, *Invariant Randers metrics on homogeneous Riemanniann manifold.* J. Phys. A **37**(2004), no. 15, 4353–4360; Corrigendum, J. Phys. A **39**(2006), no. 18, 5249–5250. http://dx.doi.org/10.1088/0305-4470/37/15/004

[11] ———, *Homogeneous Einstein-Randers spaces of negative Ricci curvature.* C. R. Math. Acad. Sci. Paris. **347**(2009), 1169–1172. http://dx.doi.org/10.1016/j.crma.2009.08.006

[12] F. Galaz-Garcia and C. Searle, *Cohomogeneity one Alexandrov spaces.* Transform. Groups **16**(2011), no. 1, 91–107. http://dx.doi.org/10.1007/s00031-011-9122-0

[13] K. Grove, L. Verdiani, B. Wilking, and W. Ziller, *Non-negative curvature obstruction in cohomogeneity one and the Kervaire spheres.* Ann. Sc. Norm. Super. Pisa Cl. Sci. **5**(2006), no. 2, 159–170.

[14] K. Grove, B. Wilking, and W. Ziller, *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry.* J. Differential Geom. **78**(2008), no. 1, 33–111.
[15] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres.* Ann. of Math. **152**(2000), no. 1, 331–367. http://dx.doi.org/10.2307/2661385
[16] ———, *Cohomogeneity one manifolds with positive Ricci curvature.* Invent. Math. **149**(2002), no. 3, 619–646. http://dx.doi.org/10.1007/s002220200225
[17] W. Y. Hsiang and B. Lawson, *Minimal submanifolds in low cohomogeneity.* J. Differential Geometry **5**(1971), 1–38.
[18] S. Kobayashi, *Transformation groups in differential geometry.* Classics in Mathematics. Springer-Verlag, Berlin, 1995.
[19] P. S. Mostert, *On a compact Lie group acting on a manifold.* Ann. of Math. **65**(1957), 447–455. http://dx.doi.org/10.2307/1970056
[20] B. O'Neill, *Semi-Riemannian geometry. With applications to relativity.* Pure and Applied Mathematics, 103, Academic Press, New York, 1983.
[21] F. Podesta and L. Verdiani, *Positively curved* 7*-dimensional manifolds.* Quart. J. Math. Oxford. Ser. **50**(1999), no. 200, 497–504. http://dx.doi.org/10.1093/qjmath/50.200.497
[22] M. Sánchez, *On the geometry of generalized Robertson-Walker spacetimes: curvature and Killing fields.* J. Geom. Phys. **31**(1999), no. 1, 1–15. http://dx.doi.org/10.1016/S0393-0440(98)00061-8
[23] C. Searle, *Cohomogeneity one and positive curvature in low dimension.* Math. Z. **214**(1993), no. 3, 491–498; Corrigendum, **226**(1997), no. 1, 165–167.
[24] L. Verdiani, *Cohomogeneity one Riemannian manifolds of even dimension with strictly positive sectional curvature. I.* Math. Z. **241**(2002), no. 2, 329–339. http://dx.doi.org/10.1007/s002090200417
[25] ———, *Cohomogeneity one manifolds of even dimension strictly positive sectional curvature.* J. Differential Geom. **68**(2004), no. 1, 31–72.
[26] H. Wang, L. Huang, and S. Deng, *Homogeneous Einstein-Randers metrics on spheres.* Nonlinear Anal. **74**(2011), no. 17, 6295–6301. http://dx.doi.org/10.1016/j.na.2011.06.008
[27] W. Ziller, *On the geometry of cohomogeneity one manifolds with positive curvature.* In: Riemannian topology and geometric structures on manifolds. Progr. Math., 271, Birkhäser Boston, Boston, MA, 2009, pp. 233–262.

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People's Republic of China
and
School of Science, Tianjin University of Technology, Tianjin 300384, People's Republic of China
e-mail: lijifu@mail.nankai.edu.cn

College of Mathematics, Tianjin Normal University, Tianjin 300387, P.R. China
e-mail: nankaitaiji@mail.nankai.edu.cn

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People's Republic of China
e-mail: dengsq@nankai.edu.cn